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**Araştırma Makalesi**

# Rough $\mathcal{I}_{(\lambda,\mu)}$-Statistical Convergence of Double Sequences in Gradual Normed Linear Spaces

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**Abstract:** The aim of this paper is to we examine the notion of gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence of double sequences in gradual normed linear spaces (GNLS). In addition, we define the concept of gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical limit set of double sequences and obtain some algebraic and topological features of this set. Theorems are proved in the light of GNLS theory approach. Results are obtained via different perspective and new examples are established to justify the counterparts and indicate existence of introduced notions. We produce significant results that present several fundamental properties of this notion. The results established in this research work supplies an exhaustive foundation in GNLS and make a significant contribution in the theoretical development of GNLS in literature. The original aspect of this study is the first wholly up-to-date and thorough examination of the features and implementations of new introduced notions in GNLS.

***Key words**:* Gradual normed linear spaces, Rough convergence, $\mathcal{I}$-convergence

## Gradual Normlu Uzaylarda Çift Dizilerin Kaba $\mathcal{I}_{(\lambda,\mu)}$-İstatistiksel Yakınsaklığı

**Öz:** Bu makalenin amacı, gradual normlu lineer uzaylarda (GNLU) çift dizilerin kaba $\mathcal{I}_{(\lambda,\mu)}$-istatistiksel yakınsaklığı kavramını incelemektir. Ayrıca, çift dizilerinin gradual kaba $\mathcal{I}_{(\lambda,\mu)}$-istatistiksel limit kümesi kavramını tanımlayacak, bu kümenin bazı cebirsel ve topolojik özelliklerini elde edeceğiz. Teoremler, GNLU teorisi yaklaşımı ışığında ispatlanacaktır. Farklı bakış açılarıyla sonuçlar elde edilecek ve karşıtları haklı çıkarmak ve tanıtılan kavramların varlığını göstermek için yeni örnekler üretilecektir. Bu kavramların bazı temel özelliklerini sunan önemli sonuçlar elde edilecektir. Bu araştırma çalışmasında elde edilen sonuçlar, GNLU'da kapsamlı bir temel sağlayacak ve literatürde GNLS'nin teorik gelişimine önemli bir katkı sağlayacaktır. Bu çalışmanın özgün yönü, GNLS'de tanımlanan yeni kavramların özelliklerinin ve uygulamalarının tamamen güncel ve kapsamlı ilk incelemesidir.

***Anahtar kelimeler**:* Gradual normed linear uzaylar, Kaba yakınsaklık, $\mathcal{I}$-yakınsaklık

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## 1. Introduction

Fuzzy theory has made significant progress on the mathematical underpinnings of fuzzy set (FS) theory, which was pioneered by Zadeh [35] in 1965. Zadeh [35] mentioned that an FS assigns a membership value to each element of a given crisp universe set from [0,1]. FSs cannot always overcome the absence of knowledge of membership degrees. These days, it has extensive applications in various branches of engineering and science. The concept "fuzzy number" is significant in the work of FS theory. Fuzzy numbers were essentially the generalization of intervals, not numbers. Indeed fuzzy numbers do not supply a couple of algebraic features of the classical numbers. So the concept "fuzzy number" is debatable to many researchers due to its different behavior. In order to overcome the confusion among the researchers, Fortin et. al. [15] put forward the concept of gradual real numbers as elements of fuzzy intervals. Gradual real numbers (GRNs) are primarily known by their respective assignment function whose domain is the interval (0,1]. So, each real number can be thought of as a GRN with a constant assignment function. The GRNs also supply all the algebraic features of the classical real numbers and have been utilized in optimization and computation problems.

Sadeqi and Azari [28] were the first to examine the idea of GNLS. They worked various properties from both the topological and algebraic points of view. Further development in this direction has been taken place due to Choudhury and Debnath [3], Ettefagh et. al. [10, 11] and many others. For an comprehensive study on GRNs, one may refer to [1, 7, 20, 34].

On the other hand, Fast [14] and Steinhaus [33] put forward the idea of statistical convergence independently utilizing the idea of natural density [16]. Afterward, it was further investigated from the sequence space point of view by Fridy [17], Salat [29] and many mathematicians across the globe. Statistical convergence was extended to λ-statistical convergence by Mursaleen [25].

Statistical convergence of real sequences has been further extended to $\mathcal{I}$-convergence by Kostyrko et al. [19]. Several studies on $\mathcal{I}$-convergence can be examined in [4, 5] and lot of others. Savaş and Das in [30] presented the idea of $\mathcal{I}^{\lambda}$-statistical convergence which is a generalisation of the conceptions of statistical convergence, $\lambda$-statistical convergence and $\mathcal{I}$-convergence. Various investigations and applications of this notion can be examined in [6, 13, 31].

Rough convergence was first given by Phu [27]. Rough convergence has been extended to rough statistical convergence by Aytar [2]. More works can be found in [21, 22]. In addition, the notion of rough statistical convergence was extended to rough $\mathcal{I}$-convergence by Pal et al. [26] utilizing ideals of $\mathbb{N}$. More investigation and application on this line can be found in [8, 9, 18, 23]. Recently, Malik et al. in [24] worked the concept of rough $\mathcal{I}$-statistical convergence in the line of Das et al. [6]. So instinctively one can hope if the new notion of gradual $\mathcal{I}^{\lambda}$-statistical convergence can be introduced in the theory of rough convergence. In this paper we do that. We mention that the results and proof techniques presented in this paper are $\mathcal{I}^{\lambda}$-statistical analogues of those in Phu's [27], Aytar's [2] and Ghosh and Malik [12] papers. Theorems are proved in the light of GNLS theory approach. Results are obtained via different perspective and new examples are produced to justify the counterparts and demonstrate existence of new introduced notions.

## 2. Material and Method

In this section, we give significant existing conceptions and results which are crucial for our findings.

**Definition 2.1.** ([15]) A GRN $\tilde{s}$ is determined by an assignment function $\mathcal{F}_{\tilde{s}}: (0,1] \to \mathbb{R}$. A GRN $\tilde{s}$ s called to be non-negative provided that for each $\gamma \in (0,1]$, $\mathcal{F}_{\tilde{s}}(\gamma) \geq 0$. The set of all GRNs and non-negative GRNs are demonstrated by $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}^*(\mathbb{R})$, respectively.

**Definition 2.2.** ([15]) Presume that $*$ be any operation in $\mathbb{R}$ and presume $\tilde{u}_1, \tilde{u}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{F}_{\tilde{u}_1}$ and $\mathcal{F}_{\tilde{u}_2}$, respectively. At that time, $\tilde{u}_1 * \tilde{u}_2 \in \mathcal{G}(\mathbb{R})$ is determined with the assignment function $\mathcal{F}_{\tilde{u}_1 * \tilde{u}_2}$ denoted by $\mathcal{F}_{\tilde{u}_1 * \tilde{u}_2}(\tau) = \mathcal{F}_{\tilde{u}_1} * \mathcal{F}_{\tilde{u}_2}$, $\forall \tau \in (0,1]$. Especially, the gradual addition $\tilde{u}_1 + \tilde{u}_2$ and the gradual scalar multiplication $p\tilde{u}(p \in \mathbb{R})$ are given as follows:

$$\mathcal{F}_{\tilde{u}^1 + \tilde{u}^2}(\tau) = \mathcal{F}_{\tilde{u}^1}(\tau) + \mathcal{F}_{\tilde{u}^2}(\tau), \forall \tau \in (0,1], \tag{1}$$

and

$$\mathcal{F}_{p\tilde{u}}(\tau) = p\mathcal{F}_{\tilde{u}}(\tau), \forall \tau \in (0,1]. \tag{2}$$

Utilizing the gradual numbers, Sadeqi and Azeri [28] developed the GNLS and determined the notion of gradual convergence as follows:

**Definition 2.3.** ([28]) Take $Y$ as a real vector space. Afterwards, the function $\|.\|_{\mathcal{G}}: Y \to \mathcal{G}^*(\mathbb{R})$ is named to be a gradual norm (GN) on $Y$, provided that for each $\rho \in (0,1]$, subsequent situations are correct for any $w, v \in Y$:

$(i)\mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau) = \mathcal{F}_{\tilde{0}}(\tau)$ iff $w = 0$;

$(ii)\mathcal{F}_{\|\sigma w\|_{\mathcal{G}}}(\tau) = |\sigma|\mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau)$ for any $\sigma \in \mathbb{R}$;

$(iii)\mathcal{F}_{\|w+v\|_{\mathcal{G}}}(\tau) = \mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|v\|_{\mathcal{G}}}(\tau)$.

Here, $(Y, \|.\|_{\mathcal{G}})$ is named GNLS.

**Example 2.1.** ([28]) Take $Y = \mathbb{R}^\alpha$ and for $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^\alpha$, $\gamma \in (0,1]$, determine $\|.\|_{\mathcal{G}}$ as

$$\mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau) = e^\gamma \sum_{j=1}^{n} |w_j| \tag{3}$$

Here, $\|.\|_{\mathcal{G}}$ is a GN on $\mathbb{R}^{\alpha}$, also $(\mathbb{R}^{\alpha}, \|.\|_{\mathcal{G}})$ denotes a GNLS.

On the other hand, Ettefagh et al. [11] were the first who determined the gradual boundedness of a sequence in a GNLS and investigated its relationship with gradual convergence.

**Definition 2.4.** ([11]) Presume that $(Y, \|.\|_{\mathcal{G}})$ be a GNLS. In that time, a sequence $(w_u)$ in $Y$ is named to be gradual bounded provided that for each $\tau \in (0,1]$, there is an $M = M(\tau) > 0$ so that $\mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau) < M, \forall u \in \mathbb{N}$.

**Definition 2.5.** Take $(w_u) \in (Y, \|.\|_{\mathcal{G}})$. At that time, $(w_u)$ is named to be gradual convergent to $w_0 \in Y$, provided that for all $\tau \in (0,1]$ and $\kappa > 0$, there is an $N = (N_\kappa(\tau)) \in \mathbb{N}$ such that $\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa, \forall u, v \geq N$.

**Definition 2.6.** Take $(w_u) \in (Y, \|.\|_{\mathcal{G}})$. Then, $(w_u)$ is named to be gradual statistically convergent to $w_0 \in Y$, provided that for all $\tau \in (0,1]$ and $\kappa > 0$,

$$\lim_{q\to\infty} \frac{1}{q} \left|\left\{u \leq q : \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| = 0. \tag{4}$$

Symbolically, $w_u \to w_0(S(\mathcal{G}))$. The set $S(G)$ indicates the set of all gradually statistical convergent sequences.

**Definition 2.7.** ([32]) Let $\lambda = (\lambda_m)$ and $\eta = (\mu_n)$ be two non-decreasing sequences of positive real numbers, each tending to $\infty$ and so that $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1$; $\mu_{n+1} \leq \mu_n + 1, \mu_1 = 1$. Let $J_m = [m - \lambda_m + 1, m]$, $J_n = [n - \mu_n + 1, n]$ and $J_{mn} = J_m \times J_n$. For any set $Q \subseteq \mathbb{N} \times \mathbb{N}$

$$\delta_{(\lambda,\mu)}(Q) = P - \lim_{m,n\to\infty} \frac{1}{\lambda_m \mu_n} |\{(p,q) \in J_m \times J_n : (p,q) \in Q\}| \tag{5}$$

is named $(\lambda, \mu)$-density of the set $Q$, provided the limit exists.

Throughout the paper, we indicate $\lambda_{mn} = \lambda_m \mu_n$, the collection of such sequences $\lambda$ will be showed by $\Delta_2$.

## 3. Results

**Definition 3.1.** A double sequence $w = (w_{uv})$ is called to be gradually rough $(\lambda, \mu)$-statistically convergent (or shortly $S^r_{(\lambda,\mu)}(\mathcal{G})$-convergent) to $w_0 \in Y$, provided that for each $\kappa > 0$ and $\tau \in (0,1]$,

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(u, v) \in J_{mn} : \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| = 0. \tag{6}$$

In that case, we write $S^r_{(\lambda,\mu)}(\mathcal{G}) - lim w_{uv} = w_0$ or $w_{uv} \to w_0 \left(S^r_{(\lambda,\mu)}(\mathcal{G})\right)$. In addition, we utilize $S^r_{(\lambda,\mu)}(\mathcal{G})$ to indicate the collection of all gradually $S^r_{(\lambda,\mu)}(\mathcal{G})$-convergent sequences in $Y$.

**Definition 3.2.** A double sequence $w = (w_{uv})$ is called to be gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent (or shortly $S\mathcal{I}^r_{(\lambda,\mu)}(\mathcal{G})$-convergent) to $w_0 \in Y$, provided that for each $\kappa, \varkappa > 0$ and $\tau \in (0,1]$,

$$K = \left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \left|\left\{(u, v) \in J_{mn} : \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{7}$$

In that case, we write $S\mathcal{I}^r_{(\lambda,\mu)}(\mathcal{G}) - lim w_{uv} = w_0$ or $w_{uv} \to w_0 \left(S\mathcal{I}^r_{(\lambda,\mu)}(\mathcal{G})\right)$.

**Note 3.1.** We examine that each gradually rough statistically convergent double sequence is gradually rough $\mathcal{I}_2$-statistically convergent and if $\lim_{m,n\to\infty} \inf \frac{\lambda_{mn}}{mn} > 0$ then all gradually rough $\mathcal{I}_2$-statistically convergent double sequence is gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent. Again if $\lim_{m,n\to\infty} \frac{\lambda_{mn}}{mn} = 1$ then gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence means gradually rough $\mathcal{I}_2$-statistical convergence and if we fixed the ideal $\mathcal{I}_{fin} = \{A \subset \mathbb{N} \times \mathbb{N} : A \text{ is finite}\}$ then gradually rough $\mathcal{I}_2$-statistical convergence becomes equivalent to rough statistical convergence. So we may say that gradually rough statistical convergence is a particular case of gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence.

Here $r$ is named the roughness degree of the gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence. When $r = 0$ we get the notion of gradually $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence. However our fundamental interest is when $r > 0$. It could happen that a sequence $w = (w_{uv})$ is not gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent in the usual sense, but there is a sequence $s = (s_{uv})$, which is gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent and supplying the condition $\mathcal{F}_{\|w_{uv}-s_{uv}\|_{\mathcal{G}}}(\tau) \leq r$ for all u, v (or for all u, v whose $\mathcal{I}_{(\lambda,\mu)}$-natural density is zero). Then, $w$ is gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent to the same limit.

It is clear that the gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical limit of a sequence is not unique. So we contemplate the set of gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical limits of a sequence $w$ and we utilize the notation $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}^r_w(\mathcal{G})$ to indicate the set of all gradually rough

$\mathcal{I}_{(\lambda,\mu)}$-statistical limits of a sequence $w$. The sequence $w$ is gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergent when $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G}) \neq \emptyset$.

We supply an example to demonstrate that there is a sequence which is neither gradually rough statistically convergent nor gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent but is gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent.

**Example 3.1.** Assume $\mathcal{I}_2$ be a strongly admissible ideal in ℕ×ℕ, which includes at least one infinite set. Select an infinite subset $Q \in \mathcal{I}_2$. We determine a sequence $w = (w_{uv})$ in the following way:

$$w_{uv} = uv, \text{for } m - \left[\sqrt{\lambda_m}\right] + 1 \leq u \leq m \text{ and } n - \left[\sqrt{\lambda_n}\right] + 1 \leq v \leq n, (m,n) \notin Q,$$

$$w_{uv} = uv, \text{for } m - \left[\sqrt{\lambda_m}\right] + 1 \leq u \leq m \; and \; n - \left[\sqrt{\lambda_n}\right] + 1 \leq v \leq n, (m,n) \in Q \tag{8}$$

$$w_{uv} = (-1)^{u+v}, \text{otherwise.}$$

Then, $w$ neither gradually rough statistically convergent nor gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically convergent but

$$\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G}) = \begin{cases} \emptyset, & \text{if } 0 \leq r < 1, \\ [1-r, r-1], & \text{if } r \geq 1. \end{cases} \tag{9}$$

**Theorem 3.1.** Take $w = (w_{uv}) \in \left(Y, \|.\|_{\mathcal{G}}\right)$. Then, $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G}) \leq 2r$. Especially when $w_{uv} \to w_0 \left(S\mathcal{I}_{(\lambda,\mu)}^r(\mathcal{G})\right)$, then

$$\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G}) = \overline{B_r(w_0)}, \tag{10}$$

where $\overline{B_r(w_0)} = \left\{\alpha \in Y : \mathcal{F}_{\|\alpha - w_0\|_{\mathcal{G}}} \leq r\right\}$ and hence

$$\text{diam}\left(\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G})\right) = 2r. \tag{11}$$

**Proof**. Suppose $\text{diam}\left(\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G})\right) > 2r$. Afterwards, there are $\alpha, \beta \in \mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}_w^r(\mathcal{G})$ so that $\mathcal{F}_{\|\alpha-\beta\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{u_0v_0} - \alpha\|_{\mathcal{G}}} > 2r$. Now, we take $\kappa > 0$ so that $\kappa < \frac{\mathcal{F}_{\|\alpha-\beta\|_{\mathcal{G}}}}{2} - r$. Let

$$K = \left\{(u, v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\} \tag{12}$$

and

$$L = \left\{(u, v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-\beta\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}. \tag{13}$$

Then

$$\begin{aligned} &\frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in K \cup L\}| \\ &\qquad \leq \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in K\}| \\ &\qquad + \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in L\}|, \end{aligned} \tag{14}$$

and so according to the feature of $\mathcal{I}_2$-convergence

$$\begin{aligned} &\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in K \cup L\}| \\ &\qquad \leq \mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in K\}| + \mathcal{I}_2 \\ &\qquad - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in L\}| = 0. \end{aligned} \tag{15}$$

So,

$$\left\{(m, n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in K \cup L\}| \geq \varkappa\right\} \in \mathcal{I}_2 \tag{16}$$

for each $\varkappa > 0$. Take

$$H = \left\{(m, n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}}|\{(u, v) \in J_{mn}: (u, v) \in K \cup L\}| \geq \frac{1}{2}\right\}. \tag{17}$$

Obviously $H \in \mathcal{I}_2$, so take $(m_0, n_0) \in (\mathbb{N} \times \mathbb{N}) \setminus H$. Afterwards,

$$\frac{1}{\lambda_{m_0 n_0}}|\{(u, v) \in J_{mn}: (u, v) \in K \cup L\}| < \frac{1}{2}. \tag{18}$$

So, we obtain

$$\frac{1}{\lambda_{m_0 n_0}}|\{(u,v) \in J_{mn}: (u,v) \notin K \cup L\}| \geq 1 - \frac{1}{2} = \frac{1}{2}, \tag{19}$$

namely, $\{(u,v) \in J_{mn}: (u,v) \notin K \cup L\}$ is a nonempty set.

Take $(u_0, v_0) \in J_{mn}$ such that $(u_0, v_0) \notin K \cup L$. Then, $(u_0, v_0) \in K^c \cap L^c$ and so $\mathcal{F}_{\|w_{u_0 v_0} - \alpha\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{u_0 v_0} - \alpha\|_{\mathcal{G}}} < r + \kappa$ and $\mathcal{F}_{\|w_{u_0 v_0} - \beta\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{u_0 v_0} - \beta\|_{\mathcal{G}}} < r + \kappa$. Hence, we acquire

$$\mathcal{F}_{\|\alpha - \beta\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{u_0 v_0} - \alpha\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|w_{u_0 v_0} - \beta\|_{\mathcal{G}}}(\tau) < 2(r + \kappa) \leq \mathcal{F}_{\|\alpha - \beta\|_{\mathcal{G}}} \tag{20}$$

which is absurd. So, $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}) \leq 2r$.

When $w_{uv} \to w_0 \left(S\mathcal{I}_{(\lambda,\mu)}^r(\mathcal{G})\right)$, then we continue as follows. Take $\kappa, \varkappa > 0$ and $\tau \in (0,1]$. Afterwards,

$$K = \left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{21}$$

Then, for $(m,n) \notin K$ we get

$$\frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| < \varkappa \tag{22}$$

namely,

$$\frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}\right| \geq 1 - \varkappa. \tag{23}$$

For each $\alpha \in \overline{B_r(w_0)}$ we get

$$\mathcal{F}_{\|w_{uv} - \alpha\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|w_0 - \alpha\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) + r. \tag{24}$$

Take

$$L_{mn} = \left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}. \tag{25}$$

Then, for $(u,v) \in L_{mn}$ we get $\mathcal{F}_{\|w_{uv} - \alpha\|_{\mathcal{G}}}(\tau) < r + \kappa$. So, we obtain

$$L_{mn}=\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)<r+\kappa\right\}. \quad (26)$$

This gives

$$\frac{|L_{mn}|}{mn}\leq\frac{1}{\lambda_{mn}}\left|\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)<r+\kappa\right\}\right| \quad (27)$$

i.e.,

$$\frac{1}{\lambda_{mn}}\left|\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)<r+\kappa\right\}\right|\geq 1-\varkappa. \quad (28)$$

Thus, for all $(u,v)\notin K$,

$$\frac{1}{\lambda_{mn}}\left|\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\}\right|<1-(1-\varkappa)=\varkappa \quad (29)$$

and so we obtain

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\left|\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\}\right|\geq\varkappa\right\}\subset K. \quad (30)$$

Since $K\in\mathcal{J}_2$, then

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}\left|\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\}\right|\geq\varkappa\right\}\in\mathcal{J}_2. \quad (31)$$

This denotes that $\alpha\in\mathcal{J}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})$. So, $\mathcal{J}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})\supset\overline{B_r(w_0)}$. Conversely, assume $\alpha\in\mathcal{J}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})$, $\mathcal{F}_{\|\alpha-w_0\|_{\mathcal{G}}}(\tau)>r$ and $\kappa=\frac{\mathcal{F}_{\|\alpha-w_0\|_{\mathcal{G}}}(\tau)-r}{2}$. Now, we take

$$H_1=\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\alpha\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\} \quad (32)$$

and

$$H_2=\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau)\geq\kappa\right\}. \quad (33)$$

Then

$$\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_1\cup H_2\}| \\ \leq \frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_1\}| \\ +\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_2\}|, \tag{34}$$

and so according to the feature of $\mathcal{I}_2$-convergence

$$\mathcal{I}_2-\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_1\cup H_2\}| \\ \leq \mathcal{I}_2-\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_1\}|+\mathcal{I}_2 \\ -\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_2\}|=0. \tag{35}$$

Now, we let

$$H=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in H_1\cup H_2\}|\geq\frac{1}{2}\right\}. \tag{36}$$

Obviously $H\in\mathcal{I}_2$, and we select $(m_0,n_0)\in(\mathbb{N}\times\mathbb{N})\setminus H$. Afterwards,

$$\frac{1}{\lambda_{m_0n_0}}|\{(u,v)\in J_{mn}:(u,v)\in H_1\cup H_2\}|<\frac{1}{2}. \tag{37}$$

So, we obtain

$$\frac{1}{\lambda_{m_0n_0}}|\{(u,v)\in J_{mn}:(u,v)\notin H_1\cup H_2\}|\geq 1-\frac{1}{2}=\frac{1}{2}, \tag{38}$$

namely, $\{(u,v)\in J_{mn}:(u,v)\notin H_1\cup H_2\}$ is a nonempty set. Take $(u_0,v_0)\in J_{mn}$ such that $(u_0,v_0)\notin H_1\cup H_2$. Then, $(u_0,v_0)\in {H_1}^c\cap {H_2}^c$ and so $\mathcal{F}_{\|w_{u_0v_0}-\alpha\|_{\mathcal{G}}}<r+\kappa$ and $\mathcal{F}_{\|w_{u_0v_0}-w_0\|_{\mathcal{G}}}<\kappa$. Hence, we acquire

$$\mathcal{F}_{\|\alpha-w_0\|_{\mathcal{G}}}(\tau)\leq\mathcal{F}_{\|w_{u_0v_0}-\alpha\|_{\mathcal{G}}}(\tau)+\mathcal{F}_{\|w_{u_0v_0}-w_0\|_{\mathcal{G}}}(\tau)<2\kappa+r\leq\mathcal{F}_{\|\alpha-w_0\|_{\mathcal{G}}} \tag{39}$$

which is absurd. Therefore, $\mathcal{F}_{\|\alpha-w_0\|_{\mathcal{G}}}(\tau)\leq r$, and so $\alpha\in\overline{B_r(w_0)}$. As a result, we get $\mathcal{I}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})=\overline{B_r(w_0)}$.

In the paper [27] Phu has already shown that for any subsequence $w' = \left(w_{m_u n_v}\right)$ of a sequence $w = (w_{uv})$, $\mathrm{LIM}_w^r \subset \mathrm{LIM}_{w'}^r$. But this fact does not hold good in case of gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical convergence. To support this we cite the following Example 3.3. To confirm Example 3.3 we first establish a set whose gradually $\mathcal{I}_{(\lambda,\mu)}$-natural density exists but natural density does not exists, as shown in the following Example 3.2.

**Example 3.2.** Suppose $\lambda \in \Delta_2$ such that $\lim\limits_{m,n\to\infty} \frac{\lambda_{mn}}{mn} = 1$. Assume $\mathcal{I}_2$ be a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$ which contains at least one infinite subset of $\mathbb{N} \times \mathbb{N}$. Let $Q \in \mathcal{I}_2$ be an infinite subset of $\mathbb{N} \times \mathbb{N}$. Now, we construct a set $R \subset \mathbb{N} \times \mathbb{N}$ as follows:

$$(m,n) \in R, \text{if } m - \left[\sqrt{\lambda_m}\right] + 1 \le u \le m \text{ and } n - \left[\sqrt{\lambda_n}\right] + 1 \le v \le n, (m,n) \in Q, \tag{40}$$

$$(m,n) \in R, \text{if } m - \left[\sqrt{\lambda_m}\right] + 1 \le u \le m \text{ and } n - \left[\sqrt{\lambda_n}\right] + 1 \le v \le n, (m,n) \notin Q. \tag{41}$$

Then, for $(m,n) \notin Q$,

$$\frac{|\{(u,v) \in J_{mn}: (u,v) \in R\}|}{\lambda_{mn}} = \frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}} \to 0 \text{ as } m,n \to \infty. \tag{42}$$

Take $\varkappa > 0$. Then, there are $m_0, n_0 \in \mathbb{N}$ so that $\frac{[\sqrt{\lambda_{mn}}]}{\lambda_{mn}} < \varkappa$ for all $m \ge m_0$, $n \ge n_0$. Therefore,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}} |\{(u,v) \in J_{mn}: (u,v) \in R\}| \ge \varkappa\right\} \subset Q \cup \{(1,1),(2,2),\ldots,(m_0-1,n_0-1)\} \in \mathcal{I}_2. \tag{43}$$

Hence, $d_{\mathcal{I}_2}^{\lambda}(Q) = 0$.

Now if possible, assume $d(Q) = 0$. Then, for any $\kappa > 0$ there is $p \in \mathbb{N}$ so that

$$\frac{|\{1 \le u \le m, 1 \le v \le n: (u,v) \in Q\}|}{mn} = \frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}} \to 1 \text{ as } m,n \to \infty. \tag{44}$$

Hence $d(Q) \neq 0$. Actually $d(Q)$ does not exist; because, if $d(Q) = r$, $0 < r \le 1$; $d_{\mathcal{I}_2}^{\lambda}(Q) = r$, which is absurd.

**Example 3.3.** Suppose $\mathcal{I}_2$ be a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$ which contains at least one infinite subset of $\mathbb{N} \times \mathbb{N}$. Select an infinite subset $A = \{j_1 < j_2 < \cdots; s_1 <$

$s_2 < \cdots\}$ whose gradually $\mathcal{I}_{(\lambda,\mu)}$-natural density is zero , on the other hand natural density does not exist. We determine a sequence $w = (w_{uv})$ in the following way:

$$w_{uv} := \begin{cases} uv, & \text{if } u = j_k, v = s_l \text{ for some } j_k, s_l \in \mathrm{A} \\ 0, & \text{if not.} \end{cases} \quad (45)$$

Then, for $r > 0$, $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}) = [-r, r]$, on the other hand $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_{w'}^r(\mathcal{G}) = \emptyset$ where $w' = \{w_{j_k s_l}\}$.

**Definition 3.3.** Suppose $w = (w_{uv})$ be a sequence in $Y$, then a subsequence $w' = \{w_{j_k s_k}\}$ of $w$ is called to be gradually $\mathcal{I}_{(\lambda,\mu)}$-dense when $\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} |\{(j_k, s_l) \in J_{mn}: k, l \in \mathbb{N}\}| = 1$.

In the subsequent theorem we demonstrate that the gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical analogue of Phu's result supplies for subsequences.

**Theorem 3.2.** If $w$ has a gradually $\mathcal{I}_{(\lambda,\mu)}$-dense subsequence $w' = \{w_{j_k s_l}\}$ then $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}) \subseteq \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_{w'}^r(\mathcal{G})$.

**Proof.** Assume $w_0 \in \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$ and let $\kappa > 0$ and $\tau \in (0,1]$ be given. Then

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| = 0. \quad (46)$$

But

$$\begin{aligned} &\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| \\ &\qquad + \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| = \frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}}. \\ \Rightarrow &\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| \\ &\qquad = -\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| + \frac{\left[\sqrt{\lambda_{mn}}\right]}{\lambda_{mn}}. \end{aligned} \quad (47)$$

Then obviously

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| = 1. \quad (48)$$

Since $w'$ is $\mathcal{I}_{(\lambda,\mu)}$-dense subsequence of $w$, we get

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} |\{(j_k, s_l) \in J_{mn}: (k,l) \in \mathbb{N}\times\mathbb{N}\}| = 1. \tag{49}$$

Let $A = \{j_1 < j_2 < \cdots ; s_1 < s_2 < \cdots\}$. Put $\mathbb{N} = (\mathbb{N} \setminus A) \cup A$. Thus

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(j_k, s_l) \in J_{mn}: \mathcal{F}_{\|w_{j_k s_l} - w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| = 1. \tag{50}$$

But

$$\begin{aligned} \frac{1}{\lambda_{mn}} &\left|\left\{(j_k, s_l) \in J_{mn}: \mathcal{F}_{\|w_{j_k s_l} - w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| \\ &\leq \frac{1}{\lambda_{mn}} \left|\left\{(k,l) \in J_{mn}: \mathcal{F}_{\|w_{j_k s_l} - w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| \leq 1. \end{aligned} \tag{51}$$

Therefore

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(k,l) \in J_{mn}: \mathcal{F}_{\|w_{j_k s_l} - w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right| = 1. \tag{52}$$

Thus

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} \left|\left\{(k,l) \in J_{mn}: \mathcal{F}_{\|w_{j_k s_l} - w_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| = 0. \tag{53}$$

Hence $w_0 \in \mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w'}(\mathcal{G})$. As a result $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w}(\mathcal{G}) \subseteq \mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w'}(\mathcal{G})$.

**Theorem 3.3.** Take $w = (w_{uv}) \in (Y, \|.\|_{\mathcal{G}})$ and $r > 0$. Then, gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical limit set of the sequence $w$, namely, the set $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w}(\mathcal{G})$ is gradually closed.

**Proof.** When $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w}(\mathcal{G}) = \emptyset$, then nothing to demonstrate.

Presume that $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w}(\mathcal{G}) \neq \emptyset$. Now, contemplate a sequence $s = (s_{uv})$ in $\mathcal{I}_{(\lambda,\mu)} - st - \text{LIM}^r_{w}(\mathcal{G})$ with $\lim_{u,v\to\infty} s_{uv} = s_0$. Take $\kappa, \varkappa > 0$ and $\tau \in (0,1]$. Afterwards, there is an $i_{\frac{\kappa}{2}} \in \mathbb{N}$ so that for all $u, v \geq i_{\frac{\kappa}{2}}$

$$\mathcal{F}_{\|s_{uv} - s_0\|_{\mathcal{G}}}(\tau) < \frac{\kappa}{2}. \tag{54}$$

Take $u_0, v_0 > i_{\frac{\kappa}{2}}$. Then $s_{u_0 v_0} \in \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$. As a result, we get

$$K = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \left| \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_{u_0 v_0}\|_{\mathcal{G}}}(\tau) \geq r + \frac{\kappa}{2}\right\} \right| \geq \varkappa \right\} \in \mathcal{I}_2. \tag{55}$$

Obviously, $T = (\mathbb{N} \times \mathbb{N}) \setminus K$ is nonempty, select $(m,n) \in T$. We get

$$\frac{1}{\lambda_{mn}} \left| \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_{u_0 v_0}\|_{\mathcal{G}}}(\tau) \geq r + \frac{\kappa}{2}\right\} \right| < \varkappa \tag{56}$$

and so

$$\frac{1}{\lambda_{mn}} \left| \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_{u_0 v_0}\|_{\mathcal{G}}}(\tau) < r + \frac{\kappa}{2}\right\} \right| \geq 1 - \varkappa. \tag{57}$$

Put

$$Q_{mn} = \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_{u_0 v_0}\|_{\mathcal{G}}}(\tau) < r + \frac{\kappa}{2}\right\} \tag{58}$$

and take $(u,v) \in Q_{mn}$. Afterwards, we obtain

$$\mathcal{F}_{\|w_{uv} - s_0\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{uv} - s_{u_0 v_0}\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|s_{u_0 v_0} - s_0\|_{\mathcal{G}}}(\tau) < r + \frac{\kappa}{2} + \frac{\kappa}{2} = r + \kappa, \tag{59}$$

and hence

$$Q_{mn} \subset \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}, \tag{59}$$

which gives

$$1 - \varkappa \leq \frac{|Q_{mn}|}{\lambda_{mn}} \leq \frac{1}{\lambda_{mn}} \left| \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\} \right|. \tag{60}$$

So

$$\frac{1}{\lambda_{mn}} \left| \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv} - s_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\} \right| < 1 - (1 - \varkappa) = \varkappa \tag{61}$$

and as a result we acquire

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} \left| \left\{ (u,v) \in J_{mn} : \mathcal{F}_{\|s_{uv}-s_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa \right\} \right| \geq \varkappa \right\} \subset K \in \mathcal{I}_2. \quad (62)$$

This denotes that $s_0 \in \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$. Hence, $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$ is gradually closed.

**Theorem 3.4.** The gradually rough $\mathcal{I}_{(\lambda,\mu)}$-statistical limit set $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$ of the sequence $w$ is a convex set.

**Proof.** Assume $\alpha_0, \alpha_1 \in \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$ and let $\kappa > 0$ and $\tau \in (0,1]$ be taken. Take

$$L_0 = \left\{ (u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv}-\alpha_0\|_{\mathcal{G}}}(\tau) \geq r + \kappa \right\} \quad (63)$$

and

$$L_1 = \left\{ (u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv}-\alpha_1\|_{\mathcal{G}}}(\tau) \geq r + \kappa \right\}. \quad (64)$$

Then, according to the Theorem 3.1, for $\varkappa > 0$ and $\tau \in (0,1]$ we get

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} |\{(u,v) \in J_{mn} : (u,v) \in L_0 \cup L_1\}| \geq \varkappa \right\} \in \mathcal{I}_2. \quad (65)$$

Now, we select $0 < \varkappa_1 < 1$ so that $0 < 1 - \varkappa_1 < \varkappa$ and take

$$L = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{mn}} |\{(u,v) \in J_{mn} : (u,v) \in L_0 \cup L_1\}| \geq 1 - \varkappa_1 \right\}. \quad (66)$$

Let $L \in \mathcal{I}_2$. For each $(m,n) \notin L$, we obtain

$$\frac{1}{\lambda_{mn}} |\{(u,v) \in J_{mn} : (u,v) \in L_0 \cup L_1\}| < 1 - \varkappa_1 \quad (67)$$

and hence

$$\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\notin L_0\cup L_1\}|\geq 1-(1-\varkappa_1)=\varkappa_1. \tag{68}$$

As a result, $\{(u,v)\in J_{mn}:(u,v)\notin L_0\cup L_1\}$ is a nonempty set. Take $(u_0,v_0)\in L_0^c\cap L_1^c$ and $0\leq\lambda\leq 1$.

$$\begin{aligned}\mathcal{F}_{\|w_{u_0v_0}-[(1-\lambda)\alpha_0+\lambda\alpha_1]\|_{\mathcal{G}}}(\tau)&=\mathcal{F}_{\|(1-\lambda)w_{u_0v_0}+\lambda w_{u_0v_0}-[(1-\lambda)\alpha_0+\lambda\alpha_1]\|_{\mathcal{G}}}(\tau)\\&\leq(1-\lambda)\mathcal{F}_{\|w_{u_0v_0}-\alpha_0\|_{\mathcal{G}}}(\tau)+\lambda\mathcal{F}_{\|w_{u_0v_0}-\alpha_1\|_{\mathcal{G}}}(\tau)\\&<(1-\lambda)(r+\kappa)+\lambda(r+\kappa)=r+\kappa.\end{aligned} \tag{69}$$

$$T=\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-[(1-\lambda)\alpha_0+\lambda\alpha_1]\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\}. \tag{70}$$

Then obviously, $L_0^c\cap L_1^c\subset T^c$. So for $(m,n)\notin L$, we get

$$\varkappa_1\leq\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\notin L_0\cup L_1\}|\leq\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\notin T\}| \tag{71}$$

and hence

$$\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in T\}|<1-\varkappa_1<\varkappa. \tag{72}$$

Therefore,

$$L^c\subset\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in T\}|<\varkappa\right\}. \tag{73}$$

Since $L^c\in\mathcal{F}(\mathcal{I}_2)$, we obtain

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in T\}|<\varkappa\right\}\in\mathcal{F}(\mathcal{I}_2) \tag{74}$$

and so

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in T\}|\geq\varkappa\right\}\in\mathcal{I}_2. \tag{75}$$

This finalizes the proof.

**Theorem 3.5.** Take $w = (w_{uv}) \in (Y, \|.\|_{\mathcal{G}})$. Then, $w_{uv} \to w_0 \left(S\mathcal{I}^r_{(\lambda,\mu)}(\mathcal{G})\right)$ if and only if there is a sequence $s = (s_{uv})$ so that $s_{uv} \to w_0 \left(S\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$ and $\mathcal{F}_{\|w_{uv}-s_{uv}\|_{\mathcal{G}}}(\tau) < r$ for each $(u,v) \in \mathbb{N} \times \mathbb{N}$.

**Proof.** Suppose $s = (s_{uv})$ be a sequence in $Y$ such that $s_{uv} \to w_0 \left(S\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$ and $\mathcal{F}_{\|w_{uv}-s_{uv}\|_{\mathcal{G}}}(\tau) < r$ for each $(u,v) \in \mathbb{N} \times \mathbb{N}$. At that time, for any $\kappa, \varkappa > 0$ and $\tau \in (0,1]$

$$Q = \left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{76}$$

Let $(m,n) \notin Q$. Then, we get

$$\frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| < \varkappa \tag{77}$$

i.e.,

$$\frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}\right| \geq 1 - \varkappa. \tag{78}$$

Now, we presume

$$B_{mn} = \left\{(u,v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}. \tag{79}$$

Afterwards, for $(u,v) \in B_{mn}$, we get

$$\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w_{uv}-s_{uv}\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa, \tag{80}$$

and so

$$B_{mn} \subset \left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}. \tag{81}$$

$$\Rightarrow \frac{|B_{mn}|}{\lambda_{mn}} \leq \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa\right\}\right|. \tag{82}$$

$$\Rightarrow \frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r+\kappa\right\}\right| \geq 1-\varkappa. \tag{83}$$

$$\Rightarrow \frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r+\kappa\right\}\right| < 1-(1-\varkappa) = \varkappa. \tag{84}$$

So, we obtain

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r+\kappa\right\}\right| \geq \varkappa\right\} \subset Q \tag{85}$$

and since $Q \in \mathcal{I}_2$, so

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r+\kappa\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{86}$$

Hence, $w_{uv} \to w_0 \left(S\mathcal{I}^r_{(\lambda,\mu)}(\mathcal{G})\right)$.

Conversely, presume that $w_{uv} \to w_0 \left(S\mathcal{I}^r_{(\lambda,\mu)}(\mathcal{G})\right)$. Then, for all $\kappa, \varkappa > 0$ and $\tau \in (0,1]$

$$Q = \left\{(m,n) \in \mathbb{N}\times\mathbb{N}: \frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r+\kappa\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{87}$$

Let $(m,n) \notin Q$. Then, we get

$$\frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \geq r+\kappa\right\}\right| < \varkappa \tag{88}$$

and so

$$\frac{1}{\lambda_{mn}}\left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r+\kappa\right\}\right| \geq 1-\varkappa. \tag{89}$$

Let

$$B_{mn} = \left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r+\kappa\right\}. \tag{90}$$

Now, we determine a sequence $s = (s_{uv})$ as follows:

$$s_{uv} := \begin{cases} w_0, & \text{if } \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \le r, \\ w_{uv} + r\dfrac{w_0 - w_{uv}}{\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau)}, & \text{if not.} \end{cases} \tag{91}$$

Then

$$\mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) = \begin{cases} 0, & \text{if } \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \le r, \\ \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) - r, & \text{if not.} \end{cases} \tag{92}$$

Take $(u, v) \in B_{mn}$. Then, we obtain $\mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) = 0$, if $\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) \le r$ and $\mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa$, if $r < \mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau) < r + \kappa$ and so

$$B_{mn} \subset \left\{(u, v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}. \tag{93}$$

This gives

$$\frac{|B_{mn}|}{\lambda_{mn}} \le \frac{1}{\lambda_{mn}}\left|\left\{(u, v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}\right|. \tag{94}$$

So, we get

$$\frac{1}{\lambda_{mn}}\left|\left\{(u, v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) < \kappa\right\}\right| \ge 1 - \varkappa \tag{95}$$

$$\Rightarrow \frac{1}{\lambda_{mn}}\left|\left\{(u, v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) \ge \kappa\right\}\right| < 1 - (1 - \varkappa) = \varkappa, \tag{96}$$

and hence

$$\left\{(m, n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}}\left|\left\{(u, v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) \ge \kappa\right\}\right| \ge \varkappa\right\} \subset Q. \tag{97}$$

As $Q \in \mathcal{I}_2$, so

$$\left\{(m, n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}}\left|\left\{(u, v) \in J_{mn}: \mathcal{F}_{\|s_{uv}-w_0\|_{\mathcal{G}}}(\tau) \ge \kappa\right\}\right| \ge \varkappa\right\} \in \mathcal{I}_2. \tag{98}$$

Hence, $s_{uv} \to w_0 \left(S\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$.

**Definition 3.4.** A double sequence $w = (w_{uv})$ is called to be gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically bounded if there is an $T > 0$ so that for any $\varkappa > 0$ the set

$$A = \left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}\|_{\mathcal{G}}}(\tau) \geq T\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{99}$$

**Theorem 3.6.** The sequence $w = (w_{uv})$ is gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically bounded iff there is an $r \geq 0$ so that $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}) \neq \emptyset$.

**Proof.** Presume $w = (w_{uv})$ be gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically bounded sequence. Then, there is an $T > 0$ so that for any $\varkappa > 0$ we acquire

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}\|_{\mathcal{G}}}(\tau) \geq T\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{100}$$

Take $A = \left\{(u,v): \mathcal{F}_{\|w_{uv}\|_{\mathcal{G}}}(\tau) \geq T\right\}$. Then

$$\mathcal{I}_2 - \lim_{m,n\to\infty} \frac{1}{\lambda_{mn}} |\{u \leq m, v \leq n: (u,v) \in A\}| = 0. \tag{101}$$

Let $r' = \sup\left\{\mathcal{F}_{\|w_{uv}\|_{\mathcal{G}}}(\tau): (u,v) \in A^c\right\}$. Then the set $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^{r'}(\mathcal{G})$ contains the origin. So, we obtain $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}) \neq \emptyset$ for $r \neq r'$.

Conversely, assume $\mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}) \neq \emptyset$ for some $r > 0$. Suppose $w_0 \in \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$. Select $\kappa = \|w_0\|$. At that time, for all $\varkappa > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{102}$$

Now, getting $T = r + 2\|w_0\|$, we get

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{\lambda_{mn}} \left|\left\{(u,v) \in J_{mn}: \mathcal{F}_{\|w_{uv}\|_{\mathcal{G}}}(\tau) \geq T\right\}\right| \geq \varkappa\right\} \in \mathcal{I}_2. \tag{103}$$

Therefore, we obtain $w$ is gradually $\mathcal{I}_{(\lambda,\mu)}$-statistically bounded.

**Definition 3.5**. A point $\lambda \in Y$ is named to be a gradually $\mathcal{I}_{(\lambda,\mu)}$-statistical cluster point of a sequence $w = (w_{uv})$ in $Y$ provided that for any $\kappa > 0$

$$d_{\mathcal{I}_2}\left(\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\lambda\|_{\mathcal{G}}}(\tau)<\kappa\right\}\right)\neq 0, \tag{104}$$

where

$$d_{\mathcal{I}_2}(Q)=\mathcal{I}_2-\lim_{m,n\to\infty}\frac{1}{\lambda_{mn}}|\{(u,v)\in J_{mn}:(u,v)\in Q\}|, \tag{105}$$

if it exists. The set of all gradually $\mathcal{I}_{(\lambda,\mu)}$-statistical cluster points of $w$ is indicated by $\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$.

**Theorem 3.7.** For any arbitrary $\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$ of a double sequence $w=(w_{uv})$ we get $\mathcal{F}_{\|w_0-\beta\|_{\mathcal{G}}}(\tau)\leq r$, for each $w_0\in\mathcal{I}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})$.

**Proof.** Presume that there is a point $\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$ and $w_0\in\mathcal{I}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})$ so that $\mathcal{F}_{\|w_0-\beta\|_{\mathcal{G}}}(\tau)>r$. Let $\kappa=\frac{\mathcal{F}_{\|w_0-\beta\|_{\mathcal{G}}}(\tau)-r}{3}$. Then,

$$\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\}\supset\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\beta\|_{\mathcal{G}}}(\tau)<\kappa\right\}. \tag{106}$$

As $\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$ we get

$$d_{\mathcal{I}_2}\left(\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-\beta\|_{\mathcal{G}}}(\tau)<\kappa\right\}\right)\neq 0. \tag{107}$$

Hence, we get

$$d_{\mathcal{I}_2}\left(\left\{(u,v)\in J_{mn}:\mathcal{F}_{\|w_{uv}-w_0\|_{\mathcal{G}}}(\tau)\geq r+\kappa\right\}\right)\neq 0 \tag{108}$$

which contradicts that $w_0\in\mathcal{I}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})$. As a result, $\mathcal{F}_{\|w_0-\beta\|_{\mathcal{G}}}(\tau)\leq r$.

**Theorem 3.8.** Assume $w=(w_{uv})\in\left(Y,\|.\|_{\mathcal{G}}\right)$.

$(i)$ If $\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$, then $\mathcal{I}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})=\overline{B_r(\beta)}$.

$(ii)$ $\mathcal{I}_{(\lambda,\mu)}-st-\mathrm{LIM}_w^r(\mathcal{G})=\bigcap_{\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)}\overline{B_r(\beta)}=\left\{w_0\in Y:\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)\subset\overline{B_r(w_0)}\right\}$.

**Proof.**

$(i)$ Take $\beta \in \Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$. Then, according to Theorem 3.7, for all $w_0 \in \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$, $\mathcal{F}_{\|w_0-\beta\|_{\mathcal{G}}}(\tau) \leq r$ and so the result follows.

$(ii)$ By (i) it is obvious that

$$\mathcal{I}_{(\lambda,\mu)} - st - LIM_w^r(\mathcal{G}) \subset \bigcap_{\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)} \overline{B_r(\beta)}. \tag{109}$$

For all $\beta \in \Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)$ and $y \in \bigcap_{\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)} \overline{B_r(\beta)}$ we get $\|y-\beta\| \leq r$. Then, obviously

$$\bigcap_{\beta\in\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right)} \overline{B_r(\beta)} \subset \left\{w_0 \in Y : \Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right) \subset \overline{B_r(w_0)}\right\}. \tag{110}$$

Now, assume $y \notin \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G})$. There is an $\kappa > 0$ such that $d_{\mathcal{I}_2}(Q) \neq 0$, where

$$Q = \left\{(u,v) \in J_{mn} : \mathcal{F}_{\|w_{uv}-y\|_{\mathcal{G}}}(\tau) \geq r + \kappa\right\}. \tag{111}$$

This means the existence of gradually $\mathcal{I}_{(\lambda,\mu)}$-statistical cluster point $\beta$ of a double sequence $w = (w_{uv})$ with $\mathcal{F}_{\|y-\beta\|_{\mathcal{G}}}(\tau) \geq r + \kappa$. This implies $\Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right) \subsetneq \overline{B_r(y)}$ and so

$$y \notin \left\{w_0 \in Y : \Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right) \subset \overline{B_r(w_0)}\right\}. \tag{112}$$

As a result

$$\left\{w_0 \in Y : \Lambda_w^S\left(\mathcal{I}_{(\lambda,\mu)}(\mathcal{G})\right) \subset \overline{B_r(w_0)}\right\} \subset \mathcal{I}_{(\lambda,\mu)} - st - \mathrm{LIM}_w^r(\mathcal{G}). \tag{113}$$

This finalizes the proof.

## 4. Conclusion and Comment

In this paper, considering rough convergence, ideal convergence and $\lambda$-statistical convergence we investigated new type of convergence in GNLS. Moreover, we obtained some interesting results. This study's findings are more generic and a natural extension of the traditional convergence of GNLS.

## Author Statement

This study was carried out in collaboration with equal responsibility. All authors read and approved the final manuscript.

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## Conflict of Interest

As the authors of this study, we declare that we do not have any conflict of interest statement.

## Ethics Committee Approval and Informed Consent

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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