

APPLICATION OF HIDDEN MARKOV CHAINS IN QUALITY CONTROL

Hanife DEMIRALP¹, Ehsan MOGHIMIHADJI²

¹Department of Industrial Engineering, Eastern Mediterranean University,
Famagusta, Mersin

²Department of Industrial Engineering, Istanbul Aydin University, Florya, Istanbul

E-mail: h_demiralp01@hotmail.com, moqimi@yahoo.com

Abstract- *The ever growing technological innovations and sophistication in industrial processes require adequate checks on quality. Thus, there is an increasing demand for simple and efficient quality control methods. In this regard the control charts stand out in simplicity and efficiency. In this paper, we propose a method of controlling quality based on the theory of hidden Markov chains. Based on samples drawn at different times from the production process, the method obtains the state of the process probabilistically. The main advantage of the method is that it requires no assumption on the normality of the process output.*

Keywords: *Quality Control, Hidden Markov Chains, Control Charts, Assignable Causes, Transition Probability*

1. INTRODUCTION

Controlling the quality of products is of paramount importance in the manufacturing as well as service sectors. Even though quality control has been practiced from time immemorial, either directly or indirectly, a quantitative approach was started only after World War II.

With the rapidly changing technologies and modern day fierce competition, the need for statistical process control (SPC) is felt more than ever. SPC is a procedure by which sample data is collected from the production process, organized, analyzed, and interpreted so as to ensure that the current level of quality of the production process is maintained or improved further. Modern day quality control experts have used several tools such as graphical methods, reliability methods, neural network methods, and control charts in their quest to control quality.

Walter A. Shewhart first proposed the general theory of control charts which is a graphical tool to monitor the activity of an ongoing process. He realized that variability is a part of any ongoing process. The process variability can be broadly classified into two categories: variability caused by assignable causes which are not part of the process affecting only some of the items. Typical examples are defective raw materials and untrained operator. The

purpose of the control charts is to detect the presence of such causes if they exist so that immediate remedial action can be carried out.

The other variability is due to what are known as chance causes which are inherent in any process and cannot be totally eliminated. Examples are small voltage fluctuations and vibration of machines. Shewhart called a process operating only under chance causes as a process under statistical control.

Normal distribution plays a crucial role in the detection of assignable causes. The crux of control charts is the fact that when the process is under statistical control, the variations in the output can be captured with probabilities.

The control charts provide a base line for initiating and measuring quality improvements. Apart from the fact that quality improves productivity, SPC can also help in cutting various costs. They provide valuable signals about why the company incurs cost and reveal opportunities to cut costs (Roth (2005)). An important assumption in the design of control charts is the assumption that the measurable quality characteristic is normally distributed or reasonably normal. However, in many situations we may have reasons to believe that the underlying distribution is far from normal. Amhemad (2010) used a positively skewed distribution to

show the effect of non normality on the significance levels of the control charts. He showed that the sign and size of error in significance level depends on the sample size. With the technological advancement and sophistication in products, there is a need for more methods of controlling their quality, although there is no single method which can be used for all products at all times. In this paper, we propose a new method of quality control based on the theory of hidden Markov chains. The proposed method uses the information provided by all the previous samples to check whether the process is under statistical control or not. We wish to state that since the method proposed is new, the determination of some of the process parameters of the method are still not clear. Much work needs to be done to bring the method to a working order. In section two, we give a brief outline of the basic ideas of hidden Markov chains that are needed in our formulation.

2.HIDDEN MARKOV CHAINS

Since Andrei Andreivich Markov introduced the concept of Markov dependence in stochastic processes in 1906, the theory of Markov processes has made great strides and is being applied in every branch of science and arts. Markov chains are discrete time, discrete state stochastic processes $\{X_n, n = 0, 1, 2, \dots\}$ which take on a finite or a countable number of states and possesses the Markov property. The A model of the preceding type in which the sequence of signals S_1, S_2, \dots are observed while the sequence of underlying Markov chains states X_1, X_2, \dots are unobserved, is called a hidden Markov chain model" (Ross (2006)). At this point we wish to observe that although the sequence $\{X_n, n = 1, 2, 3, \dots\}$ is a Markov chain, the sequence of signals transition probabilities and are basic in the study of Markov chains. The transition probabilities are generally written in the form of matrix known as one step transition probability matrix which is given below;

$$P = \begin{bmatrix} P_{00} & P_{01} & \vdots & \dots \\ P_{10} & P_{11} & \vdots & \dots \\ \vdots & \vdots & P_{ij} & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

The following relation which can be easily established shows that the matrix P together with the initial distribution completely specifies a Markov chain.

$$P\langle X_r = a, X_{r+1} = b, \dots, X_{r+n-2} = i, X_{r+n-1} = j, X_{r+n} = k \rangle = P_{ab} \dots x P_{ij} x P_{jk} x P(X_r = a) \quad (2)$$

Markov chains have been applied in diverse fields like communication, weather prediction, and production to mention a few. A good treatise on Markov chains can be found in Ross, M. Sheldon (1996). In this section, we present some basic ideas of hidden Markov chains.

Consider P_{ij} as transition probabilities for a Markov Chain $\{X_n, n = 0, 1, 2, \dots\}$ which has initial state probabilities $p_i = P\{X_1 = i\}, i \geq 0$. Consider a finite set of signals γ , and whenever a signal is emitted from γ , then the Markov chain enters state j , independently of the previous signals and states of the Markov chain, the emitted signal is s with probability $P(s|j)$, with $\sum_{s \in \gamma} P(s|j) = 1$. It means that, if the n^{th} signal emitted is represented by S_n , then

$$P\{S_1 = s | X_1 = j\} = p(s|j) \\ P\{S_n = s | X_1, S_1, \dots, X_{n-1}, S_{n-1}, X_n = j\} = p(s|j) \quad (3)$$

A model of the preceding type in which the sequence of signals S_1, S_2, \dots are observed while the sequence of underlying Markov chains states X_1, X_2, \dots are unobserved, is called a hidden Markov chain model" (Ross (2006)). At this point we wish to observe that although the sequence $\{X_n, n = 1, 2, 3, \dots\}$ is a Markov chain, the sequence of signals $\{S_n, n = 1, 2, 3, \dots\}$ is not a Markov chain.

Consider a random vector of the first n signals as $S^n = (S_1, S_2, \dots, S_n)$. Let $s_k = (s_1, s_2, \dots, s_n), k \leq n$ for a specified sequence of signals s_1, s_2, \dots, s_n . Now define $S^n = s_n$ as the conditional probability of the Markov chain state at time n . It is well known (Ross (2006), page 258) that

$$F_n(j) = P\{s_n | j\} \times \sum_i F_{n-1}(i) \times P_{ij} \quad (4)$$

where $F_n(j) = P\{S^n = s_n, X_n = j\}$, and note that

$$P\{X_n = j | S^n = s_n\} = \frac{F_n(j)}{\sum_i F_n(i)} \quad (5)$$

We can use equation (4) recursively to obtain $F_2(i), F_3(i), \dots, F_n(i)$, starting with $F_1(i) = P\{X_1 = i, S_1 = s_1\} = p_i \times p(s_1 | i)$.

In order to calculate $P\{S^n = s_n\}$ we employ the identity $P\{S^n = s_n\} = \sum_i F_n(i)$ where $F_n(i)$'s are obtained recursively using (4).

As a second prediction problem, let us consider the sequence of states as a single entity and choose our objective as to select sequence of states whose conditional probability given the sequence of signals is a maximum. For instance in signal processing, while X_1, X_2, \dots, X_n may be the actual message sent, S_1, S_2, \dots, S_n are the signals that are received so that our objective is to predict the actual message in its entirety. A similar problem arises in quality control. For instance X_1, X_2, \dots, X_n may represent the actual states of the system, while S_1, S_2, \dots, S_n are the sample values taken from the system at each of the time points. Based on the sample values the quality control engineer is interested in determining the states of the system. We present below an algorithm to find the most likely sequence of states given a prescribed sequence of signals, which is known in the literature as Viterbi Algorithm. Defining the vector of the first k states as $X_k = (X_1, X_2, \dots, X_k)$, the problem is to find the sequence of states i_1, i_2, \dots, i_n which maximizes $P\{X_n = (i_1, i_2, \dots, i_n) | S^n = s_n\}$.

In order to solve this problem, let for $K \leq n$, $V_k(j) = \max_{i_1, i_2, \dots, i_{k-1}} P\{X_{k-1} = (i_1, i_2, \dots, i_{k-1}), X_k = j, S^k = s_k\}$ (6)

It is well known (Ross (2006), page 262) that

$$V_k(j) = p(s_k | j) \times \max_i P_{ij} \times V_{k-1}(i) \quad (7)$$

In order to calculate $V_n(j)$, one can start with

$$V_1(j) = P\{X_j = j, S_1 = s_1\} = p_j \times p(s_1 | j),$$

and then use equation (7) recursively to determine $V_2(j)$; then $V_3(j)$; and so on, up to $V_n(j)$ for each j .

In the following, we will briefly mention a few interesting areas which have found applications of hidden Markov chains. In a tutorial Rabiner R. Lawrence (1989) has applied hidden Markov chains to select problems in machine recognition of speech. A similar application of the theory was made to image processing problems by Derrode *et al.* (2006). A major area of application of the theory is in bioinformatics. Chadeau-Hyam Marc *et al.* (2010) have applied hidden Markov models to predict the scale of French variant Creutzfeldt – Jakob disease epidemic in France.

3.APPLICATION

In this section, we analyze how the ideas of hidden Markov chains could be usefully applied to process control. In order to explain the motivation as well as understand the procedure, we present below an example which clearly brings out the analogy between quality control and hidden Markov chains.

Suppose that we are interested in controlling a particular quality characteristic of a production process which could be in one of the two states, under control (state 1) and out of control (state 2). Samples are drawn from the output at specific instants of time $t_0, t_1, t_2, \dots, t_n$.

We will call the sample of observations at various instants as signals. The signals are restricted to two types namely, acceptable (a) and unacceptable (u) quality. Let us assume that we are in possession of the following probabilities. (i) The probability of a transition of the process from one state to another in a time period (t_{n-1}, t_n) and (ii) The probabilities that the process produces acceptable and unacceptable product quality when it is in one of the two states. In our example that is to follow, we propose one possible way in the selection of these probabilities. If the state of the process namely under or out of control is unobservable then, we have the conditions for a hidden Markov chain model in which the state of the process

progresses according to a Markov chain. The signal is the status of the item produced (which is observed from the sample) and has the value either a or u , depending on whether the item is acceptable or unacceptable. Existing results in hidden Markov chain theory will help us in extracting the following information.

Given the vector (S_1, S_2, \dots, S_n) of the first n signals, we can also find the most likely states of the process at the n time points when the samples were drawn.

Given the sequence of signals (S_1, S_2, \dots, S_n) , by regarding the sequence of states as a single entity, we will be able to choose that sequence of states (X_1, X_2, \dots, X_n) whose probability is maximal.

In order to illustrate the above ideas as well as to propose a method for the choice of the model parameters, we present an illustration here and discuss it in detail.

Let us consider a production process whose output is normally distributed with mean $\mu = 3$ and standard deviation $\sigma = 1.5$. In case σ is unknown, one can take samples and use the estimate for population standard deviation $\hat{\sigma} = \frac{\bar{R}}{d_2}$ where \bar{R} is the mean of the sample ranges. Let samples of size $n=8$ be drawn from the output at equally spaced intervals. The 2σ control limits are given by

$$UCL = \mu + 2 \frac{\sigma}{\sqrt{n}} = 4.060$$

$$LCL = \mu - 2 \frac{\sigma}{\sqrt{n}} = 1.939$$

We have chosen to work with 2σ limits because of the reduction in the number of samples to be drawn to detect out of control points. However, the method works as well with 3σ limits. As remarked at the beginning of this section, we need to determine the probability of a transition of the process from one state to another in a time period (t_{n-1}, t_n) and the probabilities that the process produces acceptable and unacceptable product quality when it is in one of the two states. We propose that the latter probabilities can be estimated as follows: Recall the fact that the two types of errors in quality control decision making are type I and type II errors. Since the decision of out of control process is made on the basis of the sample, it is not unreasonable to assume

that the sample drawn produced unacceptable quality. Thus,

$$P(\text{Type I Error}) = P(\text{unacceptable} | X = 1) = P(\bar{x} > 4.06 \text{ or } \bar{x} < 1.939 | \mu=3) = 0.0466$$

$$\text{Hence, } P(\text{acceptable} | X=1) = 1 - P(\text{unacceptable} | X=1) = 0.9534.$$

In the computations for $P(\text{Type II Error})$, we need to know the level to which the out of control process has shifted to. One possible way will be to use the level to which the process usually shifts to, known from experience. In our example we have used the value of $\mu=3.2$ as the shifted value. Now,

$$P(\text{Type II Error}) = P(\text{under control} | X=2) = P(1.939 < \bar{x} < 4.06 | \mu=3.2) = 0.9387.$$

Using a similar logic as above after noting that the probability of a defective item being produced when the system has gone out of control must be high, we choose

$$P(\text{unacceptable} | X=2) = 0.9387 \text{ so that}$$

$$P(\text{acceptable} | X=2) = 0.0613.$$

Next, in order to generate the one-step transition probabilities of the Markov chain X_n whose state spaces are under control (state 1) and out of control (state 2), we proceed as follows: Since the output of the process under study is normally distributed with mean 3 and standard deviation 1.5, we generated 210 data points of the above distribution using the Arena input analyzer program. The data points are provided in Table I.

Table I: Generated Data Points In Order to Get the One-Step Transition Probability Matrix

4.456	3.025	2.532	2.968	1.956	3.731	2.664	2.442	5.754	0.621
3.115	2.645	3.041	2.515	3.951	1.915	3.385	0.000	3.104	4.298
2.717	1.836	2.547	5.539	5.921	2.022	3.179	1.264	2.214	6.272
3.822	2.138	2.692	3.367	2.492	4.850	1.383	1.959	5.007	4.116
0.000	4.712	6.092	3.841	4.038	3.331	1.464	1.964	1.377	1.235
2.509	2.277	1.395	5.304	1.241	2.994	3.119	2.406	2.686	1.572
0.495	2.640	1.674	0.399	1.389	4.820	0.877	3.599	3.814	4.630
4.611	3.643	3.145	0.801	2.912	3.632	0.718	2.077	0.793	5.328
6.346	0.000	2.200	3.448	4.356	4.421	3.907	3.667	2.706	2.989
3.404	0.678	4.575	1.464	2.701	3.309	2.003	2.757	2.564	2.959
3.391	2.576	3.464	2.183	4.053	4.067	1.825	4.537	3.555	3.704
5.054	4.916	4.441	4.985	4.721	2.383	3.252	3.518	2.873	4.550
5.204	4.703	3.635	4.493	3.090	3.139	0.000	3.437	0.553	2.670
3.759	2.344	5.310	3.423	6.325	1.381	4.998	3.629	3.471	4.074
5.277	3.474	1.885	2.976	2.876	3.742	1.069	2.904	3.473	2.996
4.258	2.083	5.948	0.830	1.227	2.099	1.232	4.212	0.216	2.518
1.212	2.447	1.389	6.102	4.584	2.142	2.735	2.863	2.800	5.946
3.380	2.742	0.442	3.864	4.141	2.027	2.986	2.731	4.748	6.182
5.610	2.992	5.109	5.525	3.237	3.773	1.079	2.018	2.940	4.122
1.999	3.778	2.134	3.841	4.146	3.821	1.881	1.673	0.953	2.368
2.815	4.437	4.254	0.973	3.363	5.752	0.953	2.498	2.658	2.415

We have computed the average proportion of points which lie within the 2σ limits (state 1) before an out of control point (state 2) is encountered which turns out to be 0.67262 (or 113/168). This value can reasonably be taken to be the probability P_{11} of a transition from state 1 to the same state. Consequently, the probability of a transition from state 1 to state 2 denoted by P_{12} is given by $1 - P_{11} = 0.32738$ (or 55/168). Once the process enters an out of control state it remains there until remedial action is taken so that $P_{22} = 1$. Thus, we arrive at the one-step transition probability matrix of X_n given below;

$$P = \begin{bmatrix} 0.6726 & 0.3274 \\ 0.0000 & 1.0000 \end{bmatrix}$$

Now we are in possession of all the model parameters, we proceed to explain the procedure of actual quality control.

The process controller takes samples of size 8 at various time points and uses the sample average as the signals. For instance if the sample averages fall within the 2σ limits, the signal received is that of acceptable quality. On the other hand if the sample averages fall outside the limits the signal is perceived to be of unacceptable quality. In order to generate various samples, Arena-input analyzer program was run and the respected sample

averages were calculated until an unacceptable quality signal was received. The data is shown in Table II. In the following, we will explain the various information about the process status which, can be obtained from the samples generated.

3.1. The Status Of The Process At Each Of The Instants Of Sampling

The most important question that a quality control engineer faces is the determination of the status of the process on the basis of the sample. In the case of control charts the process status is determined from only one sample which was drawn at the time of determination while the previous samples have no effect on the status. On the other hand, the process control using hidden Markov chains proposed here uses the information of all the previous samples. More specifically, using equations (4) and (5) we obtain

$$\begin{aligned} F_1(1) &= P\{X_1 = 1, S_1 = s_1\} = \\ &= P(X_1 = 1) \times p(s_1 | 1) = 0.955 \times 0.9534 \\ &= 0.910497 \\ F_1(2) &= P\{X_1 = 2, S_1 = s_1\} = \\ &= P(X_1 = 2) \times p(s_1 | 2) = 0.045 \times 0.0613 \\ &= 0.0027585 \end{aligned}$$

So that

$$P\{X_1 = 1 | s_1\} = \frac{F_1(1)}{F_1(1) + F_1(2)} = 0.9969795$$

Similarly

$$F_2(1) = p(s_2|1) \times [F_1(1) \times P_{11} + F_1(2) \times P_{21}]$$

$$= P(a|1) \times \left[0.910497 \times \frac{113}{168} + 0.0027585 \times 0 \right]$$

$$= 0.5838790$$

$$F_2(2) = p(s_2|2) \times [F_1(1) \times P_{12} + F_1(2) \times P_{22}]$$

$$= P(a|2) \times \left[0.910497 \times \frac{55}{168} + 0.0027585 \times 1 \right]$$

$$= 0.018441361$$

So that

$$P\{X_2 = 1 | s_2\} = \frac{F_2(1)}{F_2(1) + F_2(2)} = 0.9693828$$

Similar computations up to the 16th sample yield the information about the status of the process summarized in Table III.

Table III: Probabilities of the state of the process based on the samples

$P(X_1 = 1 s_1) =$	0.9969795	$P(X_9 = 1 s_9) =$	0.9665550
$P(X_2 = 1 s_2) =$	0.9693828	$P(X_{10} = 1 s_{10}) =$	0.9665550
$P(X_3 = 1 s_3) =$	0.9668246	$P(X_{11} = 1 s_{11}) =$	0.9665550
$P(X_4 = 1 s_4) =$	0.9665807	$P(X_{12} = 1 s_{12}) =$	0.9665550
$P(X_5 = 1 s_5) =$	0.9665574	$P(X_{13} = 1 s_{13}) =$	0.9665550
$P(X_6 = 1 s_6) =$	0.9665552	$P(X_{14} = 1 s_{14}) =$	0.9665550
$P(X_7 = 1 s_7) =$	0.9665550	$P(X_{15} = 1 s_{15}) =$	0.9665550
$P(X_8 = 1 s_8) =$	0.9665550	$P(X_{16} = 1 s_{16}) =$	0.08445395

From the table we see the probability that the process is under control when the 16th sample was drawn suddenly drops to a very small value indicating that the status of the process has undergone a change. Another possible indicator of the change in the state could be where $P(X_k = 1)$ becomes smaller than $P(X_k = 2)$.

In Table IV, we present the values of the probabilities of the state of the process (states 1 and 2) for each of the sample. It can be seen that for samples 1 to 15, the probability that the system is under control far exceeds the probability that the system is out of control. For sample 16, the inequality is reversed, thereby indicating the shift of the process state.

Table IV: State probabilities based on the samples

k	$P(X_k=1 S^k = s_k)$	$P(X_k=2 S^k = s_k)$	k	$P(X_k=1 S^k = s_k)$	$P(X_k=2 S^k = s_k)$
1	0.9969795	0.0030205	9	0.9665550	0.0334450
2	0.9693828	0.0306172	10	0.9665550	0.0334450
3	0.9668246	0.0033175	11	0.9665550	0.0334450
4	0.9665807	0.0334192	12	0.9665550	0.0334450
5	0.9665574	0.0334425	13	0.9665550	0.0334450
6	0.9665552	0.0334448	14	0.9665550	0.0334450
7	0.9665550	0.0334450	15	0.9665550	0.0334450
8	0.9665550	0.0334450	16	0.08445395	0.9155461

3.2. Predicting The Sequence Of States As A Single Entity

In the first part of the prediction problem, we dealt with the probability of the state of the

system at every instant of sampling. However, in this part given the information about the

first k states (X_1, X_2, \dots, X_k) . To illustrate we first compute

$$V_1(1) = P\{X_1 = 1, S_1 = s_1\} = p_1 \times p(s_1 | 1) = 0.9104970$$

$$V_1(2) = P\{X_1 = 2, S_1 = s_1\} = p_2 \times p(s_1 | 2) = 0.0027585$$

first k samples, the problem is predicting the

Using these values, $V_k(j)$ are calculated successively using equation (7) for $k = 2, 3, \dots, 16$.

These are provided in Table V.

Table V: Values of $V_k(j)$

$V_1(1) =$	0.9104970	$V_5(1) =$	0.1539770	$V_9(1) =$	0.026039526	$V_{13}(1) =$	0.004403624
$V_1(2) =$	0.0027585	$V_5(2) =$	0.00481865	$V_9(2) =$	0.000814896	$V_{13}(2) =$	0.000137809
$V_2(1) =$	0.5838790	$V_6(1) =$	0.098741598	$V_{10}(1) =$	0.016698496	$V_{14}(1) =$	0.002823934
$V_2(2) =$	0.0182723	$V_6(2) =$	0.003090079	$V_{10}(2) =$	0.000522572	$V_{14}(2) =$	0.000088373
$V_3(1) =$	0.3744270	$V_7(1) =$	0.063320518	$V_{11}(1) =$	0.010708328	$V_{15}(1) =$	0.001810918
$V_3(2) =$	0.01171755	$V_7(2) =$	0.001981597	$V_{11}(2) =$	0.000335112	$V_{15}(2) =$	0.000056671
$V_4(1) =$	0.2401107	$V_8(1) =$	0.040605865	$V_{12}(1) =$	0.006866983	$V_{16}(1) =$	0.000056761
$V_4(2) =$	0.0075142	$V_8(2) =$	0.001270744	$V_{12}(2) =$	0.000214899	$V_{16}(2) =$	0.000556517

From the above table we see that given the information about the first 16 samples namely $S^{16} = (S_1, \dots, S_{16})$, the most possible set of states $(X_1, \dots, X_{16}) = (1, 1, 1, \dots, 1, 2)$. This is in conformity with our earlier findings.

4. CONCLUDING REMARKS

This paper attempts to propose a new method for process quality control using the theory of hidden Markov chains. The features of the proposed model are as follows:

Unlike control charts, the present method of quality control does not assume the normality or near normality assumption of the output.

2) The control chart methods require several charts to monitor different process characteristics of interest. However, this difficulty is overcome using the present method.

3) **The decision on whether the process is operating under statistical control or not using traditional control chart methods is based on the last sample drawn. However, in the present method we take into consideration not only the last sample but**

also the previous ones in making our decision.

Several information on the status of the process namely under or out of control can be extracted from the hidden Markov chain model. One is the probability of the status of the process at each of the instant when the samples are drawn. Another is the most likely states of the process at various instants of sampling.

Parameters like the elements of the one-step transition probability matrix of the state of the process $\{X_n\}$ and the conditional probabilities of the signals given the state could be determined based on the experience of the quality control engineer. But such estimates are empirical and hence a more scientific way of estimation will be useful.

The method computes the probability of the state of the system only. However, one needs to know the critical level for these probabilities below which we could safely conclude that the process has gone out of control.

The illustration provided deals with a variable quality characteristic. However, the method works as well when the quality characteristic is an attribute. This is because the classification of acceptable and unacceptable output quality is made naturally as defective

or non defective output. Thus, there is no need for the use of 2σ limits for such a classification.

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