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ONE-WEIGHT CODES OVER THE RING $\mathbb{F}_q[v]/\langle v^s - 1\rangle$

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Abstract

In this study, we obtain one-Lee weight codes over a class of nonchain rings and study their structures. We give an explicit construction for one-Lee weight codes. A method to derive more one-Lee weight codes from given a one-Lee weight code is also represented. By defining and making use of a distance-preserving Gray map, we get a family of optimal one-Hamming weight codes over finite fields.

Keywords: Linear codes, one-Lee weight codes, gray map, optimal codes

1. Introduction

A linear code is called one-weight if its all nonzero codewords have the same weight. One-weight codes play an important role among error correcting codes since they have many applications such as the design of demultiplexers for nano-scale memories (see Ref. [5]) and the construction of frequency hopping lists for use in GSM networks (see Ref. [9]). There have been many studies on one-weight codes over finite fields, some of which are (see Refs. [1-3,8,11,18,19]). However, in recent years one-weight codes over finite rings become considerable since they give us another construction of one-Hamming weight codes over finite fields (see Refs. [4,6,7,12-17]). In [4], Carlet proved that there is a unique one-weight linear code over Z_4 with type $\left(4^{k_1}, 2^{k_2}\right)$ for given nonnegative integers k_1 and k_2. In [13], Shi examined the structures of one-homogeneous weight codes over the ring of integers modulo p^m for any prime p and integer $m \geq 1$. In [14], Shi *et al.* studied on one-homogeneous weight linear codes over a class of chain ring $\frac{\mathbb{F}_p[u]}{\langle u^m \rangle}$ and gave a different construction for optimal one-Hamming weight linear codes over finite fields. In [12], Sarı *et al.* extended the notion in [14] to finite chain rings. In [15], the authors obtained optimal one-weight linear codes and two-weight projective codes over F_2. In [16], Shi *et al.* produced a solution for the open problem given in [15] and proved the conditions to be nonexistence for two-Lee weight projective codes over Z_4. In [17], the authors gave a MacWilliams-type identity for the $Z_2Z_2[u,v]$-additive codes. In [6], Li *et al.* presented a few methods to construct one-weight and two-weight additive codes over Z_2R_2 where $R_2 = \frac{\mathbb{F}_2[v]}{\langle v^4 \rangle}$. In [7], Li and Shi provided a construction for two infinite families of two-weight codes over Z_{2^m} with respect to two distinct metrics. Drawing inspiration from the aforementioned studies, we examine one-Lee weight codes over a nonchain ring $\frac{\mathbb{F}_q[v]}{\langle v^s - 1 \rangle}$ in this study.

We organize this paper as follow: In Section 2, we introduce the nonchain ring $R_{q,s}$ and give the basic definitions concerning on linear codes over finite fields. We also define the Gray map from $R_{q,s}^n$ to $\mathbb{F}_q^{sn}$ and study its properties. In Section 3, we discuss the structure of one-Lee weight linear codes over $R_{q,s}$ and study their properties. We also present an explicit construction for one-Lee weight codes over $R_{q,s}$ and obtain one-Hamming weight codes over finite field $\mathbb{F}_q$ by taking Gray images of one-Lee weight codes over $R_{q,s}$, which enables us to have another construction for one-Hamming weight codes over $\mathbb{F}_q$. We also prove that these one-Hamming weight codes are optimal and illustrate the results derived in Section 3. Finally, we conclude this paper by some experimental examples.

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field of q elements where q is a prime power. A code C of length n over $\mathbb{F}_q$ is a nonempty subset of $\mathbb{F}_q^n$. A linear code C of length n over $\mathbb{F}_q$ is a subspace of $\mathbb{F}_q^n$ and so has a dimension, say k. The Hamming weight $w_H(x)$ of an element $x = (x_0, x_1, \ldots, x_{n-1})$ in $\mathbb{F}_q^n$ is defined to be $w_H(x) = \left|\{i : x_i \neq 0, i \in \{0,1,\ldots,n-1\}\}\right|$. The Hamming distance $d_H(x,y)$ between two vectors x and y in $\mathbb{F}_q^n$ is defined as $d_H(x,y) = w_H(x-y)$. The Hamming distance $d_H(C)$ of a code C is the minimum Hamming distance between two different vectors in the code C. An element of a code is called codewords. The number of the elements of a code C is called the size of the code C, denoted by $|C|$. A code C over $\mathbb{F}_q$ of length n and minimum Hamming distance d_H is denoted by $(n,|C|,d_H)_q$. A linear code C over $\mathbb{F}_q$ of length n, dimension k and minimum Hamming distance d_H is denoted by $[n,k,d_H]_q$. An $[n,k,d_H]_q$ linear code is also an $(n,q^k,d_H)_q$ code. The maximum number of the codewords of a code over $\mathbb{F}_q$ of length n and minimum distance d_H is denoted by $A_q(n,d_H)$. The following bound, called the Griesmer bound, is significant for linear codes over finite fields.

Lemma 2.1 Let C be a linear code over $\mathbb{F}_q$ of parameters $[n,k,d_H]_q$, where $k \geq 1$. Then,

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d_H}{q^i} \right\rceil.$$

Note that a linear code over finite field satisfying the Griesmer bound is called optimal.

We denote $R_{q,s}$ to be $R_{q,s} = \frac{\mathbb{F}_q[v]}{\langle v^s - 1 \rangle}$ where s is a positive integer and q is a prime power such that $\{1, 2, \ldots, s-1\} = \{q^j (\bmod s) : j \in Z^+\}$. Then, $R_{q,s}$ is a nonlocal commutative ring with identity. The nontrivial ideals of the ring $R_{q,s}$ are $\langle \vartheta_1 = v - 1 \rangle$ and $\langle \vartheta_2 = v^{s-1} + v^{s-2} + \cdots + 1 \rangle$, both of which are maximal. Denote $I_i = \langle \vartheta_i \rangle$ for $i = 1, 2$. Then, $R_{q,s} = I_1 \oplus I_2$. It is not hard to see that $I_2 = \{a(v^{s-1} + v^{s-2} + \cdots + 1) : a \in \mathbb{F}_q\}$ and so $|I_1| = q^{s-1}$ and $|I_2| = q$.

A code C over $R_{q,s}$ of length n is a nonempty subset of $R_{q,s}^n$. A linear code C over $R_{q,s}$ of length n is a submodule of $R_{q,s}$-submodule $R_{q,s}^n$. It is easy to see that any linear code C of length n over $R_{q,s}$ is permutation-equivalent to a code with the following generator matrix:

$$G_{(k_1+k_2+k_3)\times n} = \begin{pmatrix} I_{k_1} & \vartheta_2 B_1 & \vartheta_1 A_1 & \vartheta_1 A_2 + \vartheta_2 B_2 & \vartheta_1 A_3 + \vartheta_2 B_3 \\ 0 & \vartheta_1 I_{k_2} & 0 & \vartheta_1 A_4 & 0 \\ 0 & 0 & \vartheta_2 I_{k_3} & 0 & \vartheta_2 B_4 \end{pmatrix} \tag{1}$$

where I_{k_i} is $k_i \times k_i$ identity matrix, A_i and B_i are matrices over $\mathbb{F}_q$. For a code C over $R_{q,s}$ having the generator matrix $G_{(k_1+k_2+k_3)\times n}$, $|C| = q^{sk_1+(s-1)k_2+k_3}$.

Any element r in $R_{q,s}$ is of the form $r = r_0 + r_1 v + \cdots + r_{s-1} v^{s-1}$, $r_i \in \mathbb{F}_q$ for all $i \in \{0, 1, \ldots, s-1\}$. Define the map ϕ from $R_{q,s}$ to $\mathbb{F}_q^s$ by $r_0 + r_1 v + \cdots + r_{s-1} v^{s-1} \to (r_0, r_1, \ldots, r_{s-1})$ and extend the map ϕ to $\psi : R_{q,s}^n \to \mathbb{F}_q^{sn}$ componentwisely, i.e. $\psi(c_0, c_1, \ldots, c_{n-1}) = (\phi(c_0), \phi(c_1), \ldots, \phi(c_{n-1}))$. We call the map ψ as the Gray map from $R_{q,s}^n$ to $\mathbb{F}_q^{sn}$. The Lee weight of an element x in $R_{q,s}$ is defined as $w_L(x) = w_H(\psi(x))$. The Lee weight $w_L(x)$ of a vector $x = (x_0, x_1, \ldots, x_{n-1})$ in $R_{q,s}^n$ is $\sum_{i=0}^{n-1} w_L(x_i)$. The Lee distance $d_L(x, y)$ between two vectors $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ in $R_{q,s}^n$ is defined to be $d_L(x, y) = w_L(x - y)$. The following theory is direct.

Theorem 2.2 The Gray map $\psi(C)$ of a linear code C over $R_{q,s}$ of length n is a linear code over $\mathbb{F}_q$ of length sn. Moreover, the Gray map ψ is a distance-preserving map from $(R_{q,s}^n, d_L)$ to $(\mathbb{F}_q^{sn}, d_H)$.

Let C be a linear code over $R_{q,s}$ with the generator matrix $G_{(k_1+k_2+k_3)\times n}$ defined in Equation 1 and for all $1 \le i \le k_1+k_2+k_3$, let r_i be the i^{th} row of $G_{(k_1+k_2+k_3)\times n}$.

Let $G_1 = \begin{pmatrix} r_1 \\ vr_1 \\ \vdots \\ v^{s-1}r_1 \\ \vdots \\ r_{k_1} \\ vr_{k_1} \\ \vdots \\ v^{s-1}r_{k_1} \end{pmatrix}$, $G_2 = \begin{pmatrix} r_{k_1+1} \\ vr_{k_1+1} \\ \vdots \\ v^{s-2}r_{k_1+1} \\ \vdots \\ r_{k_1+k_2} \\ vr_{k_1+k_2} \\ \vdots \\ v^{s-2}r_{k_1+k_2} \end{pmatrix}$ and $G_3 = \begin{pmatrix} r_{k_1+k_2+1} \\ \vdots \\ r_{k_1+k_2+k_3} \end{pmatrix}$. Define the matrix $G' = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}$. Since $R_{q,s} = \langle \vartheta_1 \rangle \oplus \langle v\vartheta_1 \rangle \oplus \cdots \oplus \langle v^{s-2}\vartheta_1 \rangle \oplus \langle \vartheta_2 \rangle$ is an $\mathbb{F}_q$-module decomposition of $R_{q,s}$, we have the following theorem.

Theorem 2.3 Let C be a linear code over $R_{q,s}$ with the generator matrix $G_{(k_1+k_2+k_3)\times n}$ defined in Equation 1. Then the generator matrix $G_{\psi(C)}$ of $\psi(C)$ is

$$G_{\psi(C)} = \psi\left(G'\right) = \begin{pmatrix} \psi(G_1) \\ \psi(G_2) \\ \psi(G_3) \end{pmatrix},$$

where $\psi(G_i)$ is the matrix whose rows are the Gray images of the rows of the matrix G_i.

3. One-Lee weight codes over the ring $R_{q,s}$

In this section, we are going to study the structure of one-Lee weight codes over $R_{q,s}$. We firstly begin to state a well-known fact for linear codes over finite field.

Proposition 3.1 [10] Let C be a linear code over $\mathbb{F}_q$ of length n. If there is no zero column of the generator matrix of C, then $\sum_{c\in C} w_H(c) = \frac{(q-1)}{q}|C|n$.

Since $w_L(x) = w_H(\psi(x))$, $x \in R$, as similar to Proposition 3.1, we give the sum of Lee weights of the elements of the ideals in $R_{q,s}$, which is needed to prove Proposition 3.3 and we omit the proof. Denote $w_L(R_{q,s})$, $w_L(I_1)$ and $w_L(I_2)$ as the sum of Lee weights of the elements of $R_{q,s}$, I_1 and I_2, respectively.

Lemma 3.2

$$w_L\left(R_{q,s}\right)=\frac{(q-1)}{q}\left| R_{q,s}\right| s=(q-1)q^{s-1}s\ ,$$

$$w_L\left(I_1\right)=\frac{(q-1)}{q}\left|I_1\right| s=(q-1)q^{s-2}s$$

and

$$w_L\left(I_2\right)=\frac{(q-1)}{q}\left|I_2\right| s=(q-1)s.$$

In the light of Proposition 3.1 and Lemma 3.2, one can determine the sum of the Lee weights of all codewords in a linear code C over $R_{q,s}$.

Proposition 3.3 Let C be a linear code over $R_{q,s}$ of length n. If there is no zero column in the $|C|\times n$ array consisting of all codewords in C, then $\sum_{c\in C} w_L(c)=\frac{n|C|(q-1)s}{q}$.

Proof. See that in the the $|C|\times n$ array, each column has one of the following cases:

(1) the column contains all elements of the ring $R_{q,s}$ equally often,

(2) the column contains all elements of the ideal $I_1=\langle \vartheta_1\rangle$ equally often,

(3) the column contains all elements of the ideal $I_2=\langle \vartheta_2\rangle$ equally often.

For each $i\in\{1,2,3\}$, denote N_i as the number of the columns having the i^{th} case. By taking into consideration that $\sum_{i=1}^{3} N_i=n$, the sum of the Lee weights of all codewords in C is computed as following.

$$\begin{aligned}\sum_{c\in C} w_L(c)&=N_1\frac{|C|}{q^s}w_L\left(R_{q,s}\right)+N_2\frac{|C|}{q^{s-1}}w_L\left(I_1\right)+N_3\frac{|C|}{q}w_L\left(I_2\right)n\\&=\left(N_1+N_2+N_3\right)\frac{|C|(q-1)s}{q}n\\&=n\frac{|C|(q-1)s}{q}.\end{aligned}$$

We now give a construction for one-Lee weight codes over $R_{q,s}$.

Theorem 3.4 Let C be a linear code over $R_{q,s}$ and $G_{(k_1+k_2+k_3)\times n}$ be a generator matrix of C. If the columns of $G_{(k_1+k_2+k_3)\times n}$ consist of all distinct nonzero vectors

$$\left(c_1,\ldots,c_{k_1},c_{k_1+1},\ldots,c_{k_1+k_2},c_{k_1+k_2+1},\ldots,c_{k_1+k_2+k_3}\right)^T,$$

where $c_i \in R_{q,s}$ for all $i \in \{1,\ldots,k_1\}$, $c_j \in I_1$ for all $j \in \{k_1+1,\ldots,k_1+k_2\}$ and $c_z \in I_2$ for all $z \in \{k_1+k_2+1,\ldots,k_1+k_2+k_3\}$, then C is a one-Lee weight code with nonzero weight $w = \dfrac{(q-1)|C|s}{q}$ and length $n = |C|-1$.

Proof. Note that the number of columns of $G_{(k_1+k_2+k_3)\times n}$ is equal to the length of C and $|C| = q^{sk_1+(s-1)k_2+k_3}$. Without loss of generality, let $k_1 \neq 0$. Consider the first row of $G_{(k_1+k_2+k_3)\times n}$ and see that every nonzero element of $R_{q,s}$ appears $q^{s(k_1-1)+(s-1)k_2+k_3}$ times and the zero element appears $q^{s(k_1-1)+(s-1)k_2+k_3}-1$ times in the first row. Hence the number of columns is exactly

$$\left(q^s-1\right)q^{s(k_1-1)+(s-1)k_2+k_3}+q^{s(k_1-1)+(s-1)k_2+k_3}-1 = q^{sk_1+(s-1)k_2+k_3}-1 = |C|-1 = n.$$

We now prove that every row of the generator matrix $G_{(k_1+k_2+k_3)\times n}$ has the same Lee weight w. Denote $I_0 = R_{q,s}$. See that the elements of the ideal I_i appears equally often in the rows consisting of only the elements of the ideal I_i for all $i \in \{0,1,2\}$. Then the Lee weights of a row having the elements of the ideals I_0, I_1 and I_2 are as following, respectively:

(1) $w_0 = q^{s(k_1-1)+(s-1)k_2+k_3}\dfrac{(q-1)q^s s}{q} = \dfrac{(q-1)|C|s}{q} = w$.

(2) $w_1 = q^{sk_1+(s-1)(k_2-1)+k_3}\dfrac{(q-1)q^{s-1}s}{q} = \dfrac{(q-1)|C|s}{q} = w$.

(3) $w_2 = q^{sk_1+(s-1)k_2+k_3-1}\dfrac{(q-1)qs}{q} = \dfrac{(q-1)|C|s}{q} = w$.

Hence all rows of the generator matrix $G_{(k_1+k_2+k_3)\times n}$ has the same Lee weight w.

We complete the proof by showing that all codewords of the code C has the same Lee weight w. Set $m = k_1+k_2+k_3$ and define a map θ from $R_{q,s}^{k_1}\times I_1^{k_2}\times I_2^{k_3}$ to $R_{q,s}$ by

$$\theta\left(\left(x_1,\ldots,x_{k_1},x_{k_1+1},\ldots,x_{k_1+k_2},x_{k_1+k_2+1},\ldots,x_{k_1+k_2+k_3}\right)\right) = \sum_{i=1}^{m}\lambda_i x_i\ ,$$

where $\lambda_i \in R_{q,s}$ for all $i \in \{1,\ldots,m\}$. Then θ is an $R_{q,s}$-module homomorphism and so $\mathrm{Im}\theta = I_j$ for some $j \in \{0,1,2\}$. Since the generator matrix $G_{(k_1+k_2+k_3)\times n}$ has no zero column, the case $\mathrm{Im}\theta = (0)$ is possible only when $\lambda_i = 0$ for all $i \in \{1,\ldots,m\}$. Observe that each residue class of $R_{q,s}^{k_1} \times I_1^{k_2} \times I_2^{k_3}$ with respect to $Ker\theta$ corresponds to a distinct element of the ideal I_i since $\dfrac{R_{q,s}^{k_1} \times I_1^{k_2} \times I_2^{k_3}}{Ker\theta} \cong \mathrm{Im}\theta = I_i$ for some $i \in \{0,1,2\}$. This fact implies that any codeword of the code C has exactly $\dfrac{|R_{q,s}^{k_1}||I_1^{k_2}||I_2^{k_3}|}{|I_i|}$ times the nonzero elements of the ideal I_i for some $i \in \{0,1,2\}$. Since $|R_{q,s}^{k_1}||I_1^{k_2}||I_2^{k_3}| = |C|$, the Lee weight of the codewords in C is $\dfrac{|R_{q,s}^{k_1}||I_1^{k_2}||I_2^{k_3}|}{|I_i|}\dfrac{(q-1)|I_i|s}{q} = \dfrac{(q-1)|C|s}{q} = w$, which completes the proof.

Theorem 3.5 Let w be nonzero weight of a linear one-Lee weight code C over $R_{q,s}$ of length n and $\left(s, \dfrac{|C|-1}{q-1}\right) = 1$. Then, there exists a positive integer μ such that $w = \mu \dfrac{|C|}{q} s$ and $n = \mu \dfrac{|C|-1}{q-1}$.

Proof. By Proposition 3.3, we have

$$\frac{|C|(q-1)s}{q} n = (|C|-1)w.$$

Since $\left(\dfrac{|C|}{q}s, \dfrac{|C|-1}{q-1}\right) = 1$, the proof is over.

Theorem 3.5 implies that one can derive more one-Lee weight linear codes from a one-Lee weight codes having the generator matrix $G_{(k_1+k_2+k_3)\times n}$ as in Theorem 3.4. We state a definition before giving a method to obtain more one-Lee weight linear codes from a one-Lee weight codes having the generator matrix $G_{(k_1+k_2+k_3)\times n}$ given in Theorem 3.4.

Definition 3.6 Let A be any matrix and $n \in N^+$. We denote A^n by $(\underbrace{A|A|\cdots|A}_{n\ times})$.

Theorem 3.7 Let $\mu = \dfrac{t}{q-1}$, where $t = \sum_{i=1}^{q-1} t_i$ and t_i's are nonnegative integers for all $i \in \{1,\ldots,q-1\}$. Let C be a one-Lee weight code with the generator matrix $G_{(k_1+k_2+k_3)\times n}$ given in

Theorem 3.4 and $\left(s, \frac{|C|-1}{q-1}\right) = 1$. Then, there exists a family of one-Hamming weight codes over $\mathbb{F}_q$ with the parameters $\left[\mu(|C|-1)s, sk_1 + (s-1)k_2 + k_3, \mu\frac{|C|}{q}(q-1)s\right]_q$.

Proof. Label the nonzero elements of $\mathbb{F}_q$ as $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}$ and denote $\vec{a} = (a \quad a \quad \ldots \quad a)$. Since $q-1$ divides both the length $|C|-1$ of the code C and the number of nonzero elements in each ideal of $R_{q,s}$, we can partition the rows of the generator matrix $G_{(k_1+k_2+k_3)\times n}$ into $q-1$ parts $D_1, D_2, \ldots, D_{q-1}$ by the following steps:

(1) Let $a \neq 0$. Set $\begin{pmatrix} \alpha_i \vec{a} \\ \vdots \end{pmatrix}$ as a block of D_i for all $1 \le i \le q-1$ whenever $\begin{pmatrix} \vec{a} \\ \vdots \end{pmatrix}$ is a block of $G_{(k_1+k_2+k_3)\times n}$.

(2) Let $a \neq 0$. Set $\begin{pmatrix} \vec{0} \\ \vdots \\ \vec{0} \\ \alpha_i \vec{a} \\ \vdots \end{pmatrix}$ as a block of D_i for all $1 \le i \le q-1$ whenever $\begin{pmatrix} \vec{0} \\ \vdots \\ \vec{0} \\ \vec{a} \\ \vdots \end{pmatrix}$ is a block of $G_{(k_1+k_2+k_3)\times n}$.

One can see that each part D_i is a generating matrix of a one-Lee weight codes over $R_{q,s}$ with the nonzero weight $\frac{|C|}{q}s$ and the length $\frac{|C|-1}{q-1}$. Define the matrix $\bar{G}$ by ($D_1^{t_1} \big| D_2^{t_2} \big| \cdots \big| D_{q-1}^{t_{q-1}}$), where t_i's are nonnegative integers for all $i \in \{1, \ldots, q-1\}$. Let C be a linear code having the generating matrix $\bar{G}$. Then, $\bar{C}$ is a one-Lee weight code over $R_{q,s}$ of length $\mu(|C|-1)$ and nonzero weight $\mu\frac{|C|}{q}(q-1)s$. Therefore, by taking the Gray images of the code $\bar{C}$, a family of one-Hamming weight codes over $\mathbb{F}_q$ with the parameters

$$\left[\mu(|C|-1)s, sk_1 + (s-1)k_2 + k_3, \mu\frac{|C|}{q}(q-1)s\right]_q$$

is obtained.

Remark 3.8 Let C be a one-Lee weight code having the generator matrix $G_{(k_1+k_2+k_3)\times n}$ given in Theorem 3.4. If $k_3 \neq 0$ and $q = 2$, we can not split $G_{(k_1+k_2+k_3)\times n}$ into more parts than one as mentioned in the proof of Theorem 3.7 since $q-1=1$.

Theorem 3.9 The codes derived in Theorem 3.7 are optimal.

Proof. We will prove it by showing that the parameters of the codes obtained in Theorem 3.7 meet the Griesmer bound. Set $x = sk_1 + (s-1)k_2 + k_3$ and $y = \sum_{i=1}^{q-1} t_i$. See that $|C| = q^x$ and

$$\mu \frac{|C|}{q}(q-1)s = yq^{x-1}s .$$

Then,

$$\begin{aligned}\sum_{i=0}^{x-1}\left\lceil \frac{yq^{x-1}s}{q^i} \right\rceil &= yq^{x-1}s + yq^{x-2}s + \cdots + ysn \\ &= ys\left(q^{x-1} + q^{x-2} + \cdots + 1\right)n \\ &= ys\frac{q^x - 1}{q-1}n \\ &= ys\frac{|C|-1}{q-1}n \\ &= \mu\left(|C|-1\right)sn \\ &= n.\end{aligned}$$

We now give some examples to illustrate the findings in this section.

Example 3.10 Let $q = 2$ and $s = 3$. Then, $R_{2,3} = \langle v+1 \rangle \oplus \langle v^2 + v + 1 \rangle$. Let C be a linear code over $R_{2,3}$ having the generator matrix $G_{(k_1+k_2+k_3)\times n}$ where $k_1 = 1$, $k_2 = k_3 = 0$. Then,

$$G_{(k_1+k_2+k_3)\times n} = \begin{pmatrix} 1 & v & 1+v & v^2 & 1+v^2 & v+v^2 & 1+v+v^2 \end{pmatrix}_{1\times 7}$$

and by Theorem 3.4, C is a one-Lee weight code over $R_{2,3}$ with the length 7 and the nonzero weight 12. The Gray image $\psi(C)$ of C is also a one-Hamming weight code over $\mathbb{F}_2$ with the parameters $[21,3,12]$ which is optimal. Some of one-Hamming weight codes over $\mathbb{F}_2$ obtained by the Gray map ψ with respect to k_1, k_2 and k_3 are given in Table 1. Some of the generator matrices of the one-Lee weight codes given in Table 1 with respect to k_1, k_2 and k_3 are as following:

- Let $k_1 = 0$, $k_2 = k_3 = 1$. Then,

$$G_{(k_1+k_2+k_3)\times 7} = \begin{pmatrix} 0 & \mathbf{1+v} & \mathbf{v+v^2} & \mathbf{1+v^2} \\ 1+v+v^2 & G_0 & G_0 & G_0 \end{pmatrix},$$

where $G_0 = \begin{pmatrix} 0 & 1+v+v^2 \end{pmatrix}$.

- Let $k_1 = k_2 = 1$, $k_3 = 0$. Then,

$$G_{(k_1+k_2+k_3)\times 31} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{v} & \mathbf{1+v} & \mathbf{v^2} & \mathbf{1+v^2} & \mathbf{v+v^2} & \mathbf{1+v+v^2} \\ G_0 & G_1 & G_1 & G_1 & G_1 & G_1 & G_1 & G_1 \end{pmatrix},$$

where $G_0 = \begin{pmatrix} 1+v & v+v^2 & 1+v^2 \end{pmatrix}$ and $G_1 = \begin{pmatrix} 0 & 1+v & v+v^2 & 1+v^2 \end{pmatrix}$.

- Let $k_1 = k_2 = k_3 = 1$. Then,

$$G_{(k_1+k_2+k_3)\times 63} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{v} & \mathbf{1+v} & \mathbf{v^2} & \mathbf{1+v^2} & \mathbf{v+v^2} & \mathbf{1+v+v^2} \\ G_0 & G_1 & G_1 & G_1 & G_1 & G_1 & G_1 & G_1 \end{pmatrix},$$

where $G_0 = \begin{pmatrix} 0 & \mathbf{1+v} & \mathbf{v+v^2} & \mathbf{1+v^2} \\ 1+v+v^2 & G_2 & G_2 & G_2 \end{pmatrix}$, $G_1 = \begin{pmatrix} \mathbf{0} & \mathbf{1+v} & \mathbf{v+v^2} & \mathbf{1+v^2} \\ G_2 & G_2 & G_2 & G_2 \end{pmatrix}$ and

$G_2 = \begin{pmatrix} 0 & 1+v+v^2 \end{pmatrix}$.

Table 1. Some of optimal binary linear one-Hamming weight codes obtained by Gray map ψ

k_1	k_2	k_3	$\psi(C)$
0	0	1	$[3,1,3]$
0	1	0	$[9,2,6]$
1	0	0	$[21,3,12]$
0	1	1	$[21,3,12]$
1	0	1	$[45,4,24]$
1	1	0	$[93,5,48]$
1	1	1	$[189,6,96]$

The following example is important due to illustrate especially Theorem 3.7.

Example 3.11 Let $q=3$ and $s=5$. Then, $R_{3,5} = \langle \vartheta_1 \rangle \oplus \langle \vartheta_2 \rangle$ where $\vartheta_1 = v+2$ and $\vartheta_2 = v^4+v^3+v^2+v+1$. Let C be a linear code having the generator matrix $G_{(k_1+k_2+k_3)\times n}$ where $k_1 = k_2 = 0$ and $k_3 = 2$. Then,

$$G_{(k_1+k_2+k_3)\times n} = \begin{pmatrix} 0 & 0 & \vartheta_2 & \vartheta_2 & \vartheta_2 & 2\vartheta_2 & 2\vartheta_2 & 2\vartheta_2 \\ \vartheta_2 & 2\vartheta_2 & 0 & \vartheta_2 & 2\vartheta_2 & 0 & \vartheta_2 & 2\vartheta_2 \end{pmatrix}_{2\times 8}$$

and by Theorem 3.4, C is a one-Lee weight code over $R_{3,5}$ with the length 8 and the nonzero weight 30. The Gray image $\psi(C)$ of C is a one-Hamming weight code over $\mathbb{F}_3$ of the parameters $[40,2,30]_3$ which is optimal. Since $q \neq 2$, by Theorem 3.7, we can split the generator matrix $G_{(k_1+k_2+k_3)\times n}$ into two parts D_1 and D_2 as mentioned in the proof of Theorem 3.7, i.e., where

$$D_1 = \begin{pmatrix} 0 & \vartheta_2 & \vartheta_2 & \vartheta_2 \\ \vartheta_2 & 0 & \vartheta_2 & 2\vartheta_2 \end{pmatrix}_{2\times 4}$$

and

$$D_2 = \begin{pmatrix} 0 & 2\vartheta_2 & 2\vartheta_2 & 2\vartheta_2 \\ 2\vartheta_2 & 0 & \vartheta_2 & 2\vartheta_2 \end{pmatrix}_{2\times 4}.$$

See that $G_{(k_1+k_2+k_3)\times n}$ is permutation-equivalent to $(D_1|D_2)$. According to Theorem 3.7, each of D_1 and D_2 generates a one-Lee weight code over $R_{3,5}$ of length 4 and nonzero weight 15. The Gray image $\psi(D_i)$ of D_i for $i=1,2$ is a one-Hamming weight code over $\mathbb{F}_3$ of the parameters $[20,2,15]_3$ which is optimal. Let t_i be nonnegative integers for $i=1,2$ and define the matrix $\bar{G}$ by $\bar{G} = \left(D_1^{t_1} \middle| D_2^{t_2}\right)$. By Theorem 3.7, $\bar{G}$ is also a one-Lee weight code over $R_{3,5}$. Some of the parameters of one-Hamming weight codes over $\mathbb{F}_3$ with respect to k_1, k_2, k_3, t_1 and t_2 are given in Table 2.

Table 2. Some of optimal ternary linear one-Hamming weight codes obtained by Gray map ψ

k_1	k_2	k_3	t_1	t_2	$\psi(C)$
0	0	1	1	0	$[5,1,5]_3$
0	0	1	1	1	$[10,1,10]_3$
0	0	2	1	0	$[20,2,15]_3$
0	0	2	1	1	$[40,2,30]_3$
0	1	0	1	0	$[400,4,270]_3$

4. Conclusion

In this paper, we mainly focus on one-Lee weight codes over a class of nonchain rings. We study the structure of one-Lee weight codes and derive one-Lee weight codes by exploring explicit constructions in Theorem 3.4 and 3.7. It is also proven that the Gray images of one-Lee weight codes are optimal one-Hamming linear codes, some of which are tabulated in Table 1 and 2. We conclude the paper by illustrating the finding in the study.

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