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Not-ordered quasi-distance measures of generalised trapezoidal hesitant fuzzy numbers and their application to decision making problems

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Keywords
Hesitant fuzzy set, Generalised trapezoidal hesitant fuzzy numbers, Quasi-distance measure, Multi-criteria decision-making

Abstract — As an extension of the trapezoidal fuzzy number, the generalised trapezoidal hesitant fuzzy number is an effective mathematical tool for handling uncertainty and vagueness in decision-making problems. Considering that the quasi-distance measure has a strong ability to process and analyse data, we initiated some novel quasi-distance measures to measure the strength of the relationship between generalised trapezoidal hesitant fuzzy numbers in this paper. Moreover, based on the proposed measures, a new multi-criteria decision-making approach is proposed to address uncertain real-life situations. Finally, a practical application of the proposed approach is also illustrated to demonstrate the effectiveness and applicability.

Subject Classification (2020): 03E72, 94D05.

1. Introduction

Since multiple-criteria decision-making (MCDM) problem is an inevitable part of our real life under some ambiguity and imprecision environment, fuzzy sets introduced [1] is more realistic for the decision maker to provide his uncertain linguistic term. Then, Torra [2] and Torra and Narukawa [3] developed hesitant fuzzy sets which the membership degrees of an element of universe set to a given set only by crisp numbers between 0 and 1. So far, many authors have studied on the fuzzy sets and hesitant fuzzy sets in [4–13] and especially in on real number set $\mathbb{R}$. For example, Fahmi et al [14, 15] have defined the concept of triangular cubic hesitant fuzzy number. Amin et al. [16] have propound aggregation operators for triangular cubic linguistic hesitant fuzzy set. Hussain et al. [17] have defined some new operation laws for the trapezoidal linguistic cubic fuzzy numbers including Hamming distance. They have also developed a TOPSIS method to solve the MCDM problems. Peng [18] has developed a multiple attribute decision-making (MADM) approach based on Archimedean t-norm and t-conorm in which the attribute values take the form of hesitant trapezoidal fuzzy elements. Similarly, trapezoidal fuzzy hesitant numbers have been defined and applied to several practical problems, on MADM in [19], on closeness degree and defuzzification technique of hesitant

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trapezoidal fuzzy number in [20]. Moreover, Fahmi et al. [21] have introduced the idea of trapezoidal cubic hesitant fuzzy number and proposed a TOPSIS method. Fahmi et al. [22] then has proposed some new operation laws for trapezoidal cubic hesitant fuzzy numbers and their aggregation operators and Fahmi et al. [23] has defined some new operation laws for the trapezoidal linguistic cubic fuzzy number and Hamming distance of the numbers. Afterwards, Fahmi et al. [24] have developed an MADM method based on trapezoidal cubic fuzzy numbers.

Recently, Deli and Karaaslan [25] have defined the generalised trapezoidal hesitant fuzzy number as a generalisation of the hesitant fuzzy set and generalised fuzzy numbers and it permits the membership degrees of an subset of real numbers to a set to be represented as several possible fuzzy values and therefore it is easier to work on the generalised hesitant fuzzy numbers. Deli [26] then has proposed an advanced type of TOPSIS method to selecting an appropriate robot among the alternative robots under MADM problems by introducing some novel ordered distance measures including Hamming distance measure, Euclidean distance measure, λ-generalised distance measure, λ-generalised Hausdorff distance measure, λ-hybrid Hamming distance, hybrid Euclidean distance and λ-generalised hybrid distance measure on generalised trapezoidal hesitant fuzzy numbers.

In this paper, considering that the quasi-distance measure has a strong ability to process and analyse data and expanding the ordered distance measures given in Deli [26], we initiate some novel not-ordered quasi-distance measures to measure the strength of the relationship between generalised trapezoidal hesitant fuzzy numbers. Moreover, based on the proposed measures, a new MCDM approach is proposed to address uncertain real-life situations. Finally, a practical application of proposed approach is illustrated to demonstrate the effectiveness. This paper is derived from the second author's master's thesis [27].

2. Preliminary

In this section, some concepts and operations of fuzzy sets, hesitant fuzzy sets and generalised trapezoidal hesitant fuzzy numbers (GTHF-numbers) are briefly reviewed. More detailed explanations related to the fuzzy sets, hesitant fuzzy sets and generalised hesitant fuzzy numbers can be found in [1–3, 7, 21, 25, 26, 28].

Definition 2.1. [1] Let X be a non-empty set. A fuzzy set A on X is defined as:

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\}$$

where μ_A is a membership function from X to $[0,1]$.

Definition 2.2. [2, 3] Let X be a non-empty set. A hesitant fuzzy set A on X is defined as

$$A = \{\langle x, \xi(x) = \{\xi_i : i = 1,2,...,l_A(x)\}\rangle : x \in X\}$$

where $\xi(x)$ is a set of some values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set A and $\xi = \xi(x)$ is called a hesitant fuzzy element (HFE). Here, $l_A(x)$ is the number of elements in $\xi(x)$, for $x \in X$ in hesitant fuzzy set A.

Definition 2.3. [25] Let X be a space of points (objects), $\xi_i \in [0,1]$ ($i \in I = \{1,2,...,n\}$ or $\{1,2,...,m\}$ or ...) and $a,b,c,d \in R$ such that $a \leq b \leq c \leq d$. Then, in the set of real numbers $\mathbb{R}$, a generalised trapezoidal hesitant fuzzy number (GTHF-number) can be represented as

$$\xi_N = \langle(a,b,c,d);\{\xi_i : \xi_i \in \xi(x), \xi(x) \text{ is a set of some values in } [0,1]\}\rangle$$

whose membership functions can be described as follows:

$$\mu_A^i(x) = \begin{cases} (x-a)\xi_i/(b-a), & a \le x < b \\ \xi_i, & b \le x \le c \\ (d-x)\xi_i/(d-c), & c < x \le d \\ 0, & otherwise \end{cases}$$

In the paper, for focusing on GTHF- numbers, note that the set of all GTHF-number on $\mathbb{R}$ will be denoted by Φ.

Definition 2.4. [25] Let $\xi_N = \langle(a,b,c,d);\xi(x)\rangle$, $\xi_N^1 = \langle(a_1,b_1,c_1,d_1);\xi^1 = \xi^1(x)\rangle$, $\xi_N^2 = \langle(a_2,b_2,c_2,d_2);\xi^2 = \xi^2(x)\rangle \in \Phi$ and $\gamma \neq 0$ be any real number. Then,

i. $\xi_N^1 + \xi_N^2 = \langle(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2); \cup_{\xi_1^1\in\xi^1,\xi_1^2\in\xi^2}\{\xi_1^1+\xi_1^2-\xi_1^1.\xi_1^2\}\rangle$

ii. $$\xi_N^1 \cdot \xi_N^2 = \begin{cases} \langle(a_1a_2, b_1b_2, c_1c_2, d_1d_2); \cup_{\xi_1^1\in\xi^1,\xi_1^2\in\xi^2}\{\xi_1^1.\xi_1^2\}\rangle(d_1>0, d_2>0) \\ \langle(a_1d_2, b_1c_2, c_1b_2, d_1a_2); \cup_{\xi_1^1\in\xi^1,\xi_1^2\in\xi^2}\{\xi_1^1.\xi_1^2\}\rangle(d_1<0, d_2>0) \\ \langle(d_1d_2, c_1c_2, b_1b_2, a_1a_2); \cup_{\xi_1^1\in\xi^1,\xi_1^2\in\xi^2}\{\xi_1^1.\xi_1^2\}\rangle(d_1<0, d_2<0) \end{cases}$$

iii. $\gamma\xi_N = \langle(\gamma a, \gamma b, \gamma c, \gamma d); \cup_{\xi\in\xi(x)}\{1-(1-\xi)^\gamma\}\rangle(\gamma \ge 0)$

iv. $(\xi_N)^\gamma = \langle(a^\gamma, b^\gamma, c^\gamma, d^\gamma); \cup_{\xi\in\xi(x)}\{\xi^\gamma\}\rangle(\gamma \ge 0)$

Definition 2.5. [26] Let $\xi_N^1, \xi_N^2 \in \Phi$, then the distance measure between ξ_N^1 and ξ_N^2 is defined as $D_{GTHF}(\xi_N^1, \xi_N^2)$ which satisfies the following properties:

i. $0 \le D_{GTHF}(\xi_N^1, \xi_N^2) \le 1$

ii. $D_{GTHF}(\xi_N^1, \xi_N^2) = 0 \Leftrightarrow \xi_N^1 = \xi_N^2$

iii. $D_{GTHF}(\xi_N^1, \xi_N^2) = D_{GTHF}((\xi_N^1, \xi_N^2))$

Note 2.6. [26] Let $\xi_N^1 = \langle(a_1,b_1,c_1,d_1);\{\xi_i^1 : \xi_i^1 \in \xi_1(x), \xi_1(x)$ is a set of some values in $[0,1]\}\rangle$ and $\xi^2 = \langle(a_2,b_2, c_2,d_2);\{\xi_i^2 : \xi_i^2 \in \xi_2(x), \xi_2(x)$ is a set of some values in $[0,1]\}\rangle$ be two GTHF-numbers and l^1 and l^2 be number of value ξ_i^1 in $\xi_1(x)$ and ξ_i^2 in $\xi_2(x)$, respectively. Generally, since we have $l^1 \neq l^2$ we should increase the smallest one until both of them have the same number to compare them. Therefore, thought the paper, he used to add a value to the smallest one by adding the value $\xi_k^1 = \min\{\xi_i^1 : \xi_i^1 \in \xi_1(x)\}$ to compare them.

Example 2.7. Suppose that $\xi_N^1 = \langle(1,5,6,9);\{0.1,0.5, 0.3,0.4\}\rangle$ and $\xi_N^2 = \langle(1,4,6,10);\{0.1,0.5\}\rangle$ be two THF-numbers. Then, we have $l^2 = 2 < \xi_N^1 = 4$. To operate correctly, we should increase the value l_{ξ^2} until it has the same number with l^1, as $\xi_N^2 = \langle(1,4,6,10);\{0.1,0.5,0.1,0.1\}\rangle$.

Definition 2.8. [26] Let $\xi_N^1 = \langle(a_1,b_1,c_1,d_1);\{\xi_i^1 : \xi_i^1 \in \xi_1(x), \xi_1(x)$ is a set of some values in $[0,1]\}\rangle$, $\xi_N^2 = \langle(a_2,b_2, c_2,d_2);\{\xi_i^2 : \xi_i^2 \in \xi_2(x), \xi_2(x)$ is a set of some values in $[0,1]\}\rangle \in \Phi$ and $l_h = \max\{l^1, l^2\}$. Then, *i.* the Hamming distance measure between ξ_N^1 and ξ_N^2, denoted by $D^H_{GTHF}(\xi_N^1,\xi_N^2)$, is defined as;

$$D^H_{GTHF}(\xi_N^1,\xi_N^2)=\sum_{i=1}^{l_h}\Big|\Big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{8.l_h}\Big).\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big| \tag{2.1}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

ii. the Euclidean distance measure between ξ_N^1 and ξ_N^2, denoted by $D^E_{GTHF}(\xi_N^1,\xi_N^2)$, is defined as;

$$D^E_{GTHF}(\xi_N^1,\xi_N^2)=\Big(\sum_{i=1}^{l_h}\Big|\Big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{8.l_h}\Big).\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big|^2\Big)^{\frac{1}{2}} \tag{2.2}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

iii. the λ-generalised distance measure between ξ_N^1 and ξ_N^2 for $\lambda>0$, denoted by $D^\lambda_{GTHF}(\xi_N^1,\xi_N^2)$, is defined as;

$$D^\lambda_{GTHF}(\xi_N^1,\xi_N^2)=\Big(\sum_{i=1}^{l_h}\Big|\Big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{8.l_h}\Big).\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big|^\lambda\Big)^{\frac{1}{\lambda}} \tag{2.3}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

Remark 2.9. [26] Assume that ξ_N^1, $\xi_N^2\in\Phi$, $l_h=\max\{l^1, l^2\}$ and $D^\lambda_{GTHF}(\xi_N^1,\xi_N^2)$ be a λ-generalised distance measure between ξ_N^1 and ξ_N^2 for $\lambda>0$. Especially, if $\lambda=1$, then the λ-generalised distance measure reduces to the Hamming distance measure between ξ_N^1 and ξ_N^2. If $\lambda=2$, then the λ-generalised distance measure, reduces to the the Euclidean distance measure between ξ_N^1 and ξ_N^2 .

Example 2.10. Assume that $\xi_N^1=\langle(0.07,0.09,0.12,0.17);\{0.8,0.5,0.3,0.2\}\rangle$ and $\xi_N^2=\langle(0.2,0.5,0.6,0.8);\{0.4,0.1\}\rangle$ be two GTHF-numbers. Then, for $l_h=4=\max\{4,2\}$,

i. the Hamming distance measure $D^H_{GTHF}(\xi_N^1,\xi_N^2)$ between ξ_N^1 and ξ_N^2 is calculated as;

$$\begin{aligned}
D^H_{GTHF}(\xi_N^1,\xi_N^2) = {} & \textstyle\sum_{i=1}^4\Big|\Big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{8.l_h}\Big).\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big| \\
= {} & \Big|\Big(\frac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\Big).\big(0.8-0.4\big)\Big|+ \\
& \Big|\Big(\frac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\Big).\big(0.5-0.1\big)\Big|+ \\
& \Big|\Big(\frac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\Big).\big(0.3-0.1\big)\Big|+ \\
& \Big|\Big(\frac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\Big).\big(0.2-0.1\big)\Big| \\
= {} & 0.0234
\end{aligned}$$

ii. the Euclidean distance measure $D^E_{GTHF}(\xi^1_N,\xi^2_N)$ between ξ^1_N and ξ^2_N is calculated as;

$$\begin{aligned} D^E_{GTHF}(\xi^1_N,\xi^2_N) = & \Big(\sum_{i=1}^{l_h}\Big|\big(\tfrac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{8.l_h}\big).(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)})\Big|^2\Big)^{\frac{1}{2}} \\ = & \Big(\Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.8-0.4\big)\Big|^2+ \\ & \Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.5-0.1\big)\Big|^2+ \\ & \Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.3-0.1\big)\Big|^2+ \\ & \Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.2-0.1\big)\Big|^2\Big)^{\frac{1}{2}} \\ = & \ 0.0079 \end{aligned}$$

iii. the λ-generalised distance measure $D^\lambda_{GTHF}(\xi^1_N,\xi^2_N)$ between ξ^1_N and ξ^2_N is calculated as; $\lambda=0.7$,

$$\begin{aligned} D^{0.7}_{GTHF}(\xi^1_N,\xi^2_N) = & \Big(\sum_{i=1}^{l_h}\Big|\big(\tfrac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{8.l_h}\big).\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big|^{0.7}\Big)^{\frac{1}{0.7}} \\ = & \Big(\Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.8-0.4\big)\Big|^{0.7}+ \\ & \Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big)(0.5-0.1)\Big|^{0.7}+ \\ & \Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.3-0.1)|^{0.7}+ \\ & \Big|\big(\tfrac{0.12^2+0.17^2-0.07^2-0.09^2-0.6^2-0.8^2+0.2^2+0.5^2}{8.4}\big).\big(0.2-0.1\big)\Big|^{0.7}\Big)^{\frac{1}{0.7}} \\ = & \ 0.0335 \end{aligned}$$

Definition 2.11. [26] Let $\xi^1_N=\langle(a_1,b_1,c_1,d_1);\{\xi^1_i:\xi^1_i\in\xi_1(x),\ \xi_1(x)$ is a set of some values in $[0,1]\}\rangle$, $\xi^2_N=\langle(a_2, b_2,c_2,d_2);\{\xi^2_i:\xi^2_i\in\xi_2(x),\xi_2(x)$ is a set of some values in $[0,1]\}\rangle\in\Phi$. Then, based on Hausdorff metric, the λ-generalised Hausdorff distance measure between ξ^1_N and ξ^2_N for $\lambda,\gamma>0$, denoted by $DH_{GTHF}(\xi^1_N,\xi^2_N)$, is defined as;

$$DH_{GTHF}(\xi^1_N,\xi^2_N)=\max_i\Big|\frac{1}{8}\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big| \tag{2.4}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

The hybrid distances is given as;

i. Combining the Equations 2.1 and 2.4 a γ-hybrid Hamming distance, denoted by ${}_\gamma DH^H_{GTHF}(\xi^1_N,\xi^2_N)$, is defined as;

$$\begin{aligned} {}_\gamma DH^H_{GTHF}(\xi^1_N,\xi^2_N) = & \Big|\big(\tfrac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{16.l_h}\big)\Big|. \\ & \Big(\sum_{i=1}^{l_h}\gamma.\Big|\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big|+(1-\gamma).\big(\max_i\Big|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\Big|\big)\Big) \end{aligned} \tag{2.5}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

ii. Combining the Equations 2.2 and 2.4 a hybrid Euclidean distance, denoted by ${}_{\gamma}DH^{E}_{GTHF}(\xi^1_N, \xi^2_N)$, is defined as;

$$ {}_{\gamma}DH^{E}_{GTHF}(\xi^1_N, \xi^2_N) = \Big(\Big|\big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{16.l_h}\big)\Big|.\big(\sum_{i=1}^{l_h}\gamma. \Big|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big|^2+(1-\gamma).\big(\max_i\Big|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\Big|\big)^2\big)\Big)^{\frac{1}{2}} \tag{2.6}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

iii. Combining the Equations 2.3 and 2.4 a λ-generalised hybrid distance, denoted by ${}_{\gamma}DH^{\lambda}_{GTHF}(\xi^1_N, \xi^2_N)$, is defined as;

$$ {}_{\gamma}DH^{\lambda}_{GTHF}(\xi^1_N, \xi^2_N) = \Big(\Big|\big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{16.l_h}\big)\Big|.\big(\sum_{i=1}^{l_h}\gamma. \Big|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\big)\Big|^{\lambda}+\big(1-\gamma\big).\big(\max_i\Big|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\Big|\big)^{\lambda}\big)\Big)^{\frac{1}{\lambda}} \tag{2.7}$$

where $\xi^1_{\sigma(i)}$ and $\xi^2_{\sigma(i)}$ are the i. largest values in $\xi_1(x)$ and $\xi_2(x)$, respectively.

Example 2.12. Assume that $\xi^1_N = \langle(0.2, 0.4, 0.6, 0.8); \{0.7, 0.6, 0.5, 0.3, 0.2\}\rangle$ and $\xi^2_N = \langle(0.1, 0.4, 0.5, 0.7); \{0.9, 0.3, 0.4\}\rangle$ be two GTHF-numbers. Then, for $l_h = 5 = \max\{5, 3\}$,

$$\begin{aligned} DH_{GTHF}(\xi^1_N, \xi^2_N) = & \max_i\Big|\tfrac{1}{8}\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)})\Big| \\ = & \Big|\tfrac{1}{8}\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big)\Big|\max_i\Big|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}\Big| \\ = & \Big|\tfrac{1}{8}\big(0.6^2+0.8^2-0.2^2-0.4^2-0.5^2-0.7^2+0.1^2+0.4^2\big)\Big|.\max\{0.2, 0.3, 0.1, 0.1, 0.2\} \\ = & 0.0086 \end{aligned}$$

Now, we give the hybrid distance as;

i. Combining the Equations 2.1 and 2.4, 0.5-hybrid Hamming distance, denoted by ${}_{0.5}DH^{H}_{GTHF}(\xi^1_N, \xi^2_N)$, is defined as;

$$\begin{aligned} {}_{0.5}DH^{H}_{GTHF}(\xi^1_N, \xi^2_N) = & \Big|\big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{16.l_h}\big)\Big|.\Big(\sum_{i=1}^{l_h}\gamma.|\big(\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)})|+(1-\gamma).(\max_i|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}|)\Big) \\ = & 0.1504 \end{aligned}$$

ii. Combining the Equations 2.2 and 2.4 a 0.5-hybrid Euclidean distance, denoted by ${}_{0.5}DH^{E}_{GTHF}(\xi^1_N, \xi^2_N))$, is defined as;

$$\begin{aligned} {}_{0.5}DH^{E}_{GTHF}(\xi^1_N, \xi^2_N) = & \Big(\Big|\big(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{16.l_h}\big)\Big|. \\ & (\sum_{i=1}^{l_h}(0.5).|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)})|^2|+(1-0.5).(\max_i|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}|)^2)\Big)^{\frac{1}{2}} \\ = & 0.2124 \end{aligned}$$

iii. Combining the Equations 2.3 and 2.4 a 0.5-generalised hybrid distance, denoted by ${}_{0.5}DH^{0.5}_{GTHF}(\xi^1_N, \xi^2_N)$, is defined as;

$$
\begin{aligned}
{}_{0.5}DH^{0.5}_{GTHF}(\xi^1_N,\xi^2_N) = & \Big(\Big|\big(\tfrac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{16.l_h}\big)\Big|.\\
& (\textstyle\sum_{i=1}^{l_h}(0.5).|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)})|^{0.5}|+(1-0.5).(\max_i|\xi^1_{\sigma(i)}-\xi^2_{\sigma(i)}|)^{0.5})\Big)^{\frac{1}{0.5}}\\
= & \ 0.0754
\end{aligned}
$$

3. Not-Ordered Quasi-Distance Measures on GTHF-Numbers

In this section, we gave not-ordered quasi-distance measures on GTHF-numbers and their properties based on some definitions of hesitant fuzzy sets in [7] and GTHF-numbers in [25, 26, 28].

Definition 3.1. Let $\xi^1_N=\langle(a_1,b_1,c_1,d_1);\xi^1_1\rangle$ and $\xi^2_N=\langle(a_2,b_2,c_2,d_2);\xi^2_1\rangle$ be two GTHF-numbers. Then,

i. the Not-ordered Hamming quasi-distance measure between ξ^1_N and ξ^2_N, denoted by $d_{NoH}(\xi^1_N,\xi^2_N)$, is defined as;

$$d_{NoH}(\xi^1_N,\xi^2_N)=\frac{1}{4.k^2.l_p}\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big|\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).\big(\xi^1_1-\xi^2_1\big)\Big| \tag{3.1}$$

where $k=\max\{|a_1|,|a_2|,|b_1|,|b_2|,|c_1|,|c_2|,|d_1|,|d_2|\}$ and l is product of number of element ξ^1_N and ξ^2_N.

ii. the Not-ordered Euclidean quasi-distance measure between ξ^1_N and ξ^2_N, denoted by $d_{NoE}(\xi^1_N,\xi^2_N)$, is defined as;

$$d_{NoE}(\xi^1_N,\xi^2_N)=\Big(\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big|\frac{1}{4.k^2.l_p}\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).\big(\xi^1_1-\xi^2_1\big)\Big|^2\Big)^{\frac{1}{2}} \tag{3.2}$$

where $k=\max\{|a_1|,|a_2|,|b_1|,|b_2|,|c_1|,|c_2|,|d_1|,|d_2|\}$ and l is product of number of element ξ^1_N and ξ^2_N.

iii. the Not-ordered λ-generalised quasi-distance measure between ξ^1_N and ξ^2_N for $\lambda>0$, denoted by $d^\lambda_{NoG}(\xi^1_N,\xi^2_N)$, is defined as;

$$d^\lambda_{NoG}(\xi^1_N,\xi^2_N)=\Big(\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big|\frac{1}{4.k^2.l_p}\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).\big(\xi^1_1-\xi^2_1\big)\Big|^\lambda\Big)^{\frac{1}{\lambda}} \tag{3.3}$$

where $k=\max\{|a_1|,|a_2|,|b_1|,|b_2|,|c_1|,|c_2|,|d_1|,|d_2|\}$ and l is product of number of element ξ^1_N and ξ^2_N.

iv. the Not-ordered the λ-generalised not ordered Hausdorff quasi-distance measure based on Hausdorff metric, between ξ^1_N and ξ^2_N, denoted by $d^\lambda_{NoHa}(\xi^1_N,\xi^2_N)$, is defined as;

$$d^\lambda_{NoHa}(\xi^1_N,\xi^2_N)=\max_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big|\frac{1}{4.k^2}\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).\big(\xi^1_1-\xi^2_1\big)\Big| \tag{3.4}$$

where $k=\max\{|a_1|,|a_2|,|b_1|,|b_2|,|c_1|,|c_2|,|d_1|,|d_2|\}$.

Theorem 3.2. Suppose that $\xi^1_N=\langle(a_1,b_1,c_1,d_1);\xi^1_1(x)\rangle$, $\xi^2_N=\langle(a_2,b_2,c_2,d_2);\xi^2_1(x)\rangle$ and $\xi^3_N=\langle(a_3,b_3,c_3,d_3);\xi^3_1=\xi^3(x)\rangle$ three GTHF-numbers and $\gamma\neq 0$ be any real number. Then, for $d\in\{d_{NoH},d_{NoE},d^\lambda_{NoG},d^\lambda_{NoHa}\}$, d satisfies the following properties:

i. $0 \leq d(\xi_N^1, \xi_N^2) \leq 1$

ii. $d(\xi_N^1, \xi_N^2) = d(\xi_N^2, \xi_N^1)$

iii. $\xi_N^1 = \xi_N^2 \Rightarrow d(\xi_N^1, \xi_N^2) = 0$

iv. $d(\xi_N^1, \xi_N^3) + d(\xi_N^3, \xi_N^2) \geq d(\xi_N^1, \xi_N^2)$

Proof.

Let $\xi_N^1 = \langle (a_1, b_1, c_1, d_1); \xi_1^1(x) \rangle$, $\xi_N^2 = \langle (a_2, b_2, c_2, d_2); \xi_1^2(x) \rangle$ and $\xi_N^3 = \langle (a_3, b_3, c_3, d_3); \xi_1^3 = \xi^3(x) \rangle$ three GTHF-numbers and $\gamma \neq 0$ be any real number.

i. Let $k = \max\{|a_1|, |a_2|, |b_1|, |b_2|, |c_1|, |c_2|, |d_1|, |d_2|\}$ and l_p is product of number of element ξ_N^1 and ξ_N^2.

i. Since

$$0 \leq \left|\left(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2\right)\right| \leq 4k^2 \quad \text{and} \quad 0 \leq \left|\left({\xi_1}^1 - {\xi_1}^2\right)\right| \leq 1$$

we have

$$0 \leq \quad \left(\left|\left(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2\right).\left({\xi_1}^1 - {\xi_1}^2\right)\right|\right) \leq 4.k^2$$

$$\Rightarrow \quad 0 \leq \left(\sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\left(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2\right).\left({\xi_1}^1 - {\xi_1}^2\right)\right|\right) \leq 4.k^2.l_p$$

$$\Rightarrow \quad 0 \leq \frac{1}{4.k^2.l_p}\left(\sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\left(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2\right).\left({\xi_1}^1 - {\xi_1}^2\right)\right|\right) \leq 1$$

$$\Rightarrow \quad 0 \leq d_{NoH}(\xi_N^1, \xi_N^2) \leq 1$$

ii.

$$\begin{aligned}
d_{NoH}(\xi_N^1, \xi_N^2) = & \quad \frac{1}{4.k^2.l_p} \sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\left(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2\right).\left({\xi_1}^1 - {\xi_1}^2\right)\right| \\
= & \quad \frac{1}{4.k^2.l_p}\left(\sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\left(c_2^2 + d_2^2 - a_2^2 - b_2^2 - c_1^2 - d_1^2 + a_1^2 + b_1^2\right).\left({\xi_1}^2 - {\xi_1}^1\right)\right|\right) \\
= & \quad d_{NoH}(\xi_N^2, \xi_N^1)
\end{aligned}$$

iii. Since ξ_N^1 and ξ_N^2 are identical then $a = a_1 = a_2, b = b_1 = b_2, c = c_1 = c_2, d = d_1 = d_2, \xi = \xi_N^1 = \xi_N^2$. The degree of quasi-distance ξ_N^1 and ξ_N^2 are calculated as follows

$$\begin{aligned}
d_{NoH}(\xi_N^1, \xi_N^2) = & \quad \frac{1}{4.k^2.l_p} \sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\left(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2\right).\left({\xi_1}^1 - {\xi_1}^2\right)\right| \\
= & \quad \frac{1}{4.k^2.l_p} \sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\left(c + d - a - b - c - d + a + b\right).\left(\xi - \xi\right)\right| \\
= & \quad 0
\end{aligned}$$

iv.

$$\begin{aligned} d_{NoH}(\xi_N^1,\xi_N^2) = & \frac{1}{4.k^2.l_p}\sum_{\xi_1^1\in\xi^1,\xi_1^2\in\xi^2}\Big|\big(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\big).\big({\xi_1}^1-{\xi_1}^2\big)\Big| \\ = & \frac{1}{4.k^2.l_p}\sum_{\xi_1^1\in\xi_N^1,\xi_1^2,\xi_1^3\in\xi_N^2}\Big|\big(c_1^2+d_1^2-a_1^2-b_1^2+c_3^2+d_3^2-a_3^2-b_3^2 \\ & -c_2^2-d_2^2+a_2^2+b_2^2-c_3^2-d_3^2+a_3^2+b_3^2\big).\big({\xi_1}^1-{\xi_1}^2-{\xi_1}^3+{\xi_1}^3\big)\Big| \\ \le & \sum_{\xi_1^1\in\xi_N^1,\xi_1^3\in\xi_N^3}\Big|\frac{1}{4.k^2.l_p}(c_1^2+d_1^2-a_1^2-b_1^2-c_3^2-d_3^2+a_3^2+b_3^2).\big({\xi_1}^1-{\xi_1}^3\big)\Big|+ \\ & \sum_{\xi_1^3\in\xi_N^3,\xi_1^2\in\xi_N^2}\Big|\frac{1}{4.k^2.l_p}(c_3^2+d_3^2-a_3^2-b_3^2-c_2^2-d_2^2+a_2^2+b_2^2).\big({\xi_1}^3-{\xi_1}^2\big)\Big| \\ = & d_{NoH}(\xi_N^1,\xi_N^3)+d_{NoH}(\xi_N^3,\xi_N^2) \end{aligned}$$

Similarly, for $d\in\{d_{NoE},d_{NoG}^{\lambda},d_{DNoH}\}$, proof of theorem can be made.

Remark 3.3. Assume that $d_{NoG}^{\lambda}(\xi_N^1,\xi_N^2)$ be a Not-ordered λ-generalised quasi-distance measure between ξ_N^1 and ξ_N^2 for $\lambda>0$. Especially, if $\lambda=1$, then the Not-ordered λ-generalised quasi-distance measure reduces to the Not-ordered Hamming quasi-distance measure between ξ_N^1 and ξ_N^2. If $\lambda=2$, then the Not-ordered λ-generalised quasi-distance measure, reduces to the Not-ordered Euclidean quasi-distance measure between ξ_N^1 and ξ_N^2.

Example 3.4. Assume that $\xi_N^1=\langle(10,15,20,25);\{0.7,0.8,0.5\}\rangle$ and $\xi_N^2=\langle(-13,-10,-7,-5);\{0.9,0.2\}\rangle$ be two GTHF-numbers. Then,

i. the Hamming quasi-distance measure $d_{NoH}(\xi_N^1,\xi_N^2)$ between ξ_N^1 and ξ_N^2 is calculated as;

$$\begin{aligned} d_{NoH}(\xi_N^1,\xi_N^2)=\frac{1}{4.625.6} & \Big(\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.7-0.9\big)\Big|+ \\ & \Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.7-0.2\big)\Big|+ \\ & \Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.8-0.9\big)\Big|+ \\ & \Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.8-0.2\big)\Big|+ \\ & \Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.5-0.9\big)\Big|+ \\ & \Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.5-0.2\big)\Big|\Big) \\ =\frac{1}{4.625.6} & \Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big)\Big| \\ & \Big|\big(0.7-0.9)+(0.7-0.2)+(0.8-0.9)+(0.8-0.2)+(0.5-0.9\big)+(0.5-0.2\big)\Big| \\ =0.041 & \end{aligned}$$

ii. the Euclidean quasi-distance measure $d_{NoE}(\xi_N^1, \xi_N^2)$ between ξ_N^1 and ξ_N^2 is calculated as;

$$\begin{aligned}
d_{NoE}(\xi_N^1, \xi_N^2) = &\ \Big(\Big|\tfrac{1}{4.625.6}(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).\big(0.7-0.9\big)\Big|^2+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.7-0.2\big)\Big|^2+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.8-0.9\big)\Big|^2+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.8-0.2\big)\Big|^2+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.5-0.9\big)\Big|^2+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.5-0.2\big)\Big|^2\Big)^{\frac{1}{2}}\\
= &\ \Big|\tfrac{1}{4.625.6}\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big)^2\Big|\\
&\Big|\big((0.7-0.9)^2+(0.7-0.2)^2+(0.8-0.9)^2+(0.8-0.2)^2+(0.5-0.9)^2+(0.5-0.2)^2\big)\Big|^{\frac{1}{2}}\\
= &\ 0.056
\end{aligned}$$

iii. the λ-generalised quasi-distance measure $d_{NoG}^{\lambda}(\xi_N^1, \xi_N^2)$ between ξ_N^1 and ξ_N^2 is calculated as; $\lambda = 0.8$,

$$\begin{aligned}
d_{NoG}^{0.8}(\xi_N^1, \xi_N^2) = &\ \Big(\Big|\tfrac{1}{4.625.6}(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.7-0.9\big)\Big|^{0.8}+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.7-0.2\big)\Big|^{0.8}+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.8-0.9\big)\Big|^{0.8}+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.8-0.2\big)\Big|^{0.8}+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.5-0.9\big)\Big|^{0.8}+\\
&\Big|\big(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big).\big(0.5-0.2\big)\Big|^{0.8}\Big)^{\frac{1}{0.8}}\\
= &\ \Big|\tfrac{1}{4.625.6}(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2\big)^2\big((0.7-0.9)^{0.8}\\
&+(0.7-0.2)^{0.8}+(0.8-0.9)^{0.8}+(0.8-0.2)^{0.8}+(0.5-0.9)^{0.8}+(0.5-0.2)^{0.8}\big)\Big|^{\frac{1}{0.8}}\\
= &\ 0.19
\end{aligned}$$

and $\lambda = 5$,

$$
\begin{aligned}
d^5_{NoG}(\xi^1_N,\xi^2_N) = & \Big(\Big|\tfrac{1}{4.625.6}(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).(0.7-0.9)\Big|^5+\\
& \Big|(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).(0.7-0.2)\Big|^5+\\
& \Big|(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).(0.8-0.9)\Big|^5+\\
& \Big|(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).(0.8-0.2)\Big|^5+\\
& \Big|(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).(0.5-0.9)\Big|^5+\\
& \Big|(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2).(0.5-0.2)\Big|^5\Big)^{\frac{1}{5}}\\
= & \Big|\tfrac{1}{4.625.6}(20^2+25^2-10^2-15^2-(-7)^2-(-5)^2+(-13)^2+(-10)^2)^2\\
& ((0.7-0.9)^5+(0.7-0.2)^5+(0.8-0.9)^5+(0.8-0.2)^5+(0.5-0.9)^5+(0.5-0.2)^5)\Big|^{\frac{1}{5}}\\
= & \ 0.037
\end{aligned}
$$

Definition 3.5. Let ξ^1_N, ξ^2_N be two GTHF-numbers. Then, for $\lambda,\gamma > 0$, the hybrid quasi-distances are given as:

i. Combining the Equations 3.1 and 3.4 a γ- not ordered hybrid Hamming quasi-distance, denoted by $_{\gamma}DNOH^H_{GTHF}(\xi^1_N,\xi^2_N)$, is defined as;

$$
\begin{aligned}
d_{_\gamma DNOH^H}(\xi^1_N,\xi^2_N) = & \tfrac{1}{4.k^2.l_p}\Big|(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2)\Big|.\\
& \Big(\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\gamma.\Big|\xi^1_1-\xi^2_1)\Big|+(1-\gamma).l_p.(\max_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big|{\xi_1}^1-{\xi_1}^2\Big|)\Big)
\end{aligned}
$$

where $k=\max\{|a_1|,|a_2|,|b_1|,|b_2|,|c_1|,|c_2|,|d_1|,|d_2|\}$ and l_p is product of number of element ξ^1_N and ξ^2_N.

ii. Combining the Equations 3.2 and 3.4 a not-ordered hybrid Euclidean quasi-distance, denoted by $d_{_\gamma DNOH^E}(\xi^1_N,\xi^2_N)$, is defined as;

$$
\begin{aligned}
d_{_\gamma DNOH^E}(\xi^1_N,\xi^2_N) = & \Big(\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big(\tfrac{1}{4.k^2.l_p}(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\Big)^2.\Big(\gamma\Big|{\xi_1}^1-{\xi_1}^2\Big|^2+\\
& (1-\gamma).\max_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}.l_p.\Big({\xi_1}^1-{\xi_1}^2\Big)^2\Big)\Big)^{\frac{1}{2}}
\end{aligned}
$$

where $k=\max\{\Big|a_1\Big|,\Big|a_2\Big|,\Big|b_1\Big|,\Big|b_2\Big|,\Big|c_1\Big|,\Big|c_2\Big|\Big|d_1\Big|,\Big|d_2\Big|\}$ and l is product of number of element ξ^1_N and ξ^2_N.

iii. Combining the Equations 3.3 and 3.4 a λ-generalised not-ordered hybrid quasi-distance, denoted by $d_{_\gamma DNOH^\lambda}(\xi^1_N,\xi^2_N)$, is defined as;

$$
\begin{aligned}
d_{_\gamma DNOH^\lambda}(\xi^1_N,\xi^2_N) = & \Big(\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\Big(\tfrac{1}{4.k^2.l_p}(c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2\Big)^\lambda.\Big(\gamma\Big|{\xi_1}^1-{\xi_1}^2\Big|^\lambda+\\
& (1-\gamma).\max_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}.l_p.\Big({\xi_1}^1-{\xi_1}^2\Big)^\lambda\Big)\Big)^{\frac{1}{\lambda}}
\end{aligned}
$$

where $k = \max\{|a_1|, |a_2|, |b_1|, |b_2|, |c_1|, |c_2|, |d_1|, |d_2|\}$ and l_p is product of number of element ξ_N^1 and ξ_N^2.

Theorem 3.6. Suppose that $\xi_N^1 = \langle (a_1, b_1, c_1, d_1); \xi_1^1(x)\rangle$, $\xi_N^2 = \langle (a_2, b_2, c_2, d_2); \xi_1^2(x)\rangle$ and $\xi_N^3 = \langle (a_3, b_3, c_3, d_3);$ $\xi_1^3 = \xi^3(x)\rangle$ three GTHF-numbers and $\gamma \neq 0$ be any real number. Then, for $d \in \{d_{\gamma DNOH^H}, d_{\gamma DNOH^E}, d_{\gamma DNOH^\lambda}\}$, d satisfies the following properties:

i. $0 \leq d(\xi_N^1, \xi_N^2) \leq 1$

ii. $d(\xi_N^1, \xi_N^2) = d(\xi_N^2, \xi_N^1)$

iii. $\xi_N^1 = \xi_N^2 \Rightarrow d(\xi_N^1, \xi_N^2) = 0$

iv. $d(\xi_N^1, \xi_N^3) + d(\xi_N^3, \xi_N^2) \geq d(\xi_N^1, \xi_N^2)$

Proof: Proof of the theorem is clear.

Example 3.7. Assume that $\xi_N^1 = \langle (0.1, 0.2, 0.3, 0.4); \{0.9, 0.7, 0.6, 0.5, 0.1\}\rangle$ and $\xi_N^2 = \langle (0.3, 0.5, 0.7, 0.9); \{0.8, 0.1, 0.5\}\rangle$ be two GTHF-numbers. Then, for $l_p = 5.3 = 15$,

$$\begin{aligned} d_{\gamma DNOH}(\xi_N^1, \xi_N^2) = \; & \max\left\{\left|\tfrac{1}{4\cdot 0.9^2\cdot 15}(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2).(\xi_1^1 - \xi_1^2)\right|\right\} \\ = \; & \left|\tfrac{1}{48.6}(c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2)\right|.\max\left\{\left|\xi_1^1 - \xi_1^2\right|\right\} \\ = \; & \left|\tfrac{1}{48.6}(0.3^2 + 0.4^2 - 0.1^2 - 0.2^2 - 0.7^2 - 0.9^2 + 0.3^2 + 0.5^2)\right| \\ & \max\{0.1, 0.8, 0.4, 0.1, 0.6, 0.2, 0, 2, 0.5, 0.1, 0.3, 0.4, 0.0, 0.7, 0.0, 0.4\} \\ = \; & 0.012 \end{aligned}$$

Now, we give the hybrid quasi-distance as:

i. Combining the Equations 2.1 and 2.4 0.5-hybrid Hamming quasi-distance, denoted by ${}_{0.5}DNOH_{GTHF}^H(\xi_N^1, \xi_N^2)$, is defined as;

$$\begin{aligned} d_{0.5DNOH^H}(\xi_N^1, \xi_N^2) = \; & \left|\left(\frac{c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2}{4.k^2.l}\right)\right|. \\ & \left(\sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} (0.5).\left|(\xi_1^1 - \xi_1^2)\right| + (1 - (0.5)).\left(\max_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\xi_1^1 - \xi_1^2\right|\right)\right) \\ = \; & 0.042 \end{aligned} \tag{3.5}$$

ii. Combining the Equations 2.2 and 2.4 a 0.5not-ordered hybrid Euclidean quasi-distance, denoted by $d_{0.5DNOH^E}(\xi_N^1, \xi_N^2)$, is defined as;

$$\begin{aligned} d_{0.5DNOH^E}(\xi_N^1, \xi_N^2) = \; & \left(\left|\left(\frac{c_1^2 + d_1^2 - a_1^2 - b_1^2 - c_2^2 - d_2^2 + a_2^2 + b_2^2}{4.k^2.l}\right)\right|.\right. \\ & \left.\left(\sum_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} (0.5).\left|\xi_1^1 - \xi_1^2)\right|^2\right| + (1 - 0.5).\left(\max_{\xi_1^1 \in \xi_N^1, \xi_1^2 \in \xi_N^2} \left|\xi_1^1 - \xi_1^2\right|\right)^2\right)^{\frac{1}{2}} \\ = \; & 0.022 \end{aligned} \tag{3.6}$$

iii. Combining the Equations 2.3 and 2.4 a 0.5-generalised not ordered hybrid quasi-distance, denoted by ${}_{0.5}DNOH^{0.5}_{GTHF}(\xi^1_N,\xi^2_N)$, is defined as;

$$\begin{aligned} d_{{}_{0.5}DNOH^{0.5}}(\xi^1_N,\xi^2_N) = & \left(\left|\left(\frac{c_1^2+d_1^2-a_1^2-b_1^2-c_2^2-d_2^2+a_2^2+b_2^2}{4.k^2.l}\right)\right|.\right. \\ & \left(\sum_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}(0.5).\left|\xi^1_1-\xi^2_1\right|^{0.5}\right|+(1-0.5).\left(\max_{\xi^1_1\in\xi^1_N,\xi^2_1\in\xi^2_N}\left|\xi^1_1-\xi^2_1\right|\right)^{0.5}\Bigg)^{\frac{1}{0.5}} \quad (3.7) \\ = & \ 0.062 \end{aligned}$$

4. An Approach to MCDM Problems with GTHF-Numbers

In this section, we present an algorithm based on not-ordered quasi-distance measures of GTHF-numbers. The algorithm is given in Deli [26] for ordered quasi-distance measures.

Definition 4.1. [25] Let $A=\{a_1,a_2,...,a_m\}$ be a set of alternatives, $C=\{c_1,c_2,...,c_n\}$ be the set of criteria. If $A_{ij}=\langle(a_{ij},b_{ij},c_{ij},d_{ij});\xi_{ij}(x)\rangle\in\Phi$ $(i=1,2,...,m;\ j=1,2,...,n)$. Then

$$[A_{ij}]_{m\times n}=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

is called an GTHF-MCDM matrix of the decision maker or expert. Here x_{ij} denotes evaluation of the alternative a_i with respect to the criteria c_j made by expert or decision maker.

Based on the Deli [26], we now gave an orderly algorithm for TOPSIS method of GTHF-numbers as follow:

Algorithm:

Step 1. Give the GTHF-MCDM matrix x_{ij} $(i=1,2,...,m;\ j=1,2,...,n)$ as;

$$[A_{ij}]_{m\times n}=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

Step 2. Calculate the normalized GTHF-MCDM matrix n_{ij} $(i=1,2,...,m; j=1,2,...,n)$ as $n_{ij}=\langle(\frac{a_{ij}}{\eta},\frac{b_{ij}}{\eta},\frac{c_{ij}}{\eta},\frac{d_{ij}}{\eta});\xi_{ij}(x)\rangle\in\Phi$ $(i=1,2,...,m;\ j=1,2,...,n)$ where $\eta=\max_{i,j}\left\{\left|a_{ij}\right|,\left|b_{ij}\right|,\left|c_{ij}\right|,\left|d_{ij}\right|\right\}$ $(i=1,2,...,m;\ j=1,2,...,n)$

Step 3. Give the weighted vector $W=(w_1,w_2,...,w_n)$, where w_j $(j=1,2,...,n)$ is the weight of criterion c_j $(j=1,2,...,n)$ and $\sum_{j=1}^{n}w_j=1$.

Step 4. Compute the weighted normalized GTHF-MCDM matrix $n^w_{ij}=w_j\cdot n_{ij}=\langle(\widetilde{a}_{ij},\widetilde{b}_{ij},\widetilde{c}_{ij},\widetilde{d}_{ij});\widetilde{\xi}_{ij}(x)\rangle\in\Phi$ $(i=1,2,...,m;\ j=1,2,...,n)$

Step 5. Describe the GTHF-positive ideal solution A^+ and GTHF-negative ideal solution A^- for GTHF-

MCDM matrix $n_{ij}^w = w_j \cdot n_{ij} = \langle(\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij}, \tilde{d}_{ij}); \tilde{\xi}_{ij}(x)\rangle \in \Phi$ $(i = 1,2,...,m;\ j = 1,2,...,n)$ as follows;

$$A^+ = \langle(\max_{i,j}\{a_{ij}\}, \langle(\max_{i,j}\{b_{ij}\}\langle(\max_{i,j}\{c_{ij}\}, \langle(\max_{i,j}\{d_{ij}\}; \{\max_{i,j}\{\xi : \xi \in \tilde{\xi}_{ij}(x)\}\}\rangle$$

and

$$A^- = \langle(\min_{i,j}\{a_{ij}\}, \langle(\min_{i,j}\{b_{ij}\}\langle(\min_{i,j}\{c_{ij}\}, \langle(\min_{i,j}\{d_{ij}\}; \{\min_{i,j}\{\xi : \xi \in \tilde{\xi}_{ij}(x)\}\}\rangle$$

respectively.

Step 6. Compute the quasi-distance measures $r_{ij}^+ = d(n_{ij}^w, A^+)$, $(i = 1,2,...,m;\ j = 1,2,...,n)$ and $r_{ij}^- = d(n_{ij}^w, A^-)$ $(i = 1,2,...,m;\ j = 1,2,...,n)$ between n_{ij}^w and GTHF-positive ideal solution solution A^+ and GTHF-negative ideal solution A^-, respectively. ($d \in \{d_{NoH}, d_{NoE}, d_{NoG}^\lambda, d_{NoHa}^\lambda, d_{\gamma DNOH^H}, d_{\gamma DNOH^E}, d_{\gamma DNOH^\lambda}\}$) or Compute the correlation measures $r_{ij}^+ = 1 - \overrightarrow{c}_k(n_{ij}^w, A^+)$, $(i = 1,2,...,m;\ j = 1,2,...,n)$ $(k \in \{1,2,...,14\})$ and $r_{ij}^- = 1 - \overrightarrow{c}_k(n_{ij}^w, A^-)$ $(i = 1,2,...,m;\ j = 1,2,...,n)$ between n_{ij}^w and GTHF-positive ideal solution solution A^+ and GTHF-negative ideal solution A^-, respectively.

Step 7. Calculate the total quasi-distance measures d_i^+ and d_i^- $(i = 1,2,...,m)$ of each alternative R_i $(i = 1,2,...,m)$ as;

$$d_i^+ = \sum_{j=1}^{n} r_{ij}^+ \ (i = 1,2,...,m;\ j = 1,2,...,n)$$

and

$$d_i^- = \sum_{j=1}^{n} r_{ij}^- \ (i = 1,2,...,m;\ j = 1,2,...,n)$$

Step 8. Find the score values s_i ($\overline{s}_i$) of each alternative a_i as:

$$s_i = \frac{d_i^+}{d_i^+ + d_i^-} \ (i = 1,2,...,m)$$

$$(or\ \overline{s}_i = \frac{d_i^-}{d_i^+ + d_i^-} \ (i = 1,2,...,m))$$

Step 9. Rank all alternatives a_i $(i = 1,2,...,m)$ by using the score values s_i $(i = 1,2,...,m)$ of a_i $(i = 1,2,...,m)$ and determine the best alternative.

In here, for two alternatives a_k and a_l, $a_k \prec a_l$ $(k, l \in \{1,2,...,m\})$ if $s_k > s_l$, where $\prec$ is a preference relation on A. The best alternative will be the closest to the GTHF-positive ideal solution solution and farthest from GTHF-negative ideal solution (or for two alternatives a_k and a_l $a_k \prec a_l$ $(k, l \in \{1,2,...,m\})$ if $\overline{s}_k < \overline{s}_l$, where $\prec$ is a preference relation on A).

Example 4.2. Assume that among the 5 partners (R_1 to R_5) of a limited company, it is desired to choose a chairman based on 6 criteria. Six subjective criteria are considered by decision maker as:

i. Age (c_1)

ii. Foreign language (c_2)

iii. Sociability(c_3)

iv. Technological knowledge (c_4)

v. Persuasion skill (c_5)

vi. Business environment (c_6)

The calculative procedure is summarized as follows:

Step 1. The decision makers constructed the GTHF-MCDM matrix x_{ij} $(i = 1, 2, ..., 5;\ j = 1, 2, ..., 6)$ as follows:

$$[\tilde{x}_{ij}]_{5\times 6} = \left(\begin{array}{ll} \langle(0,1,1,2);\{0.9,0.8,0.7\}\rangle & \langle(0,1,2,3);\{0.6,0.3,0,9\}\rangle \\ \langle(2,4,5,6);\{0.8,0.7,0.6\}\rangle & \langle(1,2,3,4);\{0.9,0.6,0.8\}\rangle \\ \langle(0,1,2,3);\{0.6,0.3,0,9\}\rangle & \langle(3,5,6,10);\{1.0,0.8\}\rangle \\ \langle(2,4,5,6);\{0.8,0.7,0.6\}\rangle & \langle(2,2,3,4);\{0.8,0.7\}\rangle \\ \langle(2,3,4,5);\{0.8,0.6,0.9\}\rangle & \langle(1,2,3,4);\{0.9,0.6,0.8\}\rangle \end{array} \right.$$

$$\begin{array}{ll} \langle(0,1,1,2);\{0.9,0.8,0.7\}\rangle & \langle(3,4,5,7);\{0.9,0.7\}\rangle \\ \langle(2,2,3,4);\{0.8,0.7\}\rangle & \langle(3,4,5,7);\{0.9,0.7\}\rangle \\ \langle(2,3,4,5);\{0.8,0.6,0.9\}\rangle & \langle(2,2,3,4);\{0.8,0.7\}\rangle \\ \langle(2,3,4,5);\{0.8,0.6,0.9\}\rangle & \langle(0,1,1,2);\{0.9,0.8,0.7\}\rangle \\ \langle(2,2,3,4);\{0.8,0.7\}\rangle & \langle(1,2,3,4);\{0.9,0.6,0.8\}\rangle \end{array}$$

$$\left. \begin{array}{ll} \langle(0,1,2,3);\{0.6,0.3,0,9\}\rangle & \langle(2,2,3,4);\{0.8,0.7\}\rangle \\ \langle(2,2,3,4);\{0.8,0.7\}\rangle & \langle(0,1,1,2);\{0.9,0.8,0.7\}\rangle \\ \langle(2,4,5,6);\{0.8,0.7,0.6\}\rangle & \langle(2,2,3,4);\{0.8,0.7\}\rangle \\ \langle(2,4,5,6);\{0.8,0.7,0.6\}\rangle & \langle(0,1,2,3);\{0.6,0.3,0,9\}\rangle \\ \langle(2,4,5,6);\{0.8,0.7,0.6\}\rangle & \langle(2,2,3,4);\{0.8,0.7\}\rangle \end{array} \right)$$

Step 2. We calculated the normalized GTHF-MCDM matrix n_{ij} $(i = 1, 2, ..., 5;\ j = 1, 2, ..., 6)$ as;
$(n_{ij} = \langle(\frac{a_{ij}}{10}, \frac{b_{ij}}{10}, \frac{c_{ij}}{10}, \frac{d_{ij}}{10}); \xi_{ij}(x)\rangle \in \Phi$, where $\eta = \max_{i,j}\{\left|a_{ij}\right|, \left|b_{ij}\right|, \left|c_{ij}\right|, \left|d_{ij}\right|\}$ $(i = 1, 2, ..., 5;\ j = 1, 2, ..., 6))$

$$[n_{ij}]_{5\times 6} = \left(\begin{array}{ll} \langle(0.0,0.1,0.1,0.2);\{0.9,0.8,0.7\}\rangle & \langle(0.0,0.1,0.2,0.3);\{0.6,0.3,0,9\}\rangle \\ \langle(0.2,0.4,0.5,0.6);\{0.8,0.7,0.6\}\rangle & \langle(0.1,0.2,0.3,0.4);\{0.9,0.6,0.8\}\rangle \\ \langle(0.0,0.1,0.2,0.3);\{0.6,0.3,0,9\}\rangle & \langle(0.3,0.5,0.6,1.0);\{1.0,0.8\}\rangle \\ \langle(0.2,0.4,0.5,0.6);\{0.8,0.7,0.6\}\rangle & \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle \\ \langle(0.2,0.3,0.4,0.5);\{0.8,0.6,0.9\}\rangle & \langle(0.1,0.2,0.3,0.4);\{0.9,0.6,0.8\}\rangle \end{array} \right.$$

$$\begin{array}{ll} \langle(0.0,0.1,0.1,0.2);\{0.9,0.8,0.7\}\rangle & \langle(0.3,0.4,0.5,0.7);\{0.9,0.7\}\rangle \\ \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle & \langle(0.3,0.4,0.5,0.7);\{0.9,0.7\}\rangle \\ \langle(0.2,0.3,0.4,0.5);\{0.8,0.6,0.9\}\rangle & \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle \\ \langle(0.2,0.3,0.4,0.5);\{0.8,0.6,0.9\}\rangle & \langle(0.0,0.1,0.1,0.2);\{0.9,0.8,0.7\}\rangle \\ \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle & \langle(0.1,0.2,0.3,0.4);\{0.9,0.6,0.8\}\rangle \end{array}$$

$$\left. \begin{array}{ll} \langle(0.0,0.1,0.2,0.3);\{0.6,0.3,0,9\}\rangle & \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle \\ \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle & \langle(0.0,0.1,0.1,0.2);\{0.9,0.8,0.7\}\rangle \\ \langle(0.2,0.4,0.5,0.6);\{0.8,0.7,0.6\}\rangle & \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle \\ \langle(0.2,0.4,0.5,0.6);\{0.8,0.7,0.6\}\rangle & \langle(0.0,0.1,0.2,0.3);\{0.6,0.3,0,9\}\rangle \\ \langle(0.2,0.4,0.5,0.6);\{0.8,0.7,0.6\}\rangle & \langle(0.2,0.2,0.3,0.4);\{0.8,0.7\}\rangle \end{array} \right)$$

Step 3. We gave the weighted vector as; $w = (w_1 = 0.20, w_2 = 0.15, w_3 = 0.25, w_4 = 0.15, w_5 = 0.20, w_6 = 0.05)$ where w_j $(j = 1,2,...,6)$ is the weight of criterion c_j $(j = 1,2,...,6)$ and $\sum_{j=1}^{n} w_j = 1$.

Step 4. We computed the weighted normalized GTHF-MCDM matrix $n_{ij}^{w} = w_j \cdot n_{ij} = \langle(\widetilde{a}_{ij}, \widetilde{b}_{ij}, \widetilde{c}_{ij}, \widetilde{d}_{ij}); \widetilde{\xi}_{ij}(x)\rangle$ $(i = 1,2,...,5;\ j = 1,2,...,6)$ as;

$$
[n_{ij}^{w}]_{5\times 6} = \left(\begin{array}{l}
\langle(0.0000, 0.0200, 0.0200, 0.0400); \{0.3690, 0.2752, 0.2140\}\rangle \\
\langle(0.0400, 0.0800, 0.1000, 0.1200); \{0.2752, 0.2140, 0.1674\}\rangle \\
\langle(0.0000, 0.0200, 0.0400, 0.0600); \{0.1674, 0.0689, 0.3690\}\rangle \\
\langle(0.0400, 0.0800, 0.1000, 0.1200); \{0.2752, 0.2140, 0.1674\}\rangle \\
\langle(0.0400, 0.0600, 0.0800, 0.1000); \{0.2752, 0.1674, 0.3690\}\rangle \\
\\
\langle(00.0000, 0.0150, 0.0300, 0.0450); \{0.1284, 0.0521, 0.2921\}\rangle \\
\langle(0.0150, 0.0300, 0.0450, 0.0600); \{0.2921, 0.1284, 0.2145\}\rangle \\
\langle(0.0450, 0.0750, 0.0900, 0.1500); \{0.21451.0000\}\rangle \\
\langle(0.0300, 0.0300, 0.0450, 0.0600); \{0.2145, 0.1652\}\rangle \\
\langle(0.0150, 0.0300, 0.0450, 0.0600); \{0.2921, 0.1284, 0.2145\}\rangle \\
\\
\langle(0.0000, 0.0250, 0.0250, 0.0500); \{0.4377, 0.3313, 0.2599\}\rangle \\
\langle(0.0500, 0.0500, 0.0750, 0.1000); \{0.3313, 0.2599\}\rangle \\
\langle(0.0500, 0.0750, 0.1000, 0.1250); \{0.3313, 0.2047, 0.4377\}\rangle \\
\langle(0.0500, 0.0750, 0.1000, 0.1250); \{0.3313, 0.2047, 0.4377\}\rangle \\
\langle(0.0500, 0.0500, 0.0750, 0.1000); \{0.3313, 0.2599\}\rangle \\
\\
\langle(0.0450, 0.0600, 0.0750, 0.1050); \{0.2921, 0.1652\}\rangle \\
\langle(0.0450, 0.0600, 0.0750, 0.1050); \{0.2921, 0.1652\}\rangle \\
\langle(0.0300, 0.0300, 0.0450, 0.0600); \{0.2145, 0.1652\}\rangle \\
\langle(0.0000, 0.0150, 0.0150, 0.0300); \{0.2921, 0.2145, 0.1652\}\rangle \\
\langle(0.0150, 0.0300, 0.0450, 0.0600); \{0.2921, 0.1284, 0.2145\}\rangle \\
\\
\langle(0.0000, 0.0200, 0.0400, 0.0600); \{0.1674, 0.0689, 0.3690\}\rangle \\
\langle(0.0400, 0.0400, 0.0600, 0.0800); \{0.2752, 0.2140\};\rangle \\
\langle(0.0400, 0.0800, 0.1000, 0.1200); \{0.2752, 0.2140, 0.1674\}\rangle \\
\langle(0.0400, 0.0800, 0.1000, 0.1200); \{0.2752, 0.2140, 0.1674\}\rangle \\
\langle(0.0400, 0.0800, 0.1000, 0.1200); \{0.2752, 0.2140, 0.1674\}\rangle \\
\\
\langle(0.1000, 0.1000, 0.1500, 0.2000); \{0.5528, 0.4523\}\rangle \\
\langle(0.0000, 0.0500, 0.0500, 0.1000); \{0.6838, 0.5528, 0.4523\}\rangle \\
\langle(0.1000, 0.1000, 0.1500, 0.2000); \{0.5528, 0.4523\}\rangle \\
\langle(0.0000, 0.0500, 0.1000, 0.1500); \{0.3675, 0.1633, 0.6838\}\rangle \\
\langle(0.1000, 0.1000, 0.1500, 0.2000); \{0.5528, 0.4523\}\rangle
\end{array}\right)
$$

Step 5. We described the GTHF-positive ideal solution A^{+} and GTHF-negative ideal solution A^{-} for GTHF-MCDM matrix as follows:

$A^{+} = \langle(0.1000, 0.1000, 0.1500, 0.2000); \{1\}\rangle$ and $A^{-} = \langle(0.0000, 0.0150, 0.0150, 0.0300); \{0.0521\}\rangle$, respectively.

Step 6. We computed the quasi-distance measures $d_{NoH}(n_{ij}^{w}, A^{+})$, $(i = 1,2,...,5;\ j = 1,2,...,6)$ and $d_{NoH}(n_{ij}^{w}, A^{-})$ $(i = 1,2,...,5;\ j = 1,2,...,6)$ in Table 1 and Table 2, respectively.

Table 1. The quasi-distance measures $d_{NoH}(n_{ij}^{w}, A^{+})$, $(i = 1,2,...,5;\ j = 1,2,...,6)$

$d_{NoH}(n_{ij}^{w}, A^{+})$	c_1	c_2	c_3	c_4	c_5	c_6
R_1	0.1825	0.2096	0.1643	0.1517	0.1881	0.0000
R_2	0.1274	0.1872	0.1403	0.1517	0.1685	0.0888
R_3	0.1881	0.0480	0.1055	0.1958	0.1274	0.0000
R_4	0.1274	0.1958	0.1055	0.2018	0.1274	0.0465
R_5	0.1427	0.1872	0.1403	0.1872	0.1274	0.0000

Table 2. The quasi-distance measures $d_{NoH}(n_{ij}^{w}, A^{-})$ $(i = 1,2,...,5;\ j = 1,2,...,6)$

$d_{NoH}(n_{ij}^{w}, A^{-})$	c_1	c_2	c_3	c_4	c_5	c_6
R_1	0.0010	0.0012	0.0029	0.0112	0.0036	0.1171
R_2	0.0162	0.0036	0.0148	0.0112	0.0071	0.0291
R_3	0.0036	0.0765	0.0283	0.0035	0.0162	0.1171
R_4	0.0162	0.0025	0.0283	0.0000	0.0162	0.0642
R_5	0.0141	0.0036	0.0148	0.0036	0.0162	0.1171

Step 7. We calculated the total quasi-distance measures d_i^{+} and d_i^{-} $(i = 1,2,...,5)$ of each alternative R_i $(i = 1,2,...,5)$ in Table 3 based on Table 1 and Table 2, respectively;

Table 3. The total quasi-distance measures d_i^{+} and d_i^{-} $(i = 1,2,...,5)$

i	1	2	3	4	5
d_i^{+}	0.8961	0.8640	0.6649	0.8045	0.7849
d_i^{-}	0.1371	0.0819	0.2442	0.1273	0.1693

Step 8. We found the score values s_i $(i = 1,2,...,5)$ of each alternative a_i $(i = 1,2,...,5)$ in Table 4.

Table 4. The score values s_i $(i = 1,2,...,5)$ of each alternative a_i $(i = 1,2,...,5)$

i	1	2	3	4	5
s_i	0.8674	**0.9134**	**0.7314**	0.8634	0.8226

Step 9. We ranked all alternatives according to the score values s_i, in decreasing order by the rule $s_2 > s_1 > s_4 > s_5 > s_3$ and we obtained $R_2 \prec R_1 \prec R_4 \prec R_5 \prec R_3$. Therefore, the best chairman is R_3. Moreover, we ranked all alternatives according to other quasi-distance measures in Table 5 and we obtained the same results for the best chairman.

Table 5. A ranking for all alternatives (alt) according to other introduced quasi-distance measures (DM)

DM	i	1	2	3	4	5	The worst alt.	The best alt.
d_{NoE}	s_i	0.8334	**0.8461**	**0.6980**	0.8258	0.7909	R_2	R_3
$d_{NoG}^{0.5}$	s_i	0.8726	**0.9722**	**0.7499**	0.9074	0.8302	R_2	R_3
d_{NoG}^{4}	s_i	0.7815	0.7928	**0.6656**	**0.7953**	0.7562	R_4	R_3
d_{NoG}^{20}	s_i	0.7061	0.7305	**0.6342**	**0.7560**	0.7126	R_4	R_3

5. Conclusion

In this paper, we proposed some novel not-ordered quasi-distance measuresunder GTHF-numbers. Then, we applied the quasi-distance measures to TOPSIS method of GTHF-numbers in Deli [26]. Also, we gave a numerical example, to show the efficiency and the applicability of the proposed method. In future, we may study some different quasi-distance and similarity measures and aggregation operators on GTHF-numbers. Researchers can also define similarity measures based on cendroid point of the GTHF-numbers.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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