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STATISTICAL STRUCTURES AND KILLING VECTOR FIELDS ON TANGENT BUNDLES WITH RESPECT TO TWO DIFFERENT METRICS

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Abstract. Let (M, g) be a Riemannian manifold and TM be its tangent bundle. The purpose of this paper is to study statistical structures on TM with respect to the metrics $G_1^f = {}^cg + {}^v(fg)$ and $G_2^f = {}^sg_f + {}^hg$, where f is a smooth function on M, ${}^cg$ is the complete lift of g, ${}^v(fg)$ is the vertical lift of fg, ${}^sg_f$ is a metric obtained by rescaling the Sasaki metric by a smooth function f and ${}^hg$ is the horizontal lift of g. Moreover, we give some results about Killing vector fields on TM with respect to these metrics.

1. Introduction

Let (M, g) be a Riemannian manifold and TM be its tangent bundle. In [1], Abbassi and Sarih defined a general "$g-$natural" metric on TM. Some well-known examples of the $g-$natural metric are the Sasaki metric ([6], [14]), the Cheeger-Gromoll metric ([13], [15]), Cheeger-Gromoll type metrics ([4], [7]) and the Kaluza-Klein metric [2]. However, some other metrics can be defined on the tangent bundle which are not subclasses of this $g-$natural metric. As first example, in [9], Gezer and Ozkan defined a metric $G_1^f = {}^cg + {}^v(fg)$, where ${}^cg$ is the complete lift of the metric and ${}^v(fg)$ is the vertical lift of fg and f is a smooth function on M. As second example, in [8], Gezer *et al.* introduced a metric $G_2^f = {}^sg_f + {}^hg$, where ${}^sg_f$ is a metric which is obtained by rescaling the Sasaki metric with a smooth function f on M and ${}^hg$ is the horizontal lift of g. These lifts will be explained later and we will deal with these two metrics in this paper.

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Statistical manifolds were introduced by Amari [3] in view of information geometry, and they were applied by Lauritzen [10]. These manifolds have a crucial role in statistics as the statistical model often forms a geometrical manifold.

Although curvature related properties of tangent bundles are widely studied, investigating statistical structures on tangent bundles is a relatively new topic. These structures were examined with respect to various Riemannian metrics such as the Sasaki metric [5], the Cheeger-Gromoll metric and a $g-$natural metric which consists of three classic lifts of the metric g [12], the twisted Sasaki metric and the gradient Sasaki metric [11].

In this paper, we study the statistical and Codazzi structures on TM using the horizontal and complete lifts of a linear connection on M when TM is endowed with the metrics G_1^f and G_2^f, respectively. We also investigate the Killing vector fields on TM with respect to such metrics.

2. Preliminaries

Let M be an $n-$dimensional Riemannian manifold and ∇ be a linear connection on M. The tangent bundle TM of the manifold M is a $2n-$dimensional smooth manifold and it is defined by the disjoint union of the tangent spaces at each point of M. If $\{U, x^i\}$ is a local coordinate system in M, then $\{\pi^{-1}(U), x^i, u^i, i = 1, ..., n\}$ is a local coordinate system in TM, where π is the natural projection defined by $\pi : TM \to M$ and (u^i) is the local coordinate system in each tangent space in U with respect to the basis $\{\frac{\partial}{\partial x^i}\}$. We have a direct sum decomposition

$$TTM = VTM \oplus HTM$$

for the tangent bundle of TM, where the vertical subspace VTM is spanned by $\{\frac{\partial}{\partial u^i} := (\frac{\partial}{\partial x^i})^v\}$ and the horizontal subspace HTM is spanned by $\{\frac{\delta}{\delta x^i} := (\frac{\partial}{\partial x^i})^h = \frac{\partial}{\partial x^i} - u^m \Gamma_{mi}^j \frac{\partial}{\partial u^j}\}$. Here Γ_{mi}^j denote the Christoffel symbols of ∇. The vertical, horizontal and the complete lifts of a vector field $X = X^i \frac{\partial}{\partial x^i}$ are defined by, respectively

$$X^v = X^i \frac{\partial}{\partial u^i},\ X^h = X^i \frac{\partial}{\partial x^i} - y^s \Gamma_{si}^m X^i \frac{\partial}{\partial u^m},\ X^c = X^i \frac{\partial}{\partial x^i} + y^s \frac{\partial X^i}{\partial x^s} \frac{\partial}{\partial u^i},$$

where we used Einstein the summation.

The Lie brackets of the vertical lift and the horizontal lift of vector fields satisfy the following relations:

$$[X^h, Y^h] = [X, Y]^h - (R(X, Y)u)^v,\ [X^h, Y^v] = (\nabla_X Y)^v - (T(X, Y))^v,\ [X^v, Y^v] = 0,$$

where R is the curvature tensor field and T is the torsion tensor field of the linear connection ∇, [16].

For a Riemannian metric g on a smooth manifold M, the complete lift ${}^c g$, the vertical lift ${}^v g$ and the horizontal lift ${}^h g$ of g are given by

$$ {}^c g(X^h, Y^h) = {}^c g(X^v, Y^v) = 0,\ {}^c g(X^h, Y^v) = {}^c g(X^v, Y^h) = g(X, Y),$$
$$ {}^v g(X^h, Y^h) = g(X, Y),\ {}^v g(X^v, Y^v) = {}^v g(X^h, Y^v) = {}^v g(X^v, Y^h) = 0.$$

$$^{h}g(X^h,Y^h)=0,\ ^{h}g(X^v,Y^v)=0,\ ^{h}g(X^h,Y^v)=\ ^{h}g(X^v,Y^h)=g(X,Y).$$

The horizontal lift connection $\overset{h}{\nabla}$ and the complete lift connection $\overset{c}{\nabla}$ are respectively given by, [16]

$$\begin{aligned}
\overset{h}{\nabla}_{X^h}Y^h &= (\nabla_X Y)^h,\ \overset{h}{\nabla}_{X^h}Y^v=(\nabla_X Y)^v,\ \overset{h}{\nabla}_{X^v}Y^h=\overset{h}{\nabla}_{X^v}Y^v=0,\\
\overset{c}{\nabla}_{X^h}Y^h &= (\nabla_X Y)^h+(R(u,X)Y)^v,\ \overset{c}{\nabla}_{X^v}Y^h=\overset{c}{\nabla}_{X^v}Y^v=0,\\
\overset{c}{\nabla}_{X^h}Y^v &= (\nabla_X Y)^v,\ \overset{c}{\nabla}_{X^c}Y^c=(\nabla_X Y)^c,\ \overset{c}{\nabla}_{X^c}Y^v=\overset{c}{\nabla}_{X^v}Y^c=(\nabla_X Y)^v.
\end{aligned}$$

Remark 1. *The connection ∇ is a flat and torsionless linear connection if and only if $\overset{h}{\nabla}(\overset{c}{\nabla})$ is a torsionless linear connection, [16].*

In the sequel, we shall denote $\frac{\partial}{\partial x^i}$, $\frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial u^i}$ as ∂_i, δ_i and $\partial_{\bar{\imath}}$, for shortness.

The metric G_1^f on TM is defined by

$$G_1^f(X^h,Y^h)=fg(X,Y),\ G_1^f(X^h,Y^v)=G_1^f(X^v,Y^h)=g(X,Y),\ G_1^f(X^v,Y^v)=0, \tag{1}$$

where f is a strictly positive function on M, [9].

From Theorem 3.1 in [9], we can easily rewrite the Levi-Civita connection of the metric G_1^f in invariant form.

Lemma 1. *Let (M,g) be a Riemannian manifold on (TM,G_1^f) be its tangent bundle with the metric G_1^f defined by (1). The Levi-Civita connection ∇_1^f of the metric G_1^f satisfies the following relations*

$$\begin{aligned}
\nabla^f_{1X^h}Y^h &= (\nabla_X Y)^h+(R(u,X)Y+A_f(X,Y))^v,\\
\nabla^f_{1X^h}Y^v &= (\nabla_X Y)^v,\ \nabla^f_{1X^v}Y^h=\nabla^f_{1X^v}Y^v=0,
\end{aligned}$$

where X,Y are vector fields on M, ∇ is the Levi-Civita connection of g, R is the Riemannian curvature of ∇ and $A_f(X,Y)=\frac{1}{2}(X(f)Y+Y(f)X-g(X,Y)\circ(df)^*)$.

The metric G_2^f on TM is defined by

$$G_2^f(X^h,Y^h)=fg(X,Y),\ G_2^f(X^h,Y^v)=G_2^f(X^v,Y^h)=g(X,Y), G_2^f(X^v,Y^v)=g(X,Y), \tag{2}$$

where f is a strictly positive function on M, [8].

From [9], we rewrite the Levi-Civita connection of the metric G_2^f in invariant form as follows.

Lemma 2. *Let (M,g) be a Riemannian manifold on (TM,G_2^f) be its tangent bundle with the metric G_2^f defined by (2). The Levi-Civita connection ∇_2^f of the metric G_2^f satisfies the following relations*

$$\nabla^f_{2X^h}Y^h = (\nabla_X Y+\frac{1}{2(f-1)}(R(u,X)Y+R(u,Y)X)+\frac{1}{f-1}A_f(X,Y))^h$$

$$
\begin{aligned}
& -(\frac{1}{f-1}A_f(X,Y)+\frac{1}{2}R(X,Y)u+\frac{1}{2(f-1)}(R(u,X)Y+R(u,Y)X))^v, \\
\nabla^f_{2X^h}Y^v &= (\frac{1}{2(f-1)}R(u,Y)X)^h+(\nabla_X Y-\frac{1}{2(f-1)}R(u,X)Y)^v, \\
\nabla^f_{2X^v}Y^h &= (\frac{1}{2(f-1)}R(u,X)Y)^h-(\frac{1}{2(f-1)}R(u,X)Y)^v, \\
\nabla^f_{2X^v}Y^v &= 0,
\end{aligned}
$$

where X,Y are vector fields on M, ∇ is the Levi-Civita connection of g, R is the Riemannian curvature of ∇ and $A_f(X,Y)=\frac{1}{2}(X(f)Y+Y(f)X-g(X,Y)\circ(df)^*)$.

Definition 1. *Let (M,g) be a Riemannian manifold and let ∇ be a linear connection on M. The pair (g,∇) is called a Codazzi couple if the Codazzi equation are valid:*

$$(\nabla_X g)(Y,Z)=(\nabla_Z g)(X,Y),$$

for all vector fields X,Y,Z on M. The triplet (M,g,∇) is said to be a Codazzi manifold and ∇ is called a Codazzi connection. Moreover, when ∇ is torsionless, (M,g,∇) is a statistical manifold.

3. Killing Vector Fields and Statistical Structures on (TM,G_1^f)

Definition 2. *Let (M,g) be a Riemannian manifold and ∇ be a linear connection on M. A vector field X is called conformal (respectively, Killing) if $L_X g=2\rho g$ (respectively, $L_X g=0$), where ρ is a smooth function on M.*

Using this definition, we have

$$
\begin{aligned}
L_{X^v}G_1^f(Y^v,Z^v) &= 0, \\
L_{X^v}G_1^f(Y^h,Z^v) &= 0, \\
L_{X^v}G_1^f(Y^h,Z^h) &= g(\nabla_Y X,Z)+g(Y,\nabla_Z X)-g(T(Y,X),Z)-g(T(Z,X),Y)
\end{aligned}
$$

and

$$
\begin{aligned}
L_{X^h}G_1^f(Y^v,Z^v) &= 0, \\
L_{X^h}G_1^f(Y^h,Z^v) &= g(\nabla_Y X,Z)+g(T(X,Y),Z)+g(Y,T(X,Z)), \\
L_{X^h}G_1^f(Y^h,Z^h) &= X(f)g(Y,Z)+f(L_X g)(Y,Z)+g(R(X,Y)u,Z)+g(R(X,Z)u,Y).
\end{aligned}
$$

So, we have the following proposition.

Proposition 1. *Let (TM,G_1^f) be the tangent bundle of a Riemannian manifold (M,g). Then the following statements are true:*

(i) If ∇ is a torsionless linear connection on M, then the vector field X^v is Killing if and only if X is a parallel vector field on (M,g).

(ii) If ∇ is a torsionless linear connection on M, then the vector field X^h is Killing if and only if X is a $\nabla-$parallel vector field, X is a conformal vector field

such that $(L_X g)(Y,Z) = -\frac{X(f)}{f} g(Y,Z)$ *and* $R(X,Y)Z = 0$ *for all the vector fields* Y, Z *on* M.

(iii) If ∇ *is a torsionless linear connection,* f *is a constant function and* X *is a parallel vector field on* M, *then the vector field* X^h *is Killing if and only if the vector field* X *is Killing on* (M,g) *and* $R(X,Y)Z = 0$ *for all the vector fields* Y, Z *on* M.

(iv) If ∇ *is a flat connection,* X *is a* $\nabla-$*parallel vector field and* f *is a constant function on* (M,g), *then the vector field* X^h *is Killing if and only if the vector field* X *is Killing on* (M,g).

Proof. The truthfulness of the assertions are clear from the definition of the Killing vector fields. □

Now, we obtain the components of $\overset{h}{\nabla} G_1^f$. We have

$$\begin{aligned}(\overset{h}{\nabla}_{\delta_i} G_1^f)(\delta_j, \delta_k) &= \partial_i(f) g_{jk} + f(\nabla_{\partial_i} g)(\partial_j, \partial_k), \\ (\overset{h}{\nabla}_{\delta_j} G_1^f)(\delta_k, \delta_i) &= \partial_j(f) g_{ki} + f(\nabla_{\partial_j} g)(\partial_k, \partial_i), \\ (\overset{h}{\nabla}_{\delta_k} G_1^f)(\delta_i, \delta_j) &= \partial_k(f) g_{ij} + f(\nabla_{\partial_k} g)(\partial_i, \partial_j),\end{aligned} \tag{3}$$

$$(\overset{h}{\nabla}_{\partial_{\bar{i}}} G_1^f)(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0,\ (\overset{h}{\nabla}_{\partial_{\bar{i}}} G_1^f)(\partial_{\bar{j}}, \delta_k) = (\overset{h}{\nabla}_{\partial_{\bar{j}}} G_1^f)(\delta_k, \partial_{\bar{i}}) = (\overset{h}{\nabla}_{\delta_k} G_1^f)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0,$$

$$(\overset{h}{\nabla}_{\delta_i} G_1^f)(\delta_j, \partial_{\bar{k}}) = (\nabla_{\partial_i} g)(\partial_j, \partial_k),\ (\overset{h}{\nabla}_{\delta_j} G_1^f)(\partial_{\bar{k}}, \delta_i) = (\nabla_{\partial_j} g)(\partial_k, \partial_i),\ (\overset{h}{\nabla}_{\partial_{\bar{k}}} G_1^f)(\delta_i, \delta_j) = 0. \tag{4}$$

So, we can express the following theorem.

Theorem 1. *Let* (TM, G_1^f) *be the tangent bundle of a Riemannian manifold* (M,g) *and* ∇ *be a linear connection. Then the following statements are true:*

(i) If $(TM, G_1^f, \overset{h}{\nabla})$ *is a Codazzi manifold, then* f *is a constant function on* M *and* ∇ *is a metric connection.*

(ii) If $(TM, G_1^f, \overset{h}{\nabla})$ *is a statistical manifold, then* ∇ *is flat,* f *is a constant function on* M *and* ∇ *is the Levi-Civita connection of* g. *In this case, the connections* $\overset{h}{\nabla}$ *and* ∇_1^f *coincide.*

(iii) If ∇ *is the Levi-Civita connection of* g *and* f *is a constant function on* M, *then* $\overset{h}{\nabla}$ *is compatible with the metric* G_1^f. *In particular, if* ∇ *is flat, then the connections* $\overset{h}{\nabla}$ *and* ∇_1^f *coincide.*

Proof. (i) From (3) and (4) we see that if $(TM, G_1^f, \overset{h}{\nabla})$ is a Codazzi manifold, then f is a constant function on M and ∇ is a metric connection.

(ii) If $(TM, G_1^f, \overset{h}{\nabla})$ is a statistical manifold, then $\overset{h}{\nabla}$ is torsionless. From Remark 1, we see that ∇ is flat. It follows from (i) and the definition of the connections $\overset{h}{\nabla}$ and ∇_1^f.

(iii) It is clear from the definition of the Levi-Civita connection and the connections $\overset{h}{\nabla}$ and ∇_1^f.

□

Now, we repeat this process for $(TM, G_1^f, \overset{c}{\nabla})$. By direct calculations we have

$$\begin{aligned}
(\overset{c}{\nabla}_{\delta_i} G_1^f)(\delta_j, \delta_k) &= \partial_i(f) g_{jk} + f(\nabla_{\partial_i} g)(\partial_j, \partial_k) - u^s R_{sij}^t g_{kt} - u^s R_{sik}^t g_{jt}, \quad (5)\\
(\overset{c}{\nabla}_{\delta_j} G_1^f)(\delta_k, \delta_i) &= \partial_j(f) g_{ki} + f(\nabla_{\partial_j} g)(\partial_k, \partial_i) - u^s R_{sjk}^t g_{it} - u^s R_{sji}^t g_{tk},\\
(\overset{c}{\nabla}_{\delta_k} G_1^f)(\delta_i, \delta_j) &= \partial_k(f) g_{ij} + f(\nabla_{\partial_k} g)(\partial_i, \partial_j) - u^s R_{ski}^t g_{jt} - u^s R_{skj}^t g_{ti},
\end{aligned}$$

$$(\overset{c}{\nabla}_{\partial_{\bar{i}}} G_1^f)(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0,\ (\overset{c}{\nabla}_{\partial_{\bar{i}}} G_1^f)(\partial_{\bar{j}}, \delta_k) = (\overset{c}{\nabla}_{\partial_{\bar{j}}} G_1^f)(\delta_k, \partial_{\bar{i}}) = (\overset{c}{\nabla}_{\delta_k} G_1^f)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0,$$
$$(\overset{c}{\nabla}_{\delta_i} G_1^f)(\delta_j, \partial_{\bar{k}}) = (\nabla_{\partial_i} g)(\partial_j, \partial_k),\ (\overset{c}{\nabla}_{\delta_j} G_1^f)(\partial_{\bar{k}}, \delta_i) = (\nabla_{\partial_j} g)(\partial_k, \partial_i),\ (\overset{c}{\nabla}_{\partial_{\bar{k}}} G_1^f)(\delta_i, \delta_j) = 0. \quad (6)$$

Thus, we give the following theorem.

Theorem 2. *Let (TM, G_1^f) be the tangent bundle of a Riemannian manifold (M, g) and let ∇ be a torsionless linear connection. Then the following statements are true:*

i) If $(TM, G_1^f, \overset{c}{\nabla})$ is a Codazzi (respectively statistical) manifold, then ∇ is flat, f is a constant function on M. Furthermore, $\overset{c}{\nabla}$ is a metric connection (respectively, $\overset{c}{\nabla}$ becomes the Levi-Civita connection of G_1^f).

ii) If $(TM, G_1^f, \overset{c}{\nabla})$ is a statistical manifold and f is a constant function on M, then ∇ is the Levi-Civita connection of g and $\overset{c}{\nabla}$ becomes the Levi-Civita connection of G_1^f.

(iii) If ∇ is the Levi-Civita connection of g, f is a constant function on M and ∇ is a flat connection, then the connections $\overset{c}{\nabla}$ and ∇_1^f coincide.

Proof. (i) If $(TM, G_1^f, \overset{c}{\nabla})$ is a Codazzi manifold, then from (6) we obtain that ∇ is a metric connection. Differentiating $(5)_1$ with respect to u^m gives us $R_{mij}^t g_{kt} + R_{mik}^t g_{jt} = 0$. Similarly, by differentiating $(5)_2$ and $(5)_3$ with respect to u^m, we obtain $R_{mjk}^t g_{it} + R_{mji}^t g_{tk} = 0$ and $R_{mki}^t g_{jt} + R_{mkj}^t g_{ti} = 0$, respectively. So, ∇ is a flat connection. We also occur that f is a constant function on M. If $\overset{c}{\nabla}$ is torsionless, it becomes the Levi-Civita connection of G_1^f.

(ii) We get immediately from Remark 1, the definition of the Levi-Civita connection and the complete lift connection $\overset{c}{\nabla}$.

(iii) Definitions of the connections $\overset{c}{\nabla}$ and ∇_1^f give the results. □

Now, we assume that (TM, G_s, ∇_1^f) is a statistical manifold. The metric G_s is called the Sasaki metric and it is defined by

$$G_s(X^h, Y^h) = g(X, Y),\ G_s(X^h, Y^v) = G_s(X^v, Y^h) = 0,\ G_s(X^v, Y^v) = g(X, Y),$$

for all vector fields X, Y, Z on M. Using Lemma 1, we get

$$\begin{aligned}
(\nabla^f_{1\delta_i} G_s)(\delta_j, \delta_k) &= (\nabla^f_{1\delta_j} G_s)(\delta_k, \delta_i) = (\nabla^f_{1\delta_k} G_s)(\delta_i, \delta_j) = 0, \qquad (7)\\
(\nabla^f_{1\partial_{\bar{i}}} G_s)(\partial_{\bar{j}}, \partial_{\bar{k}}) &= 0,\ (\nabla^f_{1\partial_{\bar{i}}} G_s)(\partial_{\bar{j}}, \delta_k) = (\nabla^f_{1\partial_{\bar{j}}} G_s)(\delta_k, \partial_{\bar{i}}) = (\nabla^f_{1\delta_k} G_s)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0,\\
(\nabla^f_{1\delta_i} G_s)(\delta_j, \partial_{\bar{k}}) &= -u^s R^m_{sij} g_{mk} + \frac{1}{2} g_{mk}(f_i \delta^m_j + f_j \delta^m_i - g_{ij} f^m_{.}),\\
(\nabla^f_{1\delta_j} G_s)(\partial_{\bar{k}}, \delta_i) &= -u^s R^m_{sji} g_{mk} + \frac{1}{2} g_{mk}(f_j \delta^m_i + f_i \delta^m_j - g_{ji} f^m_{.}),\\
(\nabla^f_{1\partial_{\bar{k}}} G_s)(\delta_i, \delta_j) &= 0,
\end{aligned}$$

where, $f_i = \partial_i f$ and $f^m_{.} = g^{mh} f_h$. So, we have the following theorem.

Theorem 3. *Let (TM, G_1^f) be the tangent bundle of a Riemannian manifold (M, g) and let ∇_1^f is the Levi-Civita connection of the metric G_1^f. If (TM, G_s, ∇_1^f) is a statistical manifold, then ∇ is flat and f is a constant function on M.*

Proof. If (TM, G_s, ∇_1^f) is a statistical manifold, by differentiating $(7)_3$ and $(7)_4$ with respect to u^t, we occur $R^m_{tij} g_{mk} = R^m_{tji} g_{mk} = 0$. Moreover, we see that f is a constant function on M. □

4. Killing Vector Fields and Statistical Structures on (TM, G_2^f)

In this final section, we follow the same line in the previous section for the metric G_2^f. The proofs of the results will be similar.
From Definition 2, we have

$$\begin{aligned}
L_{X^v} G_2^f(Y^v, Z^v) &= 0,\\
L_{X^v} G_2^f(Y^h, Z^v) &= g(\nabla_Y X, Z) - g(T(Y, X), Z),\\
L_{X^v} G_2^f(Y^h, Z^h) &= g(\nabla_Y X, Z) - g(T(Y, X), Z) + g(\nabla_Z X, Y) - g(T(Z, X), Y)
\end{aligned}$$

and

$$\begin{aligned}
L_{X^h} G_2^f(Y^v, Z^v) &= (\nabla_X g)(Y, Z) + g(T(X, Y), Z) + g(Y, T(X, Z)),\\
L_{X^h} G_2^f(Y^h, Z^v) &= g(\nabla_Y X, Z) + g(R(X, Y)u, Z) + g(T(X, Y), Z) + g(Y, T(X, Z)),\\
L_{X^h} G_2^f(Y^h, Z^h) &= X(f) g(Y, Z) + f(L_X g)(Y, Z) + g(R(X, Y)u, Z) + g(R(X, Z)u, Y).
\end{aligned}$$

It is clear that if ∇ is a torsionless linear connection, then the vector field X^v is Killing if and only if $\nabla X = 0$. On the other hand, if ∇ is the Levi-Civita connection of g, then X^h is a Killing vector field if and only if X is $\nabla-$parallel, X is Killing, the function f is constant and ∇ is flat. More precisely, we have

Proposition 2. *Let (TM, G_2^f) be the tangent bundle of a Riemannian manifold (M, g). Then the following statements are true:*

(i) If ∇ is a torsionless linear connection on M, then the vector field X^v is Killing if and only if X is a parallel vector field.

(ii) If ∇ is a torsionless linear connection, f is a constant function and X is a $\nabla-$parallel vector field on M, then the vector field X^h is Killing if and only if X is Killing vector field on M, ∇ is the Levi-Civita connection of (M, g) and $R(X, Y)Z = 0$ for all the vector fields Y, Z on M.

(iii) If ∇ is the flat Levi-Civita connection, X is a $\nabla-$parallel vector field and f is a constant function on (M, g), then the vector field X^h is Killing if and only if the vector field X is Killing on (M, g).

Here, we compute the components of $\overset{h}{\nabla} G_2^f$. We obtain

$$
\begin{aligned}
(\overset{h}{\nabla}_{\delta_i} G_2^f)(\delta_j, \delta_k) &= \partial_i(f) g_{jk} + f(\nabla_{\partial_i} g)(\partial_j, \partial_k), \\
(\overset{h}{\nabla}_{\delta_j} G_2^f)(\delta_k, \delta_i) &= \partial_j(f) g_{ki} + f(\nabla_{\partial_j} g)(\partial_k, \partial_i), \\
(\overset{h}{\nabla}_{\delta_k} G_2^f)(\delta_i, \delta_j) &= \partial_k(f) g_{ij} + f(\nabla_{\partial_k} g)(\partial_i, \partial_j),
\end{aligned}
$$

$$
\begin{aligned}
(\overset{h}{\nabla}_{\partial_{\bar{i}}} G_2^f)(\partial_{\bar{j}}, \partial_{\bar{k}}) &= 0, \ (\overset{h}{\nabla}_{\partial_{\bar{i}}} G_2^f)(\partial_{\bar{j}}, \delta_k) = (\overset{h}{\nabla}_{\partial_{\bar{j}}} G_2^f)(\delta_k, \partial_{\bar{i}}) = 0, \\
(\overset{h}{\nabla}_{\delta_k} G_2^f)(\partial_{\bar{i}}, \partial_{\bar{j}}) &= (\nabla_{\partial_k} g)(\partial_i, \partial_j),
\end{aligned}
$$

$$
\begin{aligned}
(\overset{h}{\nabla}_{\delta_i} G_2^f)(\delta_j, \partial_{\bar{k}}) &= (\nabla_{\partial_i} g)(\partial_j, \partial_k), \ (\overset{h}{\nabla}_{\delta_j} G_2^f)(\partial_{\bar{k}}, \delta_i) = (\nabla_{\partial_j} g)(\partial_k, \partial_i), \\
(\overset{h}{\nabla}_{\partial_{\bar{k}}} G_2^f)(\delta_i, \delta_j) &= 0.
\end{aligned}
$$

From the above equations, we deduce that if $(TM, G_2^f, \overset{h}{\nabla})$ is a Codazzi manifold, then f is a constant function on M and ∇ is a metric connection. So, we can write the following theorem.

Theorem 4. *Let (TM, G_2^f) be the tangent bundle of a Riemannian manifold (M, g) and ∇ be a linear connection. Then the following statements are true:*

(i) If $(TM, G_2^f, \overset{h}{\nabla})$ is a Codazzi manifold, then f is a constant function on M and ∇ is a metric connection.

(ii) If $(TM, G_2^f, \overset{h}{\nabla})$ is a statistical manifold, then ∇ is flat, f is a constant function on M and ∇ is the Levi-Civita connection of g. In this case, the connections $\overset{h}{\nabla}$ and ∇_2^f coincide.

(iii) If ∇ is the Levi-Civita connection of g and f is a constant function on M, then $\overset{h}{\nabla}$ is compatible with the metric G_2^f. In particular, if ∇ is flat, then the connections $\overset{h}{\nabla}$ and ∇_2^f coincide.

Now, we follow this process for $(TM, G_2^f, \overset{c}{\nabla})$. By direct calculations we have

$$\begin{aligned}
(\overset{c}{\nabla}_{\delta_i} G_2^f)(\delta_j, \delta_k) &= \partial_i(f) g_{jk} + f(\nabla_{\partial_i} g)(\partial_j, \partial_k) - u^s R_{sij}^t g_{kt} - u^s R_{sik}^t g_{jt}, \quad (8)\\
(\overset{c}{\nabla}_{\delta_j} G_2^f)(\delta_k, \delta_i) &= \partial_j(f) g_{ki} + f(\nabla_{\partial_j} g)(\partial_k, \partial_i) - u^s R_{sjk}^t g_{it} - u^s R_{sji}^t g_{tk},\\
(\overset{c}{\nabla}_{\delta_k} G_2^f)(\delta_i, \delta_j) &= \partial_k(f) g_{ij} + f(\nabla_{\partial_k} g)(\partial_i, \partial_j) - u^s R_{ski}^t g_{jt} - u^s R_{skj}^t g_{ti},
\end{aligned}$$

$$(\overset{c}{\nabla}_{\partial_{\bar{i}}} G_2^f)(\partial_{\bar{j}}, \partial_{\bar{k}}) = 0, \; (\overset{c}{\nabla}_{\partial_{\bar{i}}} G_2^f)(\partial_{\bar{j}}, \delta_k) = (\overset{c}{\nabla}_{\partial_{\bar{j}}} G_2^f)(\delta_k, \partial_{\bar{i}}) = (\overset{c}{\nabla}_{\delta_k} G_2^f)(\partial_{\bar{i}}, \partial_{\bar{j}}) = 0,$$

$$\begin{aligned}
(\overset{c}{\nabla}_{\delta_i} G_2^f)(\delta_j, \partial_{\bar{k}}) &= (\nabla_{\partial_i} g)(\partial_j, \partial_k) + u^s R_{sij}^t g_{kt}, \quad (9)\\
(\overset{c}{\nabla}_{\delta_j} G_2^f)(\partial_{\bar{k}}, \delta_i) &= (\nabla_{\partial_j} g)(\partial_k, \partial_i) + u^s R_{sji}^t g_{kt},\\
(\overset{c}{\nabla}_{\partial_{\bar{k}}} G_2^f)(\delta_i, \delta_j) &= (\nabla_{\partial_k} g)(\partial_i, \partial_j).
\end{aligned}$$

If $(TM, G_2^f, \overset{c}{\nabla})$ is a Codazzi manifold, then from (9) we obtain that ∇ is a flat metric connection. We also deduce that from $(8)_1$ f is a constant function on M. Thus, we have the following theorem.

Theorem 5. *Let (TM, G_2^f) be the tangent bundle of a Riemannian manifold (M, g) and let ∇ be a torsionless linear connection. Then the following statements are true:*

i) If $(TM, G_2^f, \overset{c}{\nabla})$ is a Codazzi (respectively statistical) manifold, then ∇ is flat, f is a constant function on M. Furthermore, $\overset{c}{\nabla}$ is a metric connection (respectively, $\overset{c}{\nabla}$ becomes the Levi-Civita connection of G_2^f).

ii) If $(TM, G_2^f, \overset{c}{\nabla})$ is a statistical manifold and f is a constant function on M, then ∇ is the Levi-Civita connection of g and $\overset{c}{\nabla}$ becomes the Levi-Civita connection of G_2^f.

(iii) If ∇ is the Levi-Civita connection of g, f is a constant function on M and ∇ is a flat connection, then the connections $\overset{c}{\nabla}$ and ∇_2^f coincide.

Now, we assume that (TM, G_s, ∇_2^f) is a statistical manifold. Using Lemma 2

$$\begin{aligned}
(\nabla^f_{2\delta_i} G_s)(\delta_j, \delta_k) &= -\frac{1}{2(f-1)}(u^s R_{sij}^m + u^s R_{sji}^m + f_i \delta_j^m + f_j \delta_i^m - f_{\cdot}^{m} g_{ij}) g_{mk}\\
&\quad -\frac{1}{2(f-1)}(u^s R_{sik}^m + u^s R_{ski}^m + f_i \delta_k^m + f_k \delta_i^m - f_{\cdot}^{m} g_{ik}) g_{mj}\\
(\nabla^f_{2\partial_{\bar{i}}} G_s)(\partial_{\bar{j}}, \partial_{\bar{k}}) &= 0, \quad (10)
\end{aligned}$$

$$
\begin{aligned}
(\nabla^f_{2\partial_{\bar{\imath}}}G_s)(\partial_{\bar{\jmath}},\delta_k) &= \frac{1}{2(f-1)}u^sR^m_{sik}g_{mj}, \\
(\nabla^f_{2\delta_k}G_s)(\partial_{\bar{\imath}},\partial_{\bar{\jmath}}) &= \frac{1}{2(f-1)}(u^sR^m_{ski}g_{mj}+u^sR^m_{skj}g_{mi}), \\
(\nabla^f_{2\delta_i}G_s)(\delta_j,\partial_{\bar{k}}) &= [\frac{1}{2(f-1)}(u^sR^m_{sij}+u^sR^m_{sji}+f_i\delta^m_j+f_j\delta^m_i-f^m_{.}g_{ij}) \\
&\quad +\frac{1}{2}u^sR^m_{ijs}]g_{km}-\frac{1}{2(f-1)}u^sR^m_{ski}g_{jm}, \\
(\nabla^f_{2\partial_{\bar{k}}}G_s)(\delta_i,\delta_j) &= -\frac{1}{2(f-1)}(u^sR^m_{ski}g_{mj}+u^sR^m_{skj}g_{mi}),
\end{aligned}
$$

where $f_i=\partial_i f$ and $f^m_{.}=g^{mh}f_h$. If (TM,G_s,∇^f_2) is a statistical manifold, by differentiating $(10)_3$ with respect to u^t we occur $R^m_{tik}g_{mj}=0$ (other equations which have curvature components of ∇ is similar). Moreover, we see that f is a constant function on M. So, we have the theorem below.

Theorem 6. *Let (TM,G^f_2) be the tangent bundle of a Riemannian manifold (M,g) and let ∇^f_2 is the Levi-Civita connection of the metric G^f_2. If (TM,G_s,∇^f_2) is a statistical manifold, then ∇ is flat and f is a constant function on M.*

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