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Algebraic Thinking in Middle School Students at Different Grades: Conceptions about Generalization of Patterns

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Abstract – Algebra is generally considered as manipulating symbols, while algebraic thinking is about generalization. Patterns can be used for generalization to develop early graders' algebraic thinking. In the generalization of pattern context, the purpose of this study is to investigate middle school students' reasoning and strategies at different grades when their algebraic thinking begins to develop. First, 6 open-ended linear growth pattern problems as numeric, pictorial, and tabular representations were asked to 154 middle grade students. Next, two students from each grade (6^{th}, 7^{th}, and 8^{th} grade) were interviewed to investigate how they interpret the relationship in different represented patterns, and which strategies they use. The findings of this study showed that students tended to use algebraic symbolism as their grade level was increased. However, the students' conceptions about 'variable' were troublesome.

Key words: algebraic thinking, early algebra, generalization of patterns.

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Introduction

Algebra is essential for understanding high school mathematics and therefore students' learning of fundamental concepts of algebra is critical (Rakes, Valentine, McGatha & Ronau, 2010). The Rand Mathematics Study Panel Report (2003) indicates that algebra in elementary curriculum is a gatekeeper for K-12 schooling. Thus, it is important to focus on algebraic thinking in early grades by connecting it with students' arithmetic knowledge (Carraher &

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Schliemann, 2007; Kaput, 1999; National Council of Teachers of Mathematics (NCTM), 2000). According to Carraher and Schliemann (2007), early algebra is defined as "compass algebraic reasoning and algebra-related instruction among young learners- from approximately 6 to 12 years of age" (p. 670). In this regard, it is indicated in Principles and Standards (NCTM, 2000) that teachers can help students in middle grades and high school students by providing them experience about algebra in early grades. Thus, this study focuses on middle school students' conceptions about algebra when their algebraic thinking has begun to develop.

Patterns and Generalization

Piaget (1952) developed the theory of schema for reasoning and the schema is about conceptions in individual's mind. He proposed the terms of assimilation and accommodation. While the assimilation is the process that when a new knowledge fits with existing schemas and the schema is expanded with new knowledge; accommodation is the process when a new knowledge does not fit the existing schema and the schema is reconstructed. Tall (1991) related these concepts with generalization. He identified assimilation as expansive generalization, and accommodation as reconstructive generalization. Generalization arithmetic is one of the components of algebra (Katz, 1997; Usiskin, 1988). Lee (1996) indicated that "algebra, indeed all of mathematics is about generalizing patterns" (p.103). Generalization is important for developing the schemas about algebraic thinking. Constructing patterns shows that students have meaningful schemas (Steele & Johanning, 2004).

Hargreaves, Threlfall, Frobisher and Shorrocks-Taylor (1999) emphasize the importance of generalization of patterns in mathematics. In a detailed approach, Steele and Johanning (2004) examined 7th graders' schemas for solving algebraic problems. The researchers also applied teaching experiment to develop students' schemas. The students were asked to solve eight generalizing problems and discuss the solutions in their groups. Each student was interviewed in the process of experiment. The researchers revised the next lesson based on the lesson they observed. According to the findings of this study, students who had well-connected schemas could generalize symbolically. Kieran (1989) states that algebraic symbolization is an essential component of algebraic thinking. Although students used similar strategies (e.g. drawing tables, using smaller problems) for solving problems, successful students in verbal and symbolic generalization used more tables with diagrams. Students with well-connected schemas also checked particular cases when they reached generalization. In contrast to them, students with partial formed schemas had difficulty with generalizations, and formed unclosed symbolic generalization. Another important finding was that using diagram rather than numbers in tables provided students to interpret relationships in pattern. Warren (1996) also indicates that the students who transform numbers to table merely have difficulty in generalization. Thus, in this study, the patterns with tabular representation were used to understand how students interpret the relationship in the table and which strategies they use.

Presenting situations that requires analyzing relationship in contexts and pictures to elementary and middle school students is important to develop their algebraic thinking. Patterns as a context can provide these features for making generalizations since patterns ease students to transit from arithmetic to algebra by making generalization (English & Warren, 1998). Since algebra is considered as a way of expressing generality, generalization of patterns is one of the approach for introducing algebra to children in some countries (e.g., British, England, and Singapore) (Kendal & Stacey, 2004). To reveal students' algebraic thinking, the questions about generalization of patterns are used in this study. Particularly, presenting pictorial or figural growth patterns in elementary and middle school students is important for exploring generalization and developing algebraic thinking (Walkowiak, 2014). In pictorial growth patterns, figures change from one figure to the next one in an order and with a relationship to each other (Billings, 2008). Pictorial linear growth patterns were also used for investigating students' conceptions in this study.

Walkowiak (2014) conducted a study with different grade levels to explore how students analyzed pictorial growth patterns and observed the reasoning strategies they used such as figural and numerical. The participants were 3 students from 2^{nd}, 5^{th}, and 8^{th} grade. The researcher conducted task-based interviews. The tasks formed with two pictorial growth patterns, which require describing the next picture and some later pictures, and then generalizing the pattern. The findings showed that students used both figural and numerical reasoning for generalization; however younger students used more figural reasoning. Additionally, students used words and notations to make generalization. The notations students used could change regarding their age and knowledge. Generally, when younger students used their invented notation (e.g. using a circle for representing start), older students who had algebra course used formal notation. Only eighth graders used symbolic notation to generalize the pattern in the study. This study's findings were also supported with Moss, Beatty, McNab, and Einsband's (2006) study. They designed lessons that included figural and numerical patterns, and transition between these two types of patterns. The researchers concluded that students who participated in these lessons were better on making generalization. Similarly, Rivera and Becker (2005) studied with prospective elementary and middle school teachers, and the researchers concluded that participants who used more figural reasoning than numerical reasoning could explain the formula more explicitly. These studies suggest the use of figural and numerical patterns for developing generalization strategies. Warren and Cooper (2008a) also suggest that students should experience visual growth patterns besides repeating patterns in elementary school. Warren and Cooper (2008a) conducted a teaching experiment to investigate teaching actions that can assist developing elementary students' algebraic thinking. The researchers designed two lessons with 45 students aged about 8 years and a teacher. The task in the first lesson was about extending the pattern that was given first three steps. In the second lesson, the task was formed the pattern with the missing steps to require exploring the relationship between the position number and term. The pre-test and post-test that had growing pattern questions were applied students. The results showed that students' understandings were developed with the experiment. Barbosa and Vale (2015) obtained similar result that finding relationship in the context of visual patterns could develop students' reasoning for generalization. Thus, figural patterns, numerical and tabular patterns were also used in this study to give students opportunity to express their opinions for different representations and to explain their reasoning strategies in a broader perspective.

However, there are several studies that show high level students have difficulty with generalization of patterns. Becker and Rivera (2005) examined 9th graders analysis of patterns and functions. They used pictorial growth patterns, and found that most students could extend the patterns, but few of them could generalize it with algebraic formula. Similarly, Çayır and Akyüz (2015) found that 9th graders had difficulty in finding the generalization rule algebraically. In generalization process, older students have difficulty because they could not relate the position number and the term (MacGregor & Stacey, 1996). Harel (2001) also stated that students have difficulty in making generalization if they consider only the output values of patterns, not the relationship in the elements of the entry pattern. Jurdak and El Mouhayar (2014) emphasized the effectiveness of functional reasoning for finding a rule in pattern generalization. The researchers studied with students from 4th to 11th grade to investigate their reasoning related with the grade level, tasks, and strategy they used. Their study showed that students' level was developed across grade, far generalization task type which asks nth term was difficult for students, and students strategies were different in the same grade. Rivera (2010) also stated that middle school students had difficulty with generalizing algebraically that indicates functional relationship in far generalization type of questions. Ferrara and Sinclair (2016) proposed an approach to develop early grade students' functional reasoning related with the concept of variable between the input and output values. This approach is about a discourse related with pattern generalization based on theoretical frameworks. The researchers concluded that this approach helped students to recognize the functional relationship between variables.

Considering the difficulties students have in generalizing patterns, this study aims to investigate middle school students' generalization strategies and give suggestions for instruction to prevent students' difficulties in higher grades.

Methodology

Steele and Johanning (2004) suggested 'schema theory' oriented instruction to develop students' algebraic thinking schemas. To do this, in particular, problems with the context of size-shape, and growth-change can help students to develop algebraic thinking. Forming and generalizing patterns within these contexts can be used for introducing algebra to students. Thus, the purpose of this study is to investigate middle school students' conceptions at different grades when their algebraic thinking has begun to develop. As Walkowiak (2014) indicated the lack of research in the range of different ages about algebraic thinking; 6th, 7th, and 8th graders' conceptions about generalizing patterns are examined in this study. In this context, the following research question is framed: "how do middle school students at different grades generalize patterns?" To this end, a qualitative research design is used to get detailed information about students' reasoning and strategies.

Instrument

In this study, a pattern test was prepared and it had 6 open-ended questions that were adapted from the literature (Blanton & Kaput, 2003; Lannin, Barker & Townsend, 2006; Magiera, van den Kieboom & Moyer, 2013; Moss, Beatty, Barkin, & Shillolo, 2008; Stacey & MacGregor, 2001; Warren & Cooper, 2008b). While preparing this test, the researchers considered different types of pattern for the purpose of the study: patterns with numerical, pictorial, and tabular representations. The time for solving the questions was set based on the level of the students. Finally, the instrument was tested before the actual study to ensure the usability and validity issues.

The questions in the test consisted of numerical, pictorial, and tabular representations of linear growth patterns, since the aim of the study was to investigate students' interpretations and strategies for generalization of different types of patterns. At the beginning, the test included more than 10 questions. Then, the test was revised and only 6 questions were included as students spent long time in each question. The questions were open-ended and students were asked to explain their solutions by writing on the paper. One class hour was given to students to solve problems. Each question was implemented and tested in the context of the studies about teaching generalization of patterns to the students. To ensure the validity of questions, the opinions of an experts in mathematics education were taken about the test. The rationale of the selection of questions is explained in the following, and the test is included in the appendix.

In the test, the first and fourth questions are pictorial patterns. The first pattern is "the lunchroom table problem" that is explained by Blanton and Kaput (2003). The aim of the problem is to explore the relationship between the independent and dependent variable. The researchers asserted that many students had difficulty while solving this problem (Moss et al., 2008). Thus, they used this problem in their instructional design to develop students' functional thinking. The fourth question belongs to Magiera et al.'s (2013) study. They proposed that the task could provide students to use algebraic thinking features such as describing and justifying the rule based on Driscoll's (2001) description.

The second and fifth questions are tabular representations of patterns. The second question type is suggested by Stacey and MacGregor (2001) as a task to investigate students' use of algebraic rules rather than express the relationship between the terms. The researchers stated that students should be encouraged to use algebraic thinking to develop the rules (Lannin et al., 2006). This task was adapted by changing the numbers in the cost columns to challenge the students. In the presented task, the numbers were continuing as 9, 15, 27 … as three times of the number of t-shirt. These numbers were changed as 10, 16, 28 ... . On the other hand, the fifth question was suggested by Magiera et al. (2013) to develop algebraic thinking as in the fourth question. This question was adapted by changing the number of people that came in each time (2 was replaced with 3). It was essay type of problem, and the researchers changed it for this study by adding a table that consists of the number of bell and the number of coming people in columns.

The third and sixth questions are numerical patterns. These patterns were exemplified as growing pattern represented with numbers by Warren and Cooper (2008b). They assert that the constant difference between the terms is important to develop elementary students' understanding in order to see the relationship between input and output values. The pattern in the sixth question was selected with the numbers that are decreasing by 5 from one term to the next term instead of increasing in order to make it unusual for students. The aim was to examine how students generalize this pattern.

Data Collection

The participants were 6th, 7th, and 8th grade students who were taught pattern generalization. The students were selected from a middle socio-economic level school based on its availability to the researcher. The questions were asked to 154 students in total (48 sixth grade students, 59 seventh grade students, and 47 eighth grade students) to determine students' generalization strategies of patterns. Then, the students who had different and detailed solution strategies from other students' solutions were interviewed since deep and detailed information was needed to investigate students' reasoning strategies in generalization process. Thus, six volunteered students, including two students from each grade, were selected for task-based interviews according to their solutions. These students were selected by considering their detailed solutions, their volunteering, and the opinions of their mathematics teacher. Their mathematics teacher stated that based on students' performance in mathematics classes these students could explain their opinions and solutions explicitly. The researcher conducted task-based interviews with these students. Task-based interviews present mathematical problems and require participants to explain and justify their solutions. Task-based interviews give opportunities to understand participants' knowledge, understanding, and strategies (Goldin, 2000). In this study, the questions were designed as tasks with sub questions to require explanation and justification, and they were used in the interviews. The interviews lasted about 15-20 minutes. In the findings part, 6th grade students are named as A1 and A2, 7th grade students are named as B1 and B2, and 8th grade students are named as C1 and C2.

Data Analysis

Creswell (2007) describes the data analysis process as "consists of preparing and organizing the data for analysis, then reducing the data into themes through a process of coding and condensing the codes, and finally representing the data in figures, tables, or a discussion" in a qualitative research (p. 148). In this study, the data included sources from students' answers for open-ended questions and interviews. In data analysis process, first, the students' solutions for six questions were categorized regarding their similarities and differences based on the generalization strategies that Walkowiak (2014) proposed in the framework. After forming the categories as themes that were using descriptive words, and using notation; sub-categories were deducted based on the data. These sub-categories were writing verbal statement, assuming the next term as nth term, and stating the growth of the figures under the heading of descriptive words/statements strategy; and writing algebraic expressions under the heading of formal notation strategy. For these categories and sub-categories, frequencies were also found in order to give the whole picture regarding the frequency of students' using of the generalization strategies (see Table 1). After getting general overview about students' strategies for generalization of patterns, interviews were examined to understand why and how students think for generalization in order to reveal their reasoning. The recorded interviews were transcribed and read by the researchers. The revealed reasoning strategies for generalization of patterns in the interviews were analyzed by using Walkowiak's (2014) conceptual framework. The data from interviews were examined based on two reasoning strategies, numerical and figural reasoning.

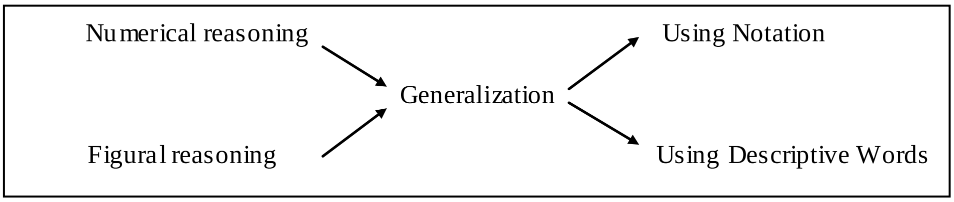

Figure 1 Conceptual Framework for Analyzing Students' Generalization Strategies

Walkowiak (2014) explained the conceptual framework within two themes based on the findings of the study. These themes are "an intersection between reasoning figural and numerical, and making generalizations using symbols and/or words " (p. 67). The first theme indicates that students use both numerical and figural reasoning to get a generalization. The researcher also explains that students could understand generalization of patterns by using these two types of reasoning strategies in a better way. The second theme states that students use notation or/and descriptive words for describing generalizations The notations used by students can be both invented and written algebraically. The themes used in this study based on the framework are shown in Figure 1.

Trustworthiness

In qualitative research, trustworthiness as a term is used to explain validity and reliability issues (Lincoln & Guba,1985). Merriam (2009) states that ensuring validity and reliability is necessary in data collection, analysis process, and interpretation of the findings. While validity can be described as "the extent to which research findings are credible", reliability can be described as "the extent to which there is consistency in the findings"

(Merriam, 2009, p. 234). In order to provide the trustworthiness of the study, several methods (e.g. triangulation, member checking, peer examination, and cross-checking) can be used. To provide the trustworthiness of this study, triangulation in collecting data for validity, and cross-checking method for reliability were used. "Triangulation is the process of corroborating evidence from different individuals (e.g., a principal and a student), types of data (e.g., observational field notes and interviews), or methods of data collection (e.g., documents and interviews) in descriptions and themes in qualitative research" (Creswell, 2012, p. 259). In this study, data were collected from the solutions of the open-ended questions as well as interviews with the students. On the other hand, cross checking is the agreement on the codes for the same data (Creswell, 2009). In this study, both an expert in mathematics education, and a researcher were asked to code the data. The researcher and the expert discussed the codes of sub-themes and agreed on them during the process.

Findings

In this section, the students' solution strategies are examined by using Walkowiak's (2014) framework and the table below represents the frequencies of 154 students' strategies. The strategies are categorized in two groups based on the framework, and the subcategories are extracted from the data.

The Frequencies of Students' Used Strategies (Categories)

Table 1 The Frequencies (%) of Students' Used Strategies

Strategy (Categories)	**Descriptive words/statements**			**Formal notation**
Sub-Categories / Grade	Writing verbal statement	Assuming the next term as n^{th} term	Stating the growth of the figures	Writing algebraic expressions
6^{th} grade	17	47	14	7
7^{th} grade	26	10	4	42
8^{th} grade	26	13	5	39

According to Table 1, descriptive words/statements strategy is examined in three categories, and formal notation strategy is examined in one category. It can be said that, as grade level is increased, students prefer to use algebraic expressions as formal notation for generalizations. 6^{th} graders generally express generalization verbally using descriptive words and statements. Especially, 6^{th} graders have the conception is that n^{th} figure/number is the next term after given terms in the pattern. This tendency is less seen in 7^{th} and 8^{th} graders solutions. 6th graders also state the growth of the figures as general rule in pictorial patterns. This category is also not seen frequently in higher grades. The percentages of 7th and 8th graders' using descriptive words/statements and formal notation' categories are about similar.

In the next part, these categories are explained with examples in detail regarding to each grade. Additionally, students' reasoning strategies are analyzed based on the dialogues from interviews.

Sixth Grade Students' Generalization and Reasoning Strategies

Table 2 shows sixth grade students' solution strategies with categories. Additionally, the examples from students' answers are given for different categories.

Table 2 The Categories of 6th Grade Students' Solutions

Students' solution strategies (Categories)	Sub-categories	Representative students' writings
Descriptive words	Writing verbal statements	Chairs increase by 3, tables increase by 1 It increases as 3 times and plus 1 The numbers in both columns increase Increases by 4 Increases by 3, the difference between the number of bell and people increases by 2 Decreases by 5
	Assuming n^{th} term as next term after existing figures/numbers (and finding corresponding number regarding pattern rule)	There are 4 tables and 14 chairs (there are 3 figures in the question) Decreases by 5 and 7th number is 30 Increases by 4 and for n=7, it is 27 Decreases by 5 and for n=8, it is 25 Finding n based on the row number
	Stating the growth of the figures	Increases by 2 like "v" letter
Formal Notation	Writing algebraic expressions*	n.3, *3n+2*, n.4+2 *3n+1*, n.3 *4n-1*, n+4 n+2, 2n-1, n.2+1 n+3, 3n-2, n.16 n-5
Other	No solution	
	Having difficulty with variable concept	Since it is n, it is indefinite

*Italics are correct answer.

Students' solutions were analyzed within three strategies (descriptive words, formal notation, other) based on Walkowiak's framework. According to the findings, these strategies were categorized with the examples from students' answers in Table 2. Especially, the first strategy was examined within three categories. Students mostly used the first category, and they wrote verbal statements such as chairs "increase by 3, tables increase by 1, increases by

4, decreases by 5" etc. Then, in the second category, they assumed n^{th} figure/number as next figure/number after existing terms and found corresponding figure/number regarding pattern rule. For example, if there are 3 figures in the pictorial pattern, students wrote there were 4 tables and 14 chairs in the 4^{th} picture as n^{th} picture. Another example is finding the number as n^{th} term considering the last given number in the question. If the pattern has 5 terms like this "3, 7, 11, 15, 19…" students assumed that dotted place was 6^{th} term and then answered 7^{th} term for n^{th} term (e.g. increases by 4 and for n=7, it is 27). Few of the students used formal notation and wrote algebraic expressions as in the second strategy such as n.3, n.4+2, 4n-1, n+2, etc. for generalization. There were students who had no solution for some questions. Interestingly, one student of 6^{th} graders had difficulty with variable concept and so this student had the idea of unknown was indefinite.

In terms of pattern types, students had different ways to find the general rule of the pattern. In the 1^{st} and 4^{th} question (pictorial patterns), students wrote verbal statements by stating the growth of the figures in the pattern. For example; some students stated the growth of the figures such as increasing by 2 like v letter. They also assumed n^{th} figure as next figure after the existing figures. If there were 3 figures as in the first question, students considered 4^{th} figure as n^{th} figure. In number patterns (3^{rd} and 6^{th} question), students wrote verbal statements (e.g. increases by 4, decreases by 5) and most of them gave a number for n assuming the next number such as for n=8, it was 25. For particularly tabular patterns (2^{nd} and 5^{th} question), students generally found row numbers to n given in the table. In general for each type of patterns, there were only few students who wrote algebraic expressions for generalizations.

Most of the sixth graders expressed generalization of patterns verbally indicating the difference between consecutive terms in the pattern. For example, in the first question which included a pictorial pattern with tables and chairs, the students generalized as 'chairs increase by 3, or tables increase by 1'. In number and tabular patterns, they used similar statements. An example is 3, 7, 11, 15, 19… pattern and they stated "increases by 4", or they express "decreases by 5" for 60, 55, 50, 45... pattern. The following excerpt shows how sixth graders generally explained their reasoning:

R (researcher): How do you explain your solution for the first question?

A1: For the first table, there are 1, 2, 3, 4, 5 (A1 is counting chairs). 3 chairs increase for the second table. 6 chairs increase for the third table. Because, 3 chairs are put each time, first time 3 and then add 3 more for the second time, 6 chairs increase. 3 chairs are put for each table.

R: What is the relation between the numbers of chairs and tables?

A1: When the number of tables increases by 1, the number of chairs increases by 3. So, the difference is 2.

R: How do you write an expression for this relation?

A1: I can write (A1 is writing 'when a table is put, 3 chairs are put')

In the generalization process, the students first tried to understand how the picture was. They investigated what they were and how the relationship was between different units (e.g. chair and table) in the pictures. They considered how the picture grew as in different steps. Thus, they used figural reasoning. Then, the students generally stated generalizations verbally using descriptive words or statements. They focused on the difference between consecutive terms and indicated the generalization rule as increment or decrement. Few of the 6th grade students could write symbolic generalization using 'n' (e.g. n+3, 3n, n+4), but very few of them could reach correct algebraic generalization. Other students tried to find a number for n. For example, if a number pattern was given to the 5th term, students found the 6th term and explained the 7th term as nth term. If the pattern had four pictures, they had tendency to draw the 5th picture as the nth picture in pictorial patterns. In tabular patterns, they first found the blank rows and then determined a number for 'n' regarding the previous position number (see Figure 2). A1 explained her strategy to find the nth term as in the following:

A1: When the bell rings once, 1 person comes. The bell rings the second time, 4 people come. So, it increases by 3. When the bell rings for the sixth time, the number of the coming people is 16.

R: How do you find 16?

A1: It increases by 3. The fifth number is 13. And then the sixth

number is 16.

R: How many people come when the nth bell rings?

A1: For n, a number is determined and the other column

(in table) is increased.

R: While increasing, where do you stop?

A1: For example, I can continue how many people there are in the class.

If there are 30 people, we can count up to 30.

Zil sayısı	Gelen kişi sayısı
1	1)3
2	4)3
3	7)3
4	10
: 5	: 13
: 6	: 16
n 7	? 19

Figure 2 A1's Solution

This excerpt suggests that she did not have the variable concept conceptually. She could not consider finding a rule using 'n'. Instead of this, she tried to give a number for n. Thus, she did not consider any invented or formal notation for generalization. In the generalization process of the number and tabular patterns, she used numerical reasoning to explore a relationship between the numbers in the process.

Based on the Walkowiak's (2014) framework, it was observed that the sixth graders used mostly numerical reasoning and descriptive word strategy for different type of pattern questions. Only in pictorial patterns, they preferred to use figural reasoning more than numerical reasoning.

Seventh Grade Students' Generalization and Reasoning Strategies

Table 3 shows seventh grade students' solution strategies with categories. Additionally, the examples from students' answers are given for different categories.

Table 3 The Categories of 7^{th} Grade Students' Solutions

Students' solution strategies (Categories)	Sub-Categories	Representative students' writings
Descriptive words	Writing verbal statements	Chairs increase by 3, tables increase by 1 It increases as 3 times and plus 1 The numbers in both columns increase Increases by 4 Increases by 2 Increases by 3, the difference between the number of bell and people increases by 2 Decreases by 5
	Assuming n^{th} term as next term after existing figures/numbers and finding corresponding number regarding pattern rule	There are 4 tables and 14 chairs (there are 3 figures in the question) 6^{th} number as 23 Increases by 4 and for n=7, it is 27 Decreases by 5 and for n=8, it is 25 Decreases by 5 and 7^{th} number is 30 Increases by 3 and for n=7, it is 19 Finding n based on the row number
	Stating the growth of the figures	Increases by 2 like "v" letter
Formal Notation	Writing algebraic expressions*	*3n+2*, n+3, n+1 6n-8, n.2-1 *4n-1*, n+4, 3n+1 n+2, 2n-1, 2n+1 n+3, *3n-2*, 2n+1, n+2, n.2+2 n-5, 5n+55
Other	No solution	

*Italics are correct answer.

Students' solutions were analyzed within two (descriptive words and formal notation) strategies based on Walkowiak's (2014) framework. According to the findings, these strategies were categorized based on students' answers as in Table 3. Especially, the first strategy was examined within three categories. Slightly more than half of the students used the first strategy. Similar to 6^{th} graders, most of them were in the first category and they used descriptive words for generalization. These students wrote verbal statements such as "chairs increase by 3, tables increase by 1, increases by 4, decreases by 5" etc. When compared to 6th graders in the second category, there were less students in 7th graders as they assumed nth term as next figure after existing terms and found corresponding number regarding pattern rule. For example, if the pattern has 6 terms such as "60, 55, 50, 45, 40, 35 …" students assumed 7th term for dotted place and then answered 8th term for nth term (e.g. decreases by 5 and for n=8, it is 25). In 7th graders, there were more students who used formal notation as a second strategy. They wrote algebraic expressions such as 3n+2, 6n-8, 4n-1, n-5 for generalization. There were also students who had no solution for some questions. Unlike 6th graders, there were not any students who had difficulty with variable concept.

In terms of pattern types, 7th grade students had different ways to find the general rule of the pattern. In the 1st and 4th question (pictorial patterns), students wrote verbal statements by stating the growth of the figures in the pattern. For example; some students stated the growth of the figures such as increasing by 2 like v letter. Different from 6th graders, more students used formal notation for pictorial pattern generalizations. In number patterns (3rd and 6th question), different from 6th graders, about half of the students wrote verbal statements (e.g. increases by 4, decreases by 5) . They gave a number for n assuming the next number such as for n=8, it was 25. The other students preferred to express the general rule algebraically (e.g. 4n-1, n+4, 3n+1, n-5, 5n+55). Particularly for tabular patterns (2nd and 5th question), students generally had difficulty to find a rule. Most of them had no solution for this type of patterns. The students who did this question preferred to write the rule algebraically. In general for each type of patterns, there were more students who wrote algebraic expressions for generalizations than 6th graders.

Algebraic expressions were seen more in 7th graders' generalizations. However, 7th graders could not reach correct expressions since they did not take into consideration the position number of the terms in the pattern. For example, B2 directly wrote 'n-5' by considering the difference between the consecutive terms for the sixth question. Students who considered the position number could write correct algebraic expressions. However, they explained their strategies by trial and error. The following dialogue shows how B1 explained her strategy for generalization (see Figure 3):

R: Let's look at the second question.

B1: … Again I can do this way, n times 3 and plus 1, 1 times 3 and I add 1, 4. But it is 10. Starting with 1 does not work. I multiply 3 by 3, 9, add 1, 10. I multiply 3 by 5, 15, add 1, 16. I multiply 9 by 3, 27, add 1, 28.

R: You put the number of t-shirts instead of n and you multiply 3 and add 1. Am I right?

B1: Yes. The number of t-shirts and n are the same thing.

R: If I want to buy 100 t-shirts, how much will I pay?

B1: Again, I multiply 100 by 3, and add 1, 301.

R: How do you find this relation?

B1: I tried.

R: What do you try? What comes first to your mind?

B1: First, I think what I can do with 3 to find 10. I could try multiplying by 2. I did not remember. Maybe 3. Because generally we multiply something and add something. Then, I tried for other terms in the term. I think, if it works for others, it is right.

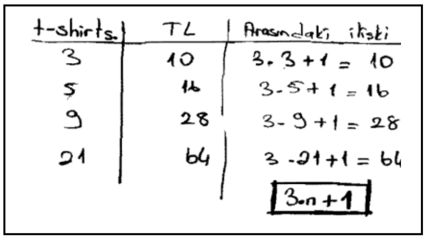

Figure 3 B1's Solution

This student reached the correct algebraic generalization. She first started with position number as 1. When she realized 1 did not work, then she started with 3 as the number of t-shirts. Then she tried multiplying by 3 and added 1 to get 10 as the first term. She indicated that she could try multiplying by 2 before 3. Similar to B1, most students tried to reach an algebraic generalization. They used formal symbols in their solutions. The students who did not use algebraic expressions used verbal statements similar to 6th graders. However, 7th grade students differed from 6th graders for pictorial patterns. In the generalization process, 7th grade students counted the number of units in pictures in pictorial patterns and they looked for the relationship in these numbers. They did not consider how the picture grew in different steps. Thus, they drew the picture asked for the next step by considering how many units there would be. They thought the shape of picture such as the shape of 'v' letter as figural reasoning. They already considered numbers in number patterns and tabular patterns that had numbers as terms in patterns. Thus, they used numerical reasoning.

The reasoning of seventh grade students was different regarding representations of patterns. In general, based on Walkowiak's (2014) framework, for number and tabular patterns, they used numerical reasoning. They used numerical reasoning mostly in pictorial patterns and sometimes figural reasoning. Students generally preferred to generalize by using formal notations and descriptive words. Particularly number patterns, similar to 6th graders, they mostly used descriptive words for generalizations.

Eighth Grade Students' Generalization and Reasoning Strategies

Table 4 shows eighth grade students' solution strategies with categories and the examples from students' answers for different categories.

Students' solutions were analyzed within two (descriptive words and formal notation) strategies based on Walkowiak's (2014) framework. According to the findings, these strategies were categorized with representative students' writings in Table 4. About half of the students were in the first category and they used descriptive words for generalization.

Table 4 The Categories of 8^{th} Grade Students' Solutions

Students' solution strategies (Categories	Sub-Categories	Representative students' writings
Descriptive words	Writing verbal statements	Chairs increase by 3, tables increase by 1 It increases as 3 times and plus 1 Increases by 4 Increases by 2 Increases by 3, the difference between the number of bell and people increases by 2 Increases by 3 and for n=7, it is 19 Decreases by 5
	Assuming n^{th} term as next term after existing figures/numbers (and finding corresponding number regarding pattern rule)	There are 4 tables and 14 chairs (there are 3 figures in the question) 6th number as 23 Increases by 4 and for n=7, it is 27 Decreases by 5 and for n=8, it is 25 Decreases by 5 and 7^{th} number is 30
	Stating the growth of the figures	Increases by 2 like "v" letter
Formal Notation	Writing algebraic expressions*	*3n+2*, n+3, 3n *3n+1*, 8n+1, 2n+1 *4n-1*, n+4 n+2, 2n-1, n+3 n-5, 5n, *-5n+65* n+3, *3n-2*, n+2
Other	No solution	

*Italics are correct answers.

There were also students who assumed n^{th} term as next term after existing figures. Similar to 7^{th} graders, 8^{th} graders also used formal notation and wrote the general rule algebraically. Additionally, their generalizations were more correct than 7^{th} graders' generalizations. They wrote algebraic expressions such as 3n+2, 3n+1, 4n-1, 3n-2. There were also students who had no solution for some questions.

In terms of pattern types, 8^{th} grade students had different ways to find the general rule of the pattern. In the 1^{st} and 4^{th} question (pictorial patterns), students wrote verbal statements by stating the growth of the figures in the pattern. Different from 6th graders, more students used formal notation for figural pattern generalizations. In number patterns (3rd and 6th question), similar to 7th graders, about half of the students wrote verbal statements (e.g. increases by 4, decreases by 5) and gave a number for n assuming the next number such as for n=8, it would be 25. Other half of the students preferred to express the general rule algebraically (e.g. 4n-1, n+4, 3n+1, n-5, 5n+55). For particularly tabular patterns (2nd and 5th question), although there were students who had no solution for these questions, there were more answers than 7th graders' answers. About half of the students who did this question wrote the rule algebraically and others used the descriptive statements. In general for each type of patterns, the students who wrote algebraic expressions for generalizations are about same percentage of 7th graders. However, 8th graders' generalizations were more correct.

Almost all eight graders tried to generalize algebraically; but the students who could not do algebraic generalization used verbal statements similar to the seventh graders. They reached more correct algebraic expressions (e.g. for the pattern 3, 7, 11, 15, 19… 4n-1), since eight grade students considered the position number of terms in the pattern while generalizing. Students generally found the difference between the consecutive terms. They used this difference as the coefficient for n. Then, to find the constant number in the expression, they applied trial and error strategy. In the following excerpt, C1 indicated this strategy:

C1: The number of chairs increases by 3 for each table. For this increment, it is 3n.

R: Do you write '3' in 3n considering the difference?

C1: Yes. Then I apply for the second picture to check.

When the student (C1) was asked to further explain this strategy, he indicated that his mathematics teacher taught it with this way. He focused on numbers in this pictorial pattern and used numerical reasoning considering which number he could multiply and then he could add. The other interviewed student (C2) generally did not reach the correct generalization. This student tried to find a different rule for each term (Figure 4):

C2: When the bell rings once, only 1 person comes. Then, when the bell rings for the second time, the number of people is 2 times the number of bells. When the third bell rings, 2 times and plus 1. When the fourth bell rings, 2 times and plus 2. When the fifth bell rings, 2 times and plus 2, so 12 people come. Then when the sixth bell rings, 6 times and plus 2,

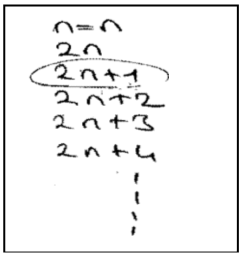

Figure 4 C2's Solution

> 14 people …? I did wrong. It is 4 more than 12, 16. 2 times and plus 1, 2 times and plus 2, 2 times and plus 3 ... goes like this.

This student tried to generalize algebraically by using ‘n’ as formal notation. However, this student’s understanding about generalization was troublesome because she found a different rule for each term in the pattern.

In general, based on Walkowiak’s (2014) framework, the reasoning of eighth grade students was numerical in different representations of patterns. Figural reasoning was not seen generally in their solutions. Students generally preferred to generalize by using formal notations. In tabular patterns, when they could not use formal notations, they used descriptive words.

Discussion and Conclusion

It has been observed that students tend to use algebraic symbolism as their grade level is increased. This finding is supported with Walkowiak’s (2014) study. 6^{th} graders generally use verbal statements to explain the relationship in the pattern. Healy and Hoyles (1999) define this strategy as a recursive rule that students can explain the relationship focusing on the difference among consecutive output values in the pattern. 6^{th} grade students do not consider using algebra for generalizations. In contrast, 7^{th} and 8^{th} graders use algebra more often. This strategy is related to using explicit rules in which students find a rule relating input and output values in the pattern (Healy & Hoyles, 1999; Lannin et al., 2006). Particularly, when 7^{th} graders do not generalize algebraically, they explain the relationship verbally. 8^{th} graders use both symbolic notations and verbal representations for generalizations.

In general, most of the students who use algebraic symbols do not reach correct generalization. As MacGregor and Stacey (1996) indicate that students do not consider the position number in the pattern, they focus on the difference between the consecutive terms. Students who generalize algebraically and correctly explain their method as follows: first, find the difference between the consecutive terms and use this difference as the coefficient for n; then, to find the constant number in the expression, add or subtract different numbers to get the first term. Additionally, they check for the second and third terms and sometimes other terms. Several studies also explore this strategy and it is called as chunking. It is about multiplying by the common difference and adding number to find the first term (Lannin et.al, 2006; Yeap & Kaur, 2008). To illustrate; for 5, 9,13,17,.. pattern, the difference is 4, and students multiply n by 4 and get 4n. Then, they substitute 1 for n, and they add 1 to get 5 as a first term. Students focus on finding a rule rather than understanding the generalization approach (Harel, 2001). In the interviews, students stated that they learnt this method from their mathematics teacher.

The conceptions about ‘n’ as a variable is troublesome. Students have a tendency to give a specific number for n. In tabular patterns, they fill blank rows and then they give the number as one more than the last number. This strategy is observed other pattern types as well. In number patterns, they extend the pattern one more into the blank space, and then they consider the next term as the n^{th} term. They also think similar in pictorial patterns. They draw a picture for blank space after the given pictures and they call the next picture as n^{th} picture. This strategy is seen more in 6^{th} graders and sometimes in 7^{th} graders. Particularly, the sixth grade student (A1) extends the decreasing pattern to zero and state n is zero. It can be explained with early graders not understanding variable as a varying quantity conceptually (Asquith, Stephens, Knuth & Alibali, 2007; Küchemann, 1978; MacGroger & Stacey, 1997). Particularly early grade students have difficulty to understand the variable concept. Beginning patterns for teaching algebra can be useful to give the idea of variable (Kendal & Stacey, 2004). Thus, the teacher can give more time to students to work on patterns, as studies indicate more experience with patterns develops students’ algebraic thinking (Lannin et al., 2006; Warren & Cooper, 2008a).

In this study, it was asked to generalize a decreasing pattern (60, 55, 50…) in the last question. Students indicate that they are not familiar with this type of pattern, so that they have difficulty with it. They say that they get used to multiply something and add something for getting a rule since they think patterns have increasing numbers. Thus, different types of patterns can be presented to students to develop students’ generalization strategies.

In the generalization process, 6^{th} graders use figural reasoning in pictorial patterns by extending the pattern. 7^{th} and 8^{th} graders use numerical reasoning to explore how many units there are in given pictures. This finding is consistent with one of the findings of El Mouhayar and Jurdak’s (2016) study. Similar to their results, we found that the numerical reasoning was more dominant in students’ generalizations while using recursive strategy . Healy and Hoyles (1999) indicate relating numerical reasoning to pictures can develop students’ generalization abilities. Since students have different reasoning for generalization, teachers can use different types and representations of patterns. While 6^{th} graders can be encouraged to use symbolism, 7th and 8th graders can be provided to understand the meaning of generalization rather than finding a rule.

Based on our study's results, we can suggest that teachers should give more emphasis on exploring the relationship between the position number and the term instead of focusing on only algebraic rule. Students should be given different opportunities to explore the relationship between pattern and variable. To do this, students might be given more time by guiding them such as asking how the 50th, 100th or 1000th term could be found; and exposed with the tasks that include questions to support them reach a general rule. Teachers also need to use different patterns that include different representations such as pictorial, tabular and numerical. Different representations regarding pattern questions can also be included in textbooks for supporting teachers to use them in their lessons. Teachers can also use the pattern test in this study or adapt the questions based on their students' levels while teaching generalization of patterns. They can also give this pattern test to the students as homework. The findings including representative examples of students' solutions, misconceptions, errors and difficulties in generalizing patterns can help the teachers to develop their lesson designs by considering students' thinking. Additionally, the findings of this study can be used in method courses to improve pre-service teachers' pedagogical content knowledge.

To enrich the findings, the test given in this study can be used to understand teachers' reasoning of the students' solutions. In the future studies, researchers might develop the test to use in quantitative studies and this might allow to get more general conclusions with applying larger samples.

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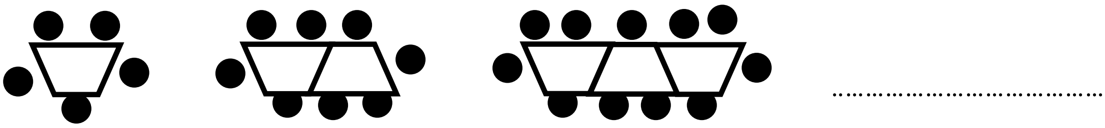
Appendix: Pattern test

ÖRÜNTÜ TESTİ

1.



1.şekil 2.şekil 3.şekil n.şekil

Yukarıdaki yamuk şeklindeki masalara şekildeki gibi sandalyeler yerleştirilecektir. Her şekil, önceki şekle bir yamuk masa daha eklenerek devam etmektedir. Bu örüntüde 1.,2., ve 3., şekiller, örüntünün ilk üç şeklidir. Örüntüdeki n.şekil ile çevresine yerleştirilebilecek sandalye sayısı arasında nasıl bir ilişki vardır?

Çözümünüzü açıklayınız:

2. Aşağıdaki tablo belli sayıdaki t-shirtin ne kadar olduğunu göstermektedir.

t-shirt sayısı	TL
3	10
5	16
9	28
21	64
: :	: :
n	?

Yukarıda verilen tabloya göre t-shirt sayısı ile değeri arasında bir ilişki vardır. Bu ilişkiye göre, n tane t-shirt için kaç TL ödenmelidir?

Çözümünüzü açıklayınız:

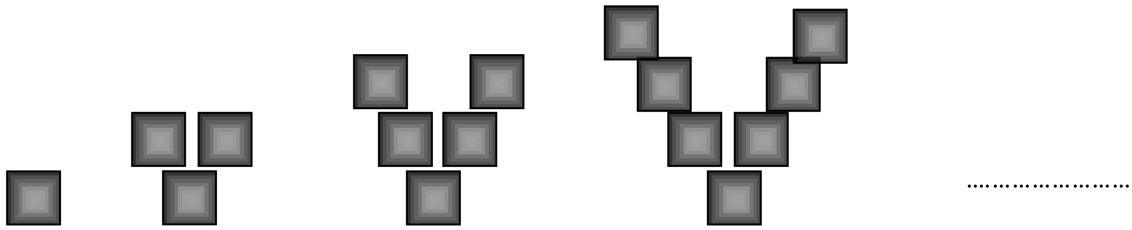
3.

1.sayı	**2.sayı**	**3.sayı**	**4.sayı**	**5.sayı**	**………**	**n.sayı**
3	**7**	**11**	**15**	**19**	**………**	**?**

Yukarıdaki örüntüde yer alan sayılar bir kuralla oluşturulmuştur. Örüntüdeki sayıların oluşum kuralını bulunuz.

Çözümünüzü açıklayınız:

4.



1.şekil 2.şekil 3.şekil 4.şekil n.şekil

Yukarıdaki örüntüde "V" harfinin değişik boyutları küçük kareler kullanarak oluşturulmuştur. Örüntüdeki herhangi bir "V" harfi ile küçük kare sayıları arasında nasıl bir ilişki vardır?

Çözümünüzü açıklayınız:

5. Zeynep'in doğum günü partisinde, zil ilk kez çaldığında bir arkadaşı gelmiştir. Bundan sonra çalan her zilde, gelen gruptaki kişi sayısı, bir önceki gelen gruptan 3 kişi fazladır. Aşağıdaki tabloda gelen kişi sayısı gösterilmiştir. Herhangi bir zil çalışını n kabul edersek, gelen kişi sayısını bulmak için kullanılacak genel ifade ne olmalıdır?

Zil sayısı	Gelen kişi sayısı
1	1
2	4
3	7
4	10
: :	: :
n	?

Çözümünüzü açıklayınız:

6.

1.sayı	2.sayı	3.sayı	4.sayı	5.sayı	6.sayı	………..	n.sayı
60	55	50	45	40	35	………..	?

Yukarıdaki örüntüde yer alan sayılar bir kuralla oluşturulmuştur. Örüntüdeki sayıların oluşum kuralını bulunuz.

Çözümünüzü açıklayınız:

Farklı Sınıf Seviyelerindeki Ortaokul Öğrencilerinde Cebirsel Düşünme: Örüntülerde Genelleme Hakkındaki Algıları

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Özet - Cebir, genel olarak sembolleri manipüle etmek olarak görülürken, cebirsel düşünmenin genelleme ile ilgili olduğu kabul edilir. Örüntüler, erken yaşlardaki çocukların cebirsel düşünmelerini geliştirmek için genelleme ile kullanılabilir. Örüntüleri genelleme bağlamında, bu çalışmanın amacı cebirsel düşünmenin geliştiği ortaokul yıllarındaki farklı sınıf seviyelerindeki öğrencilerin akıl yürütme ve çözüm stratejilerini araştırmaktır. Öncelikle, 154 ortaokul öğrencisine sayı, şekil ve tablo şeklinde temsil edilen farklı tipte örüntü soruları sorulmuştur. Sonra, her bir sınıf seviyesinden (6., 7. ve 8.sınıf) iki öğrenci ile, öğrencilerin farklı temsillerle gösterilen örüntülerdeki ilişkiyi nasıl yorumladıkları ve hangi stratejileri kullandıklarını incelemek için görüşmeler yapılmıştır. Çalışmanın bulguları, sınıf seviyeleri arttıkça, öğrencilerin cebirsel sembolleri kullanmaya daha eğilimli olduğunu göstermektedir. Bununla birlikte, öğrencilerin değişken kavramı ile ilgili algılarında sıkıntılar olduğu görülmüştür.

Anahtar Kelimeler: cebirsel düşünme, erken cebir, örüntülerde genelleme

Genişletilmiş Özet

Cebir, lise matematik anlayışını geliştirmek için bir temeldir ve öğrencilerin cebirin temel kavramlarını öğrenmesi önemlidir. Bu yüzden, ilköğretim müfredatındaki cebirin, ortaöğretim matematik eğitimi için bir geçiş sağladığı belirtilir. Dolayısıyla, erken yaşlarda öğrencilerin cebirsel düşünmelerini geliştirmek için aritmetik ile ilişkilendirme yapmak önemlidir. Erken cebir, yaklaşık 6 ila 12 yaş grubundaki öğrencilerin cebirsel akıl yürütmesi ve bu öğrencilere cebirle ilişkili öğretim olarak tanımlanır. Bu çalışmada da cebirsel düşünmenin geliştiği ortaokul seviyesindeki öğrencilerin cebirle ilgili algıları incelenmiştir.

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İlkokul ve ortaokul öğrencilerinin cebirsel düşünmelerini geliştirmek için onlara bir konu ve şekil bağlamında ilişkileri analiz etmeyi gerektiren problem durumlarının sunulmasının önemli olduğu belirtilir. Örüntüler bir durum olarak sunulduğunda genelleme yapmak için bu özellikleri sağlayabilir. Ayrıca, örüntülerde genelleme yapmak aritmetikten cebire geçişi kolaylaştırır. Özellikle, genellemeyi keşfettirmek ve böylece cebirsel düşünmeyi geliştirmek için ilkokul ve ortaokul öğrencilerine şekil ya da geometrik örüntüler sunmak önemlidir. Şekil örüntülerinde, şekiller bir ilişki ve bir düzen içinde sonraki şekilde değişir. Bu bağlamda, şekil örüntüleri öğrencilerin algılarını araştırmak için bu çalışmada kullanılmıştır. Ayrıca bu çalışmada, öğrencilerin farklı temsil biçimlerinde düşüncelerini açıklamalarını sağlamak için şekil örüntülerinin yanında, sayısal ve tablo olarak sunulmuş örüntüler de kullanılmıştır. Böylece, bu çalışmada farklı temsil biçimlerinde öğrencilerin akıl yürütme stratejilerini daha geniş çerçevede anlamak da amaçlanmıştır.

Yöntem

Genelleştirme, öğrencilerin cebirsel düşünme şemalarını geliştirmek için önemlidir. Öğrencilerin örüntülerdeki genellemeleri yapılandırmaları, onların anlamlı şemalara sahip olduğunu göstermektedir. Öğrencilerin cebirsel düşünme şemalarını geliştirmek için önceki bilgilerini geliştirebilecek ya da yeni bilgilerle yeni oluşturulacak şema odaklı öğretime vurgu yapılır. Bunun için, öğrencilerin cebirsel düşünmelerini geliştirmeye yardımcı olabilecek, özellikle, boyut-şekil ve büyüme-değişim bağlamında problemler ve bu bağlamlarda örüntülerin oluşturulmasının ve genelleştirilmesinin öğrencilere cebiri tanıtmak için kullanılabileceği vurgulanmaktadır. Dolayısıyla, bu çalışmada da öğrencilerden verilen örüntüleri genellemeleri istenmektedir ve böylece cebirsel düşünme algıları araştırılmaktadır.

Bu çalışmanın amacı, farklı sınıf seviyelerindeki ortaokul öğrencilerinin örüntülerde genelleme yaparken kullandıkları stratejileri araştırmaktır. Çalışmanın katılımcıları, örüntüler konusunu görmüş, bir okuldaki 6., 7. ve 8.sınıf öğrencileridir. Bu kapsamda, toplamda 154 olmak üzere, 48 tane 6.sınıf öğrencisi, 59 tane 7.sınıf öğrencisi ve 47 tane 8.sınıf öğrencisine 6 sorudan oluşan bir örüntü testi uygulanmıştır. Bu sorular, alan yazından adapte edilmiş olup; sayı, şekil ve tablo şeklinde sunulan sabit değişen örüntülerden oluşmaktadır. Daha sonra her sınıf seviyesinden seçilen iki öğrenciyle etkinlik temelli görüşmeler yapılmıştır. Etkinlik temelli görüşmeler, katılımcıların çözümlerini açıklamaları ve gerekçelendirmelerini gerektiren problem durumlarının sunulduğu görüşmelerdir. Bu görüşmeler, araştırmacının, katılımcının bilgisini ve stratejisini anlamasına olanak sağlar. Bu çalışmada, örüntü testinde kullanılan sorular alt sorularla birlikte etkinlik şeklinde düzenlenerek görüşmelerde kullanılmıştır.

Çalışmada, toplanan veriler Walkowiak'ın (2014) çalışmasından elde ettiği bulgulara göre oluşturduğu kavramsal çerçevede analiz edilmiştir. Bu kavramsal çerçeveye göre, öğrenciler genellikle sayısal ve şekilsel olmak üzere iki tip akıl yürütme stratejisi kullanarak genellemeye ulaşmaya çalışmaktadır. Öğrenciler genelleme aşamasında ise kendi keşfettikleri ya da bildikleri sembolik gösterim ve durumu açıklayan tanımlayıcı kelimeler kullanmaktadır.

Bulgular

Çalışmadan elde edilen bulgulara göre, 6.sınıf öğrencileri örüntünün genel ifadesini, terimler arasındaki farkı dikkate alarak sözel cümlelerle açıklamaktadırlar. Örneğin, 3,7,11,15,19,.. örüntüsünde "sayılar 4'er artmıştır" şeklinde genelleme yapmaktadırlar. Öğrencilerin çok azı cebirsel olarak n'yi kullanarak (örn. n+4 gibi) genelleme yapmaktadır. Fakat bu genellemeyi yaparken, öğrenciler genellikle terim sırasını dikkate almadan, örüntüdeki terimler arasındaki farka odaklanmaktadır. 7.sınıf öğrencilerinin genellemelerinde cebirsel ifadeler daha çok görülmektedir. 7.sınıf öğrencileri de genellikle terim sırasını dikkate almadan genelleme yaptıklarından çoğu doğru ifadeye ulaşamamaktadır. Genellemede cebirsel ifade kullanmayan öğrenciler 6.sınıf öğrencileri gibi sözel cümleler kullanmaktadır. 8.sınıf öğrencilerinin çoğu cebirsel ifadelerle genelleme yapmaya çalışmaktadır; ancak cebirsel gösterim olarak ifade etmeyen öğrenciler ise sözel cümlelerle genellemeyi ifade etmektedir. 8.sınıf öğrencileri cebirsel olarak ifade ettikleri genellemeleri, terim sırasına göre yaptıklarından (örneğin 3,7,11,15,19,.. örüntüsü için 4n-1 gibi) doğru ifadeler daha çok görülmektedir. Öğrencilerin seviyesi arttıkça daha çok cebirsel sembol kullanmalarına rağmen, öğrencilerin çoğunun değişken kavramı ile ilgili sıkıntı yaşadıkları görülmüştür. Öğrenciler, örüntülerde verilen terimlerden sonra gelen terimi yazarak genellikle bir sonraki terimi de n.terim olarak kabul etmekte ve bu terime karşılık gelen sayıyı bulmaktadırlar. Örüntüde ilişki soran soruları bu şekilde cevaplamaktadırlar.

Sonuç ve Tartışma

Bu çalışmada farklı gösterimlerle ifade edilen (sayı, şekil, ve tablo olarak gösterilen) örüntüler kullanılmıştır. Bu kapsamda, özellikle tablo şeklindeki örüntülerde öğrencilerin çoğunda n yerine bir sayı koyarak karşılık gelen sonucu bulma eğilimi olduğu görülmektedir. Bu örüntülerde n'den önceki boş bırakılan satıra bir sayı geleceğini düşünerek, n yerine konulacak sayıya karar vermektedirler. Sayı örüntülerinde de verilen terimlerden sonra gelmesi gereken terimi de bularak bir sonraki terimi n.terim olarak kabul etmekte ve bu terime karşılık gelen sayıyı, örüntünün kuralı olarak düşünmektedirler. Şekil örüntülerinde ise öğrenciler, genellikle şekle odaklanarak, örüntünün ilerleyen adımlarında şekli büyütme eğilimde olmaktadırlar ve cebirsel ifade olarak genellemeyi tercih etmemektedirler. Bununla birlikte, araştırmalar ise genellleme stratejilerini geliştirmek için sayı ve şekil örüntülerini kullanmayı önermektedirler. Bu çalışma için öğrencilerin şekil örüntülerindense sayı örüntülerinde cebirsel genelleme yaptıkları gözlemlenmiştir. Sayı örüntülerinde ise, öğrencilerin örüntüdeki terimin sırasını dikkate almadan, sadece terimlere odaklanarak genelleme yapma eğiliminde oldukları fark edilmiştir. Araştırmanın sonuçlarına göre, öğretmenlere öğrencileri cebirsel düşünmeye alıştırmak için değişken kavramı algılarını geliştirmeleri ve bunun için farklı temsillerle sunulan örüntüler kullanmaları önerilmektedir.