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Finitely-cosmall Quotients

Berke KALEBOĞAZ*[1]

Abstract

In this paper, we first define the notion of finitely-cosmall quotient (singly-cosmall quotient) morphisms. Then we give a characterization of this new concept. We show that an epimorphism $p: Y \rightarrow U$ is a finitely-cosmall quotient (singly-cosmall quotient) if and only if for any right R-module Z any morphism $g: Z \rightarrow Y$ such that pg is a finitely-copartial isomorphism (singly-copartial isomorphism) from Z to Y with codomain U is a finitely (singly) split epimorphism. We also investigate the relation between pure-cosmall quotient and finitely-cosmall quotient (singly-cosmall quotient) morphisms. We prove that over a right Noetherian ring R, an epimorphism $p: Y \rightarrow U$ is a pure-cosmall quotient morphism if and only if p is a finitely-cosmall quotient (singly-cosmall quotient) morphism. Moreover, we obtain an example of right minimal morphisms by using finitely-cosmall quotient (singly-cosmall quotient) morphisms.

Keywords: Pure-cosmall quotient morphisms, finitely-cosmall quotient morphisms, right minimal morphisms

1. INTRODUCTION

Partial morphisms were first introduced in 1984 by Ziegler by using model theoretical language (see in [1]). Then, Monari Martinez studied partial morphisms with algebraic methods by using matrix-theoretic reformulations (see in [2]). In 2020, Cortés-Izurdiaga, Guil Asensio, Kaleboğaz and Srivastava developed partial morphisms in categorical aspect (see in [3]). In [3], they gave a categorical definition of partial morphisms by using pushout and developed a general theory of partial morphisms in any additive category (in the sense of Quillen). First of all, they defined $\mathcal{F}$-partial morphisms with respect to an additive exact substructure $\mathcal{F}$ of an exact structure in an additive category. Then, they showed that $\mathcal{F}$-partial morphisms with respect to a pure-exact substructure $\mathcal{F}$ in the category of left modules over a ring is exactly the partial morphisms that were defined by Ziegler. So they called them *Ziegler partial morphisms*.

Later on, in [4], Kaleboğaz defined $\mathcal{F}$-copartial morphisms with respect to an additive exact substructure $\mathcal{F}$ of an exact structure in an additive category as a dual version of $\mathcal{F}$-partial morphisms. This definition reduces to the definition of copartial morphisms in the specific case of the pure-exact structure in the category of right modules over a ring. In [4], Kaleboğaz also studied another application of $\mathcal{F}$-copartial morphisms to a finite (single) pure-exact substructure $\mathcal{F}$ on the category of left

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modules over a ring. And she called them *finitely (singly) copartial morphisms* (see in Definition 2.2). In [4], Kaleboğaz investigated the relations between copartial morphisms and finitely (singly) copartial morphisms. Moreover, she gave new characterizations of finitely (singly) pure projective modules, flat modules and finitely (singly) projective modules by using copartial morphisms and finitely (singly) copartial morphisms. This new characterizations allowed her to obtain categorical proofs of some results of Azumaya [5] and Mao [6].

Then, in [7], Kaleboğaz and Keskin Tütüncü first introduced $\mathcal{F}$-cosmall quotient morphisms with respect to an additive exact substructure $\mathcal{F}$ of an exact structure in an additive category by using $\mathcal{F}$-copartial morphisms and they gave an application of this definition to a pure-exact substructure $\mathcal{F}$ in the category of right modules over a ring and they called that kind of morphisms *pure-cosmall quotients*. In this paper, we first give the definiton of *finitely-cosmall quotients (singly-cosmall quotients)* as an another application of $\mathcal{F}$-cosmall quotients to the finite (single) pure-exact substructure $\mathcal{F}$ in the category of right modules over a ring (see in Definition 2.9). We also give a new characterization of finitely-cosmall quotient (singly-cosmall quotient) morphisms (see in Proposition 2.11). Moreover, the relation between pure-cosmall quotient morphisms and finitely-cosmall quotient (singly-cosmall quotient) morphisms are investigated in Theorem 2.13.

A morphism $p: M \to N$ is called *right minimal* if any endomorphism $g: M \to M$ with $pg = p$ is an isomorphism (see in [8, page 6]). In [9], right minimal morphisms were defined by Keskin Tütüncü as a dual definition of left minimal morphisms which were studied by Cortés-Izurdiaga et al. in [10]. In [7], Kaleboğaz and Keskin Tütüncü gave an example of right minimal morphisms by using pure-cosmall quotients. One of the main purposes of this paper is to give an another example of right minimal morphisms. In Theorem 2.14, we show that every finitely-cosmall quotient (singly-cosmall quotient) $f: P \to M$ is right minimal for a right R-module P which is projective with respect to finitely (singly) split epimorphisms.

Throughout this paper, all rings are associative with unit and all modules are unitary right modules. Given a ring R, Mod-R is the category of right modules over R whose objects are all right modules over R and morphisms are all module homomorphisms between right R-modules.

2. RESULTS

At the beginning of this work, I will recall that the definitions of pure-exact sequences and finitely (singly) pure-exact sequences that are used frequently from now on.

Let R be a ring, M, Y, Z be right R-modules and $f: Y \to Z$ be an epimorphism. Recall that Azumaya called an epimorphism f M-*pure* if $Hom_R(M, f): Hom_R(M, Y) \to Hom_R(M, Z)$ is an epimorphism or in other words for each homomorphism $g: M \to Z$ there exists a homomorphism $h: M \to Y$ such that $fh = g$ (see in [5]). If an epimorphism $f: Y \to Z$ is M-pure for all finitely presented right R-modules M , f is called *pure epimorphism*. Let X be the kernel of $f: Y \to Z$ with the inclusion $u: X \to Y$. Then it is known that, f is pure epimorphism if and only if X is pure in Y (u is a pure monomorphism) in the sense that the natural homomorphism $X \otimes_R N \to Y \otimes_R N$ derived from the inclusion map $X \to Y$ is a monomorphism for all left R-modules N by the theorem of Fieldhouse [11] and Warfield [12]. Moreover, by Cohn's theorem in [13], this is equivalent to the condition that every finite system of linear equations over X which is solvable in Y is also solvable in X. Then the short exact sequence $X \to Y \to Z$ is said to be pure exact sequence.

In [5], Azumaya replaced the class of finitely presented modules with the class of finitely generated modules in the definition of purity and he got a lot of meaningful results. He called an epimorphism $f: Y \to Z$ *finitely split* if f is M-pure for all finitely generated right R-modules M while f is called *singly split* if f is M-pure for all cyclic right R-modules M. That means, an epimorphism $f: Y \to Z$ is finitely (singly) split if $Hom_R(M, f): Hom_R(M, Y) \to Hom_R(M, Z)$ is an epimorphism for all finitely generated (cyclic) right R-modules M. It is clear that every finitely split epimorphism is both pure and singly split.

For a right R-module M and for any index set I, M^I means I-times direct product and $M^{(I)}$ means I-times direct sum of M. Each element of M^I is denoted by (x_i) whose i-th entry is x_i for each $i \in I$ (can be regarded as row vector) and each element of $M^{(I)}$ is denoted by $[x_i]$ whose i-th entry is x_i for each $i \in I$ (can be regarded as a column vector). If n is a positive integer, M^n is defined to be $M^{(I)} = M^I$ where $I = \{1, 2, \dots, n\}$. Let $\mu = [a_{ij}]_{I \times J}$ be a row-finite matrix over R for two index sets I and J. There exists a right R-homomorphism $\mu: R^{(I)} \to R^{(J)}$ with the mapping $(r_i) \to (r_i)\mu = \sum_i r_i a_{ij}$. For a right R-module M, in [5] Azumaya called μ is a *defining matrix of* M (or μ *defines* M) if $Coker(\mu) \cong M$, i.e., if there is an exact sequence;

$$R^{(I)} \xrightarrow{\mu} R^{(J)} \xrightarrow{\theta} M$$

where θ is an epimorphism.

Let M be any right R-module. For an index J, let $[u_j | j \in J]$ be a system of generators of M. Then there exists an epimorphism $R^{(J)} \to M$ with $(s_j) \to (s_j)[u_j]$. Let $[\mu_{ji} | i \in I]$ be a system of generators of the kernel of this epimorphism for an index set I, and let μ be the (row-finite) $I \times J$ matrix whose i-th row is μ_i for each $i \in I$. Then the mapping $(r_i) \to (r_i)\mu = \sum_i r_i \mu_i$ gives an epimorphism from $R^{(I)}$ onto the kernel. Therefore we have an exact sequence

$$R^{(I)} \xrightarrow{\mu} R^{(J)} \to M$$

and so μ is a defining matrix of M. The matrix depends on the choice of generators $[u_j]$ and $[\mu_i]$. Thus defining matrices of M are not necessarily unique. It is obvious that M is finitely generated or cyclic if and only if M has a defining matrix of finite columns or of single column, respectively, while M is finitely presented if and only if M has a defining matrix of finite rows and columns, i.e., a finite matrix.

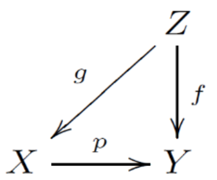
Let V be a right R-module and $\mu = [a_{ij}]_{I \times J}$ be any row-finite matrix over R. By a system of linear equations for μ in V we mean a system of linear equations of the form $\sum_i a_{ij} x_j = v_i$ for $i \in I$, where $[v_i]$ is a given vector in V^I. Let A be a right R-module and B a submodule of A with inclusion $u: B \to A$ and μ be a defining matrix of any module. In [5], Azumaya called that B is *μ-pure in A* (or u is μ-pure monomorphism) if a system of linear equations for μ in B is solvable in B whenever it is solvable in A.

Let M be a right R-module, μ be a defining matrix of M and $f: Y \to Z$ be an epimorphism with kernel X. In [5, Proposition 1], Azumaya proved that f is M-pure if and only if X is μ-pure in Y. In [5, page 119], he had the theorem of Cohn, Fieldhouse and Warfield that f is pure epimorphism if and only if X is pure in Y if and only if X is μ -pure in Y for all finite matrices μ over R. Then in [5, Theorem 3], Azumaya also proved that f is finitely (singly) split epimorphism if and only if X is μ-pure in Y for all matrices μ of finite columns (single column) over R if and only if X is finitely (singly) split in Y. Then, in [4], Kaleboğaz called the short exact sequence $X \to Y \to Z$ is *μ-pure-exact sequence* if u is μ-pure monomorphism (or f is M-pure epimorphism). μ-pure-exact sequences for all finite matrices μ over R in

Mod-R are coincide with pure-exact sequences. In [4], Kaleboğaz also called a short exact sequence *finite (single) pure-exact sequence* if it is a μ-pure exact sequences for all matrices μ of finite columns (one column) over R in Mod-R and she called the class of all finite (single) pure-exact sequences, *finite (single) pure-exact structure* on Mod-R.

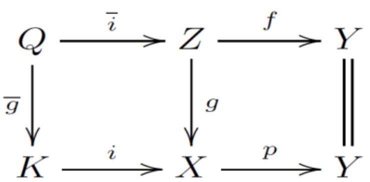
After giving basic definitions and constructions, now we will give our results. Next lemma is a special version of the dual of the Obscure Axiom. This result is used in the rest of this study. It is given by Kaleboğaz as an application in [4, Corollary 3.10] and also proved by Mao in [6, Lemma 2.6 (b)], in another way.

Lemma 2.1 If a finitely (singly) split epimorphism $f: Y \to Z$ factors through an epimorphism $p: X \to Y$ as follows



then p is a finitely (singly) split too.

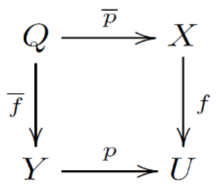
Proof. Let $i: K \to X$ be a kernel of p. If we take the pullback g along i, we get the following commutative diagram:



Since f is a finitely (singly) split epimorphism and $\bar{i}$ is the kernel of f, then $\bar{i}$ is an μ-pure monomorphism for all matrices μ of finite columns (one column) over R. Since the left square is pushout by [14, Example 3, page 93], i is also μ-pure monomorphism for all matrices μ of finite columns (one column) over R. Thus p is a finitely (singly) split epimorphism.

Let us recall the definitions of copartial morphisms (copartial isomorphisms) and finitely (singly) copartial morphisms (finitely (singly) copartial isomorphisms) which were defined by Kaleboğaz in [4, Definition 3.2].

Definition 2.2 Let X, Y be right R-modules and $p: Y \to U$ be an epimorphism. Let $f: X \to U$ be a morphism and consider the pullback of f along the quotient map p:



Then;

1. f is called a *copartial morphism* from X to Y with codomain U if $\overline{p}$ is a pure epimorphism.
2. f is called a *copartial isomorphism* from X to Y with codomain U if both $\overline{p}$ and $\overline{f}$ are pure epimorphisms.
3. f is called a *finitely (singly) copartial morphism* from X to Y with codomain U if $\overline{p}$ is a a finitely (singly) split epimorphism.
4. f is called a *finitely (singly) copartial isomorphism* from X to Y with codomain U if both $\overline{p}$ and $\overline{f}$ are finitely (singly) split epimorphisms.

After this definitions, the relations between them is given by Kaleboğaz in [4, Remark 3.6], as in the following:

Remark 2.3 Let X, Y be right R-modules and U be a quotient of Y. Every finitely (singly) copartial morphism from X to Y with codomain U is copartial morphism from X to Y with codomain U.

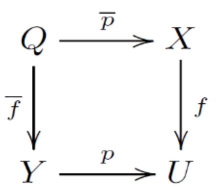
We can extend this corollary to the copartial isomorphisms and finitely (singly) copartial isomorphisms.

Corollary 2.4 Let X, Y be right R-modules and U be a quotient of Y. Every finitely (singly) copartial isomorphism from X to Y with codomain U is copartial isomorphism from X to Y with codomain U. But the converse is not true (see in Example 2.5).

Example 2.5 Let F be a field, $R = \prod_{i=1}^{\infty} F$ and $I = \bigoplus_{i=1}^{\infty} F$. Then R/I is a flat R-module (every epimorphism onto it, is pure) but it is not finitely projective (every epimorphism onto it, is finitely (singly) split) by [15, page 1611]. Thus the identitiy map $1_{R/I}$ is a copartial isomorphism from R/I to R with codomain R/I which is not finitely copartial morphism from R/I to R with codomain R/I.

Lemma 2.6 Let X, Y, U be right R-modules and $p: Y \to U$ be a finitely (singly) split epimorphism. A morphism $f: X \to U$ is a finitely (singly) split epimorphism if and only if f is a finitely (singly) copartial isomorphism from X to Y with codomain U.

Proof. Assume that $f: X \to U$ be a finitely (singly) split epimorphism. If we take the pullback of f along p, we get the following commutative diagram:



with $f\overline{p} = p\overline{f}$. Since pullback preserves the finitely (singly) split epimorphisms $\overline{p}$ and $\overline{f}$ are finitely (singly) split epimorphisms. Therefore, f is finitely (singly) copartial isomorphism from X to Y with codomain U.

For the converse, let f be a finitely (singly) copartial isomorphism from X to Y with codomain U. If we take the pullback of f along p, we get the above commutative diagram. Since f is a finitely (singly) copartial isomorphism from X to Y with codomain U, $\overline{p}$ and $\overline{f}$ are finitely (singly) split epimorphisms. So the composition $p\overline{f}$ and also $f\overline{p}$ are finitely (singly) split epimorphisms. Therefore, f is finitely (singly) split epimorphism from Lemma 2.1.

Corollary 2.7 Let Y, Z be right R-modules and let $g: Z \to Y$ be any morphism. g is a finitely (singly) split epimorphism if and only if g is a finitely (singly)-copartial isomorphism from Z to Y with codomain Y.

Proof. Let us take the pullback of g along 1_Y. Since 1_Y is a finitely (singly) split epimorphism, g is a finitely (singly) copartial isomorphism if and only if g is a finitely (singly) split epimorphism from Lemma 2.6.

Pure-cosmall quotient morphisms are first defined by Kaleboğaz and Keskin Tütüncü [7, Definition 4], as an application of $\mathcal{F}$-cosmall morphisms to a pure-exact substructure $\mathcal{F}$ in the category of right R-modules over a ring R as follows:

Definition 2.8 Let Y, U be right R-modules and $p: Y \to U$ be an epimorphism. Y is *pure-cosmall in* U if for any copartial morphism $g: Z \to Y$ from a right R-module Z to Y with codomain Y, the following holds:

pg is a copartial isomorphism from Z to Y with codomain U implies that g is a copartial isomorphism from Z to Y with codomain Y.

An epimorphism $p: Y \to U$ is called *pure-cosmall quotient* if Y is pure-cosmall in U.

Now we will give another application of $\mathcal{F}$-cosmall quotients to a finite (single) pure-exact substructure $\mathcal{F}$ in the category of right R-modules over a ring R.

Definition 2.9 Let X, Y and U be right R-modules, $p: Y \to U$ and $p': X \to Y$ be epimorphisms.

1. We shall say that Y is *finitely (singly) cosmall in U over X* if for any finitely (singly) copartial morphism $g: Z \to Y$ from a right R-module Z to X with codomain Y, the following holds:
   pg is a finitely (singly) copartial isomorphism from Z to X with codomain U implies that g is a finitely (singly) copartial isomorphism from Z to X with codomain Y.
2. We shall say that Y is *finitely (singly) cosmall in U* if Y is finitely (singly) cosmall in U over Y.

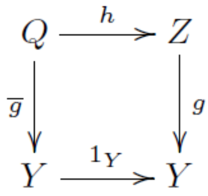
With the notion of finitely (singly)-cosmall R-module which is defined above, we can define finitely (singly)-cosmall quotient morphisms as in the following:

Definition 2.10 *A finitely-cosmall quotient (singly-cosmall quotient)* is an epimorphism $p: Y \to U$ such that Y is finitely-cosmall (singly-cosmall) in U.

Here we will give a characterization of finitely-cosmall quotient (singly-cosmall quotient) which will be used in the rest of the paper.

Proposition 2.11 Let Y, U be right R-modules and $p: Y \to U$ be an epimorphism. p is a finitely-cosmall quotient (singly-cosmall quotient) if and only if for any right R-module Z any morphism $g: Z \to Y$ such that pg is a finitely-copartial isomorphism (singly-copartial isomorphism) from Z to Y with codomain U is a finitely (singly) split epimorphism.

Proof. Let Z be a right R-module and $g: Z \to Y$ be a morphism such that pg is a finitely-copartial isomorphism (singly-copartial isomorphism) from Z to Y with codomain U. We will show that g is a finitely (singly) split epimorphism. If we take pullback of g along 1_Y, then we get the following commutative diagram:



Since 1_Y, is a finitely (singly) split epimorphism, h is also a finitely (singly) split epimorphism. Therefore, g is a finitely-copartial morphism (singly-copartial morphism) from Z to Y with codomain Y. As p is a finitely-cosmall quotient (singly-cosmall quotient), g is also a finitely-copartial isomorphism (singly-copartial isomorphism) from Z to Y with codomain Y. Then, by Corollary 2.7, g is a finitely (singly) split epimorphism.

For the converse, to show that p is a finitely-cosmall quotient (singly-cosmall quotient), let us take a finitely (singly) copartial morphism $g: Z \to Y$ from Z to Y with codomain Y such that pg is a finitely (singly) copartial isomorphism from Z to Y with codomain U. By assumption, g is a finitely (singly) split epimorphism. By Corollary 2.7, g is a finitely (singly) copartial isomorphism from Z to Y with codomain Y. Therefore, p is a finitely-cosmall quotient (singly-cosmall quotient).

In [4, Proposition 3.8], Kaleboğaz proved that over a right Noetherian ring every copartial morphism is finitely and also singly copartial morphism. We can extend this result to the copartial isomorphisms and and finitely copartial isomorphisms as in the following:

Corollary 2.12 Let R be a ring. The followings are equivalent:

1. R is right Noetherian ring.
2. Every copartial isomorphism is finitely copartial isomorphism.
3. Every copartial isomorphism is singly copartial isomorphism.

Now we will investigate the relation between pure-cosmall quotients and finitely (singly)-cosmall quotients:

Theorem 2.13 Let R be a right Noetherian ring. Let Y and U be right R-modules. For the epimorphism $p: Y \to U$ the followings are equivalent:

1. p is a pure-cosmall quotient morphism.
2. p is a finitely-cosmall quotient (singly-cosmall quotient) morphism.

Proof. (1)⇒(2) Let R be a right Noetherian ring. Let $p: Y \to U$ be a pure-cosmall quotient. To show that p is a finitely-cosmall quotient (singly-cosmall quotient), let us take any morphism $g: Z \to Y$ for any right R-module Z such that pg is a finitely (singly) copartial isomorphism from Z to Y with codomain U. By Corollary 2.4, pg is copartial isomorphism from Z to Y with codomain U. By assumption, g is a pure epimorphism by [7, Corollary 2]. Then g is finitely (singly) split epimorphism by [5, Proposition 6]. So p is finitely-cosmall quotient (singly-cosmall quotient).

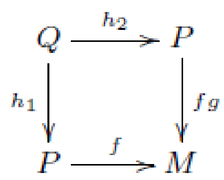
(2)⇒(1) Let R be a right Noetherian ring. Let $p: Y \to U$ be a finitely-cosmall quotient (singly-cosmall quotient). To show that p is a cosmall quotient, let us take any morphism $g: Z \to Y$ for any right R-module Z such that pg is a copartial isomorphism from Z to Y with codomain U. By Corollary 2.12, pg is finitely (singly) copartial isomorphism from Z to Y with codomain U. By assumption, g is finitely (singly) split epimorphism. So g is a pure epimorphism. Therefore, p is a cosmall quotient.

A morphism $p: M \to N$ is called *right minimal* if any endomorphism $g: M \to M$ with $pg = p$ is an isomorphism (see in [8, page 6]). Right minimal morphisms are studied by Keskin Tütüncü in [9]. In [9], the author dualized some results of Cortés-Izurdiaga in [10] and got several useful results by investigating the relationship between $End_R(N)$ and $End_R(M)$ when there is a right minimal epimorphism $p: M \to N$. In [7], Kaleboğaz and Keskin Tütüncü gave an example of right minimal morphisms. They proved in [7, Corollary 3] that every pure-cosmall quotient $f: P \to M$ with P an pure-projective right R-module is right minimal. This result is the dual version of [10, Proposition 1.6] proved by Cortés-Izurdiaga et al.

Now we will show that finitely-cosmall quotient (singly-cosmall quotient) morphisms are also right minimal morphisms under a condition. So the following theorem gives us an example of right minimal morphisms.

Theorem 2.14 Let P be a right R-module which is projective with respect to a finitely (singly) split epimorphisms. Every finitely-cosmall quotient (singly-cosmall quotient) $f: P \to M$ is right minimal.

Proof. Let $f: P \to M$ be finitely-cosmall quotient (singly-cosmall quotient) with P be a right R-module which is projective with respect to finitely (singly) split epimorphisms. Let $g: P \to P$ be a morphism such that $fg = f$. Now we will show that g is an isomorphism. If we consider the pullback of fg along f we get the following commutative diagram;



Since $fg = f$ the identity map 1_P satisfies that $fg1_P = f1_P$. Then by the universal property of pullback, there exist $\alpha: P \to Q$ such that $h_1\alpha = 1_P$ and $h_2\alpha = 1_P$. Since 1_P is finitely (singly) split epimorphism, h_1 and h_2 are both finitely (singly) split epimorphisms, by Lemma 2.1. Therefore, fg is a finitely (singly)-copartial isomorphism from P to P with codomain M. Since f is a finitely (singly)-cosmall quotient, by Proposition 2.11, g is a finitely (singly) split epimorphism.

Now, since $g: P \to P$ is a finitely (singly) split epimorphism and P is a right R-module which is projective with respect to finitely (singly) split epimorphism, then there exists $h: P \to P$ such that $gh = 1_P$. We get;
$f = f1_P = fgh = fh$

So by using the previous argument, h is also an epimorphism. Then as $hgh = h = 1_P h$, we get that $hg = 1_P$. Therefore, g is a monomorphism. So g is an isomorphism.

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