Arbitrage in the Term Structure of Interest Rates: a Bayesian Approach

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ABSTRACT

This work presents an analysis of the presence of arbitrage opportunities in the term structure of interest rates, through the estimation of the affine generalized Nelson-Siegel model with correction for no-arbitrage. We challenge the necessity of the condition of no-arbitrage using the Brazilian term structure of interest rates by observing the interbank deposits (DI) contracts traded in the Mercantile and Futures Exchange (BM & FBOVESPA) in Brazil between 2007 and 2009.
To verify the necessity of imposing no-arbitrage restrictions, we propose an analysis using Bayesian methods of estimation and testing of this model. We also discuss the estimation procedure in the presence of an irregular maturity structure. Our chosen methodology is especially relevant for emerging markets, where the liquidity of emerging markets varies substantially over time, especially in periods of crises. The results of our analysis indicate that the no-arbitrage corrections are not necessary and that this model is an appropriate specification for this term structure of interest rates.

Key words: *Arbitrage, Term Structure of Interest Rates, Latent Factors, Bayesian Inference, Nelson-Siegel Dynamic Models*
JEL Classifications: G12, C22, C11

1. INTRODUCTION

Among the models studied in the literature of term structure of interest rates there is a fundamental difference between purely statistical models and models that are based on no-arbitrage conditions. The main reference among statistical models is the decomposition proposed by Litterman and Scheinkman (1991), which uses the method of principal components to obtain an orthogonal decomposition of the movements in the term structure of interest rates. The techniques used in these models allow for adjustments and forecasts of the yield curve, but these models are not consistent with the imposition of no-arbitrage conditions given by the theory of asset pricing.

The imposition of no-arbitrage conditions not only ensure consistency with the pricing in the sense proposed by Harrison and Kreps (1979), Harrison and Pliska (1981), Delbaen and Schachermayer (1994), but also affect the fit and model predictions. Ang and Piazzesi (2003) and Almeida and Vicente (2008) indicate that the imposition of no-arbitrage conditions leads to an improvement in the forecasts for the term structure.

An alternative interpretation for this problem can be found in Duffee (2011), which indicates that in correctly specified affine models the imposition of no-arbitrage corrections is

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irrelevant, and so the imposition of the no-arbitrage condition should not represent an improvement in the predictive potential or model fit. A similar interpretation is discussed in Joslin et al. (2011). In this interpretation no-arbitrage conditions are only relevant in the presence of misspecification problems, and thus the presence of arbitrage opportunities can be interpreted as diagnosis for misspecification problems in the models for the term structure of interest rates.

Due the relevance of the topic, the purpose of this work is to dynamically model the term structure of interest rates in Brazil, based on the affine generalized dynamic Nelson-Siegel (AFDNS) proposed by Christensen et al. (2009), where the best of both traditional yield curve modeling techniques are combined: the affine no-arbitrage models and a version of the Nelson-Siegel model developed by Diebold and Li (2006). Through estimation of models with and without the no-arbitrage restrictions for this class of models, we propose a test based on Bayes Factors to check the validity of no-arbitrage conditions. This test is made possible by the use of a Bayesian estimation using Markov Chain Monte Carlo method, which enables mechanisms for obtaining finite sample distributions of parameters, latent factors and predictions of this model. We also discuss estimation procedures in the presence of an irregular maturity structure, which is necessary to avoid problems of contamination by arbitrage opportunities artificially created by interpolation procedures and temporal aggregation.

This paper is structured into seven sections, including this introduction. The second section is a brief review of related works in term structure modeling. The next section presents the affine dynamic Nelson-Siegel analyzed. In the fourth section is presented a descriptive analysis of the data used and some stylized aspects of the term structure of interest rates in Brazil. In the following section we present the estimation results and the analysis of the results of this estimation. In section six we show the proposed test for the presence of no-arbitrage. Finally, in section seven, are the final conclusions.

2. MODELING THE TERM STRUCTURE OF INTEREST RATES

Most models for the term structure of interest rates assume by construction that the observed rates are free of arbitrage opportunities. This hypothesis, however, is only consistent when these financial instruments are traded in large and liquid markets, for example, in the case of U.S. government debt securities. In this type of market, rational agents rapidly exploit arbitrage opportunities, and so the overall result is the consistency of observed prices with no-arbitrage assumption.

According to Diebold and Li (2006), in the last 25 years great advances in theoretical models for the term structure of interest rates have been made. Two major approaches for term structure models are rooted in the assumption of no-arbitrage and general equilibrium. Still, from the perspective of forecasting models, according to Diebold and Li (2006), despite the theoretical advances, little attention has been devoted to the practical problem of forecasting the yield curve. The no-arbitrage literature on the term structure models has little to infer on the dynamics of the yield curve or prediction, given that majority of the literature is focused primarily on fitting the term structure of forward rates at a given moment of time.

The general equilibrium literature on the term structure of models generally focuses on the dynamics derived from the short-term interest rates, but the empirical results of general equilibrium models, such as models of Vasicek (1977) and Cox et al. (1985) indicate that the in-sample fit and forecasting derived using these models are rather poor.

In general, basically, it should be easy for a correctly specified model of yield curve dynamics to reproduce the stylized facts concerning the term structure of interest rates. Thus, as stated by Diebold and Li (2006), term structure models should summarize some of the most important stylized facts of the existing yield curves, which are: (i) the shape of the average yield curve is increasing and concave; (ii) the yield curve has various formats over time, due to variation of the estimated parameters for each factor; (iii) the dynamics of yields are persistent and the spreads less persistent; (iv) greater volatility of the yield curve in the short term;(v) higher persistence for long term yields.

Nelson and Siegel (1987) proposed a basic static model for the term structure of interest rates now widespread in the financial market. Due to its ease of implementation and predictive power the model is also used by central banks, enabling an efficient way to fit the cross-section part of the yield curve.

The representation of this model is given by equation (2.1) below:

$$y(\tau) = \beta_0 + \beta_1\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_2\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \tag{2.1}$$

where $y(\tau)$ is the vector with interest rates, observed at time t for the maturities τ and β_0, β_1, β_2 and λ are model parameters.

To understand the evolution of the yield curve over time, i.e. the term structure, Diebold and Li (2006) made the previous model, given by equation (2.1), dynamic, by replacing the parameters with time-varying factors. Similar to the principal components developed by Litterman and Scheinkman (1991), the new parameters or factors assumed represent the level, slope, and curvature of the yield curve.

The new proposal generates a dynamic yield curve modeled through the relationship of a three-dimensional parameter, which is represented by equation (2.2) below:

$$y(\tau) = \beta_{0t} + \beta_{1t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \tag{2.2}$$

$$\begin{bmatrix}\beta_{1t}\\ \beta_{2t}\\ \beta_{3t}\end{bmatrix} = \begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix} + \Phi\begin{bmatrix}\beta_{1t-1}\\ \beta_{2t-1}\\ \beta_{3t-1}\end{bmatrix} + \varepsilon_{\beta t}$$

where $y_t(\tau)$ is the vector with the nominal rates of a security that pays no coupons in function of maturities τ and β_{0t}, β_{1t}, β_{2t} and λ are the factors and may vary over time. Among them, β_{0t}, β_{1t} and β_{2t} represent, respectively, the components of level, slope, and curvature of interest rates.

The level factor of the yield curve denotes the average of the rates charged for the different maturities. The second factor, the slope, shows if the shapes of the curve indicates an increasing or decreasing function of maturity, while the third factor, curvature, reflects the speed with which the curve is increasing or decreasing. Additionally, the decay parameter λ, which controls the shape (decay) of the curve, is usually kept constant but can vary over time in order to capture the instability of curves.

Other studies using the approach proposed by equation (2.2) accede to the widely recognized limitation of the model, whose fit is satisfactory only for shorter maturities. For longer-term rates, the model often provides a yield curve that becomes stable in the long term, a fact that is not consistent with the observed dynamic of long-term interest rates.

An extension to the model characterized by equation (2.2) seeks to solve the above problem. This model produced by Svensson (1994) is a generalization of the model made by Nelson and Siegel (1987) and added one more factor of curvature to better adjust interest rates for long-term maturities. The representation of the extended version proposed by Svensson (1994), according to Christensen et al. (2009), now including the temporal dynamics is given by equation (2.3) below:

$$y_t(\tau) = \beta_{0t} + \beta_{1t}\left(\frac{1-e^{-\lambda_{1t}\tau}}{\lambda_{1t}\tau}\right) + \beta_{2t}\left(\frac{1-e^{-\lambda_{1t}\tau}}{\lambda_{1t}\tau} - e^{-\lambda_{1t}\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_{2t}\tau}}{\lambda_{2t}\tau} - e^{-\lambda_{2t}\tau}\right) \tag{2.3}$$

where $y_t(\tau)$ is the vector with the nominal rates, now with the factors β_{0t}, β_{1t}, β_{2t}, β_{3t}, λ_{1t} and λ_{2t}.

Other versions including additional factors are also used in literature, in particular the model with five factors used in Christensen et al. (2009). Several other generalizations and applications of these models have been proposed in the literature after the formulation of the dynamic Nelson-Siegel family presented in Diebold and Li (2006). Examples include the use of models with macroeconomic variables (macro-finance models), models with stochastic volatility structures, the use of these models to extract inflation expectations using indexed bonds, and many other applications. A detailed discussion of these topics can be found in the monograph by Diebold and Rudebusch (2013).

3. THE AFFINE GENERALIZED DYNAMIC NELSON-SIEGEL MODEL

The model of Nelson and Siegel (1987) is not consistent with no-arbitrage assumption, as pointed out by Filipovic (1999). There is only one restriction on the model of Svensson that is consistent with no-arbitrage, but this structure is too restrictive for practical use. However, as shown in Christensen et al. (2011) and Christensen et al. (2009), some modifications allow a similar class of models with the no-arbitrage property for the Nelson-Siegel and Svensson family of models, respectively.

To obtain this family of arbitrage-free models, Christensen et al. (2009) use the structure of Affine Term Structure Models. This structure is useful, because it presents interesting analytical properties, such as the existence of analytical solutions for the pricing of assets and is characterized by a common framework that allows representing various other models studied in the literature, as shown by Dai and Singleton (2000). A detailed and didactic derivation of arbitrage-free Nelson-Siegel family models can be seen in the monograph of Diebold and Rudebusch (2013), in chapters 3 and 5. To enhance the interpretive power of the model they propose a simple function with just two factors, but discuss in detail the no-arbitrage conditions and restrictions involved in this family of models with the imposition of conditions of consistency.

Following Duffie and Kan (1996), we set the price of a zero coupon bond in period t with maturity T in the equivalent martingale measure Q to characterize the structure of affine models of the term structure (ATSM), given by:

$$P(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right]$$

where $r(t)$ is the instantaneous interest rate (short rate). In this class of models $r(t)$ is an *affine* function of a vector of state variables (latent factors) $X_i(t)$:

$$r(t) = \delta_0 + \sum_{i=1}^{N} \delta_y' X_i(t)$$

where δ_s are parameters and each $X_i(t)$ is an affine diffusion given by the following stochastic differential equation:

$$dY_i(t) = \kappa(\theta - Y_i(t))dt + \sum \sqrt{S(t)}dW(t)$$

with parameters κ and θ. $dW(t)$ represents a standard Brownian motion, and $S(t)$ is a diagonal matrix with i-th element given by:

$$S(t)_{ii} = \alpha_i + \beta_i' X_i(t)$$

Duffie and Kan (1996) show this class of models, where the bond price is an again an affine function given by:

$$P(t,\tau) = e^{A(\tau) - B(\tau)' X(t)} \tag{3.4}$$

where $A(\tau)$ and $B(\tau)$ are derivations from the following system of Riccati ordinary differential equations:

$$\frac{dA(\tau)}{dt} = -\theta' \kappa' B(\tau) + \frac{1}{2}\sum_{i=1}^{N}\left[\sum B(\tau)\right]_i^2 \alpha_i - \delta_0$$
$$\frac{dB(\tau)}{dt} = -\kappa' B(\tau) + \frac{1}{2}\sum_{i=1}^{N}\left[\sum B(\tau)\right]_i^2 \beta_i - \delta_y$$

To obtain an arbitrage-free Svensson family, Christensen et al. (2009) modify this affine structure by assuming that the short rate is given as the sum of the first three latent factors:

$$r(t) = X_t^1 + X_t^2 + X_t^3$$

and X_t^1, X_t^2, X_t^3, X_t^4, X_t^5 latent factors evolve through the following system of stochastic differential equations:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \\ dX_t^4 \\ dX_t^5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & -\lambda_1 & 0 \\ 0 & 0 & \lambda_1 & 0 & -\lambda_2 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \\ \theta_4^Q \\ \theta_5^Q \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \\ X_t^5 \end{pmatrix} \right]$$

In this model according to equation (3.4), prices of zero-coupon bonds are given by the following expression:

$$P(t,T) = E_t^Q[e^{\int_t^T r_u du}] = exp(B^1(t,T)X + B^2(t,T)X_t^2 + B^3(t,T)X_t^3 + B^4(t,T)X_t^4 + B^5(t,T)X_t^5 + C(t,T))$$

where the terms $B^i(t,T)$ and $C(t,T)$ are defined as the solutions to the following systems of ordinary differential equations:

$$\begin{pmatrix} \frac{dB^1(t,T)}{dt} \\ \frac{dB^2(t,T)}{dt} \\ \frac{dB^3(t,T)}{dt} \\ \frac{dB^4(t,T)}{dt} \\ \frac{dB^5(t,T)}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{bmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & -\lambda_1 & 0 & \lambda_1 & 0 \\ 0 & 0 & -\lambda_2 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} B^1(t,T) \\ B^2(t,T) \\ B^3(t,T) \\ B^4(t,T) \\ B^5(t,T) \end{pmatrix} \end{bmatrix}$$

$$\frac{dC(t,T)}{dt} = -B(t,T)' \kappa^Q \theta^Q - \frac{1}{2} \sum_{j=1}^{5} \left(\sum' B(t,T) B(t,T)' \sum \right)_{j,j}$$

In this formulation now the zero-coupon yields are given by: (3.5)

$$y(t,T) = X_t^1 + \frac{1-e^{-\lambda_1(t-T)}}{\lambda_1(t-T)} X_t^2 + \frac{1-e^{-\lambda_2(t-T)}}{\lambda_2(t-T)} X_t^3 + \left[\frac{1-e^{-\lambda_1(t-T)}}{\lambda_1(t-T)} - e^{-\lambda_1(t-T)} \right] X_t^4 + \left[\frac{1-e^{-\lambda_2(t-T)}}{\lambda_2(t-T)} - e^{-\lambda_2(t-T)} \right] X_t^5 - \frac{C(t,T)}{T-t}$$

which can be interpreted as the augmented Svensson curve, replacing notation β_{it} to X_t^i for latent factors, with the addition of a no-arbitrage correction factor $-C(t,T)/(T-t)$ given by the following expression:

$$-\frac{C(t,T)}{T-t} = -\frac{1}{2} \frac{1}{T-t} \sum_{j=1}^{5} \left(\sum B(s,T) B(s,T)' \sum ' \right) \tag{3.6}$$

where Σ denotes the covariance matrix of the latent factors. This correction factor (3.6) is a function of the variances of the latent factors and also of decay parameters of the model, all assumed to be constant[1].

An important point is that a no-arbitrage formulation needs a structure with five latent factors, implying additional slope and curvature factors in the Nelson-Siegel model. This specification also allows correcting the empirical failure in adjusting the long-term maturities using Nelson-Siegel class of models. According to Christensen et al. (2009), most models get a good fit only for short maturities, failing to adjust the longer maturities. Therefore, the initiative to include more latent factors, besides allowing the imposition of no-arbitrage correction, represents a gain in model fit.

To verify the validity of no-arbitrage conditions, we estimate two versions of the model proposed by Christensen et al. (2009). The first is the model *with* the no-arbitrage correction, denoted by:

$$y(t,T) = \beta_{1t} + \frac{1-e^{-\lambda_1(t-T)}}{\lambda_1(t-T)} \beta_{2t} + \frac{1-e^{-\lambda_2(t-T)}}{\lambda_2(t-T)} \beta_{3t} + \left[\frac{1-e^{-\lambda_1(t-T)}}{\lambda_1(t-T)} - e^{-\lambda_1(t-T)} \right] \beta_{4t} + \left[\frac{1-e^{-\lambda_2(t-T)}}{\lambda_2(t-T)} - e^{-\lambda_2(t-T)} \right] \beta_{5t} - \frac{C(t,T)}{T-t}$$

$$\beta_{it} = \mu_i + \Phi \beta_{it-1}$$

In this expression the last equation describes the vector of autoregressive latent factors. The second version is the model *without* the no-arbitrage correction, and so not included in the

[1] The analytical expression for this correction term is found in the appendix in Christensen et al. (2009), and is omitted due to space constraints.

model is the $C(t,T)/(T-t)$ factor. We analyze the validity of the no-arbitrage correction estimation and compare the two models using a Bayesian methodology.

An alternative would be to estimate the three-factor model proposed in Christensen et al. (2011), which corresponds to the arbitrage-free version of the original Nelson-Siegel model. The three-factor model, however, is unable to capture the full complexity presented in the curve of DI contracts. Hence, we adopted the more general model with five factors, which include the three factors as a particular case.

4. BAYESIAN ESTIMATION BY MCMC

Due to the complexity of the model, it is important to discuss the estimation methodology adopted. Following the approach proposed by Laurini and Hotta (2010), we chose to use Bayesian estimation methods in this paper. The Bayesian approach is advantageous since it eliminates the need to assume the usual restrictions usually adopted in classical estimations of models of term structure of interest rates. The inefficient two-step ordinary least squares OLS estimation method, used in Diebold and Li (2006), suffers short-comings in constructing confidence intervals for both parameters and latent factors, and the Kalman Filter methods are affected by multiple local maximum in likelihood function. Instead of these two alternatives, we opted to perform our estimation by using a Bayesian methodology based on the Markov Chain Monte Carlo (MCMC) method, which allowed us to obtain accurate credibility intervals for both latent factors and parameters, as well as for the forecasts generated by the model.

Applying Bayesian estimation by using the MCMC method allows replacing the numerical maximization procedure, which is notoriously unstable in this class of models, for expectations of the posterior distributions. Note that there is a problem of identification and local maxima in this family of models, when employing models with five factors. This identification problem arises due to the use of two similar factors of slope and curvature, which cannot become identified for some vectors of parameters, and thus can generate multiple (infinite) maxima in the likelihood function. In this respect, the use of Bayesian framework can avoid this problem by using a local identification imposed by the information in the prior distributions, which circumvent global non identification in the five factor Nelson-Siegel family. The Bayesian inference allows finding the posterior distribution of the parameters conditional on the observed sample. This distribution is the result of updating the prior distribution assumed for the parameters using the likelihood function. This posterior distribution, denoted by $p(\Theta\,|\,y)$, is the result of updating the prior distribution assumed for the parameters with the existing information in the sample, given by the likelihood function.

To get the posterior distribution of the parameters conditioned sample, we use Bayes' theorem:

$$p(\Theta\,|\,y) = p(\Theta, y)/p(y) = p(y\,|\,\Theta)\,p(\Theta)/p(y)$$

where $p(\Theta|y)$ is the likelihood of the model, $p(\Theta)$ denotes the prior distribution assumed for the parameter, and $p(y)$ is the marginal distribution of the observed sample, known at a constant of integration:

$$p(\Theta\,|\,y) = p(\Theta, y)/p(\Theta) = p(y\,|\,\Theta)\,p(\Theta)/c$$

and, thus the posterior distribution is proportional to the product of the likelihood by the prior distribution:

$$p(\Theta\,|\,y) \propto p(y\,|\,\Theta)\,p(\Theta)$$

After obtaining the posterior distribution, a summary of the results can be obtained, for example, by calculating the expected value and variance of the posterior distribution, as demonstrated below:

$$E(\theta_k \mid y) = \int \theta_k p(\Theta \mid y) d\theta$$
$$Var(\theta_k \mid y) = \int \theta_k^2 p(\Theta \mid y) d\theta - [E(\theta_k \mid y)]^2$$

We evaluate the marginal density of parameter θ_j with the following expression:

$$p(\theta_j \mid y) = \int p(\Theta \mid y) d\theta_1 d\theta_2 ... d\theta_d$$

Except in some specific cases, generally using conjugate distributions (the prior distribution is from the same family of posterior distributions), there are no analytical forms for these posteriors. In these situations it is possible to apply techniques of numerical integration using Monte Carlo methods.

A fundamental Monte Carlo methodology for Bayesian estimation is the method of Markov Chain Monte Carlo (MCMC). The fundamental idea of MCMC methods (e.g. Robert and Casella, 2005) is to simulate a Markov chain whose stationary distribution converges to the posterior distribution of interest. A key result is that the joint estimation can be factored by sampling the conditional distributions of the parameters. These conditionals are smaller in size and can be simulated in simpler forms. The procedure can be summarized by the sequence of iterations:

$$\begin{gathered} p(\Theta_1 \mid \Theta_2, \Theta_3, ..., \Theta_n, y) \\ p(\Theta_2 \mid \Theta_1, \Theta_3, ..., \Theta_n, y) \\ \vdots \\ p(\Theta_n \mid \Theta_2, \Theta_3 ., ..., \Theta_{n-1}, y) \end{gathered}$$

When all the conditional distributions are known, one of the simplest MCMC algorithms is the Gibbs Sampling algorithm, where the estimation is performed by sampling from known conditional distributions. If it is not possible to sample from the analytical conditional distribution, sampling can be performed using the Metropolis-Hastings algorithm, which consists of a rejection method for sampling conditional distributions with simulated random variables.

Under certain regularity conditions this set of sampled conditional distributions converges for posterior distribution, a result known as the Hammersley-Clifford theorem. One advantage of this methodology is that it involves no numerical maximization methodology, and thus avoids the numerical problems involved in maximizing nonlinear functions and problems of multiple local maxima existing in the estimation of term structure models of interest rates, as pointed out by Duffee (2002).

To completely characterize our model a discussion of the prior distributions used is necessary. For the autoregressive parameters determining the dynamic for the latent factors β_{it} we assume Normal-Inverse Gamma distributions with an inverse-Wishart matrix for the variance of these latent factors, and for a λ_i we use a log-normal prior[2]. The model specification is

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[2] Details on the values used in the priors can be obtained from the authors.

completed assuming a multivariate normal likelihood for the observed term structure. In Table A.5 in the appendix we show how to implement the five-factor model using the WinBUGS/JAGS software. This structure avoids the problems of identification observed in the five factor Nelson-Siegel models. We can observe truly distinct factors of level and curvature in the estimated models, as can be seen in Table 6.2 by the distinct λ_1 and λ_2 parameters estimated in the Bayesian procedure.

To obtain the predictive distribution of the model one-step ahead we use the relation:

$$p(y_{t+1} \mid y_t) = \int p(y_{t+1} \mid \Theta)\, p(\Theta \mid y_t)\, p(\Theta) d\Theta$$

which is the predictive likelihood weighted by the posterior distribution of the parameters, where y_t are the observations up to period t.

In the MCMC procedure, we used 15'000 simulations, discarding the first 5'000 as burn-in period. The diagnostic tools (Gelman-Rubin) indicate convergence of the chains in the two estimated models.

Due to the widespread success of Nelson-Siegel family models in practical and market applications, these models are already implemented in a number of distinct softwares. We can find methods of building spot and forward curves using Nelson-Siegel and Svensson models in the free/open-source library for quantitative finance Quantlib (www.quantlib.org). Various libraries for estimation of these models in R software, as YieldCurve, DEnss and termstrc packages, are available at http://cran.r-project.org/web/packages/. Also there are already several implementations of these models in Matlab, Eviews, and other econometric softwares.

Since the Nelson-Siegel model can be formulated as a state space model, it is possible to perform Bayesian estimation of this model using the usual tools of Bayesian estimation of dynamic models (e.g. Robert and Casella, 2005; Gamerman and Lopes, 2005). Detailed discussion on estimation methods of Nelson-Siegel models by MCMC can be found in Laurini and Hotta (2010) and Hautsch and Yang (2012). The MCMC Bayesian estimation can be easily implemented using the MCMC softwares WinBUGS or JAGS. We show in Appendix A.2 a prototype of a Nelson-Siegel in WinBUGS / JAGS model language.

An alternative form of Bayesian estimation of Nelson-Siegel models using Integrated Nested Laplace Approximations can be found at Laurini and Hotta (2014). Another possible extension would be to implement the Bayesian estimation of the dynamic Nelson-Siegel model using the methods of Hamiltonian Monte Carlo software available in STAN (www.mc-stan.org).

5. Database

We collected data from the daily bulletin with the statistical summary of trading on the Brazilian Mercantile and Futures Exchange (BM&F BOVESPA), located in the city of São Paulo, SP. Among the products supplied and traded by the institution, an important financial instrument in Brazil financial market is the DI contract. The DI One Day Futures Contract, whose trading market code is DI1, is the interest rate future for interbank deposits, quoted in an annual effective interest rate, using a basis of 252 working days. DI is the most important interest rate reference in Brazil and a widely used reference for the term structure models and pricing of derivative contracts.

The data can be obtained via the FTP system of BM&F BOVESPA, located at ftp://ftp.bmf.com.br within the ContratosPregaoFinal folder. The system provided a file for each session conducted. The reported price from contract DI1 is the unitary price (PU), and yields for specific maturities are calculated using the relation yield = ((10000000/PU)^(252/Mat))-1. Table 5.1 presents a sample of some closing quotes of DI1 contract for two consecutive trading sessions.

Date	ID	PU	Mat
2009-03-06	DI1	9921174	18
2009-03-06	DI1	9836551	38
...			
2009-03-06	DI1	3241502	2466
2009-03-06	DI1	2295908	3221
...			
2009-04-06	DI1	9929303	17
2009-04-06	DI1	9854147	37
..			
2009-04-06	DI1	3273032	2445
2009-04-06	DI1	2313061	3200

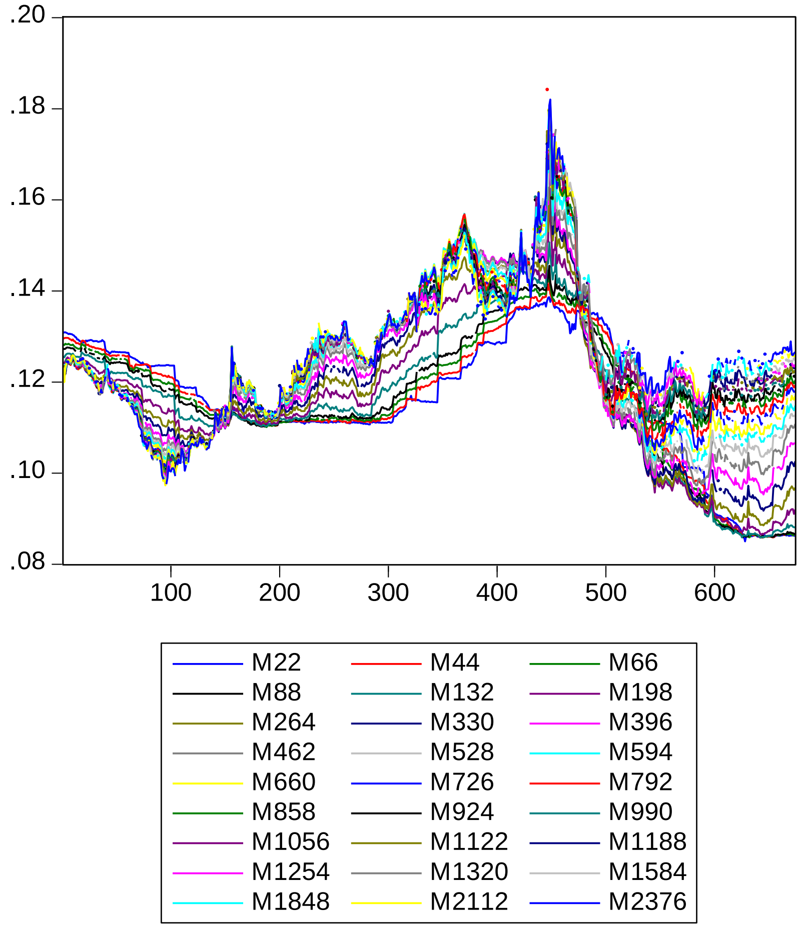
Table 5.1 Data from DI1 Contract.

The period analyzed from 2 January 2007 to 30 September 2009 contains 674 daily observations. The increase in the availability of observed rates for existing maturities beginning in 2007 greatly influenced the choice of the time range observed in this paper. This stylized fact reflects the recent developments in the country and its institutions, indicating that, over the years, investors have greater confidence in domestic financial products, thus, impacting the horizon and trading fees of the DI1 contract. About ten years ago, for example, it was possible to perform this kind of transaction with a maximum of two years ahead for traded maturities.

The sample used can affect the validity of the no-arbitrage conditions, as discussed in Duffee (2011). The limitation in the time span of our sample poses some possible problems. The estimation method we have chosen in this study allows estimating the same model using irregular maturities, and thus helps capture, in part, the possible misspecification problem generated by this feature of our data. In the estimation procedure we just work with vertices of DI observed in the market, hence each day of the yield curve can have different maturities since a certain specific maturity may not have been transacted that same day. Thus we are not performing any interpolation procedure to generate a curve with fixed maturities Furthermore, as pointed out by Laurini and Moura (2010), yield curves generated using interpolation procedures generally may result in arbitrage conditions. Hence, in our paper we refrain from performing this procedure to avoid altering the test results for the presence of no-arbitrage conditions. The data used contain maturities ranging from 22 to 2376 days; their descriptive statistics are shown in Table A.4 in the appendix, and a view of this data can be seen in Figure 5.1.

The use of a data structure with irregular maturities creates a number of difficulties in the estimation and analysis of models of the term structure of interest rates. For example the estimation by maximum likelihood using the Kalman filter, often used in the estimation of models of the Nelson-Siegel family (e.g. Diebold and Rudebusch, 2013) requires an equation of observation for each maturity observed which is infeasible in the presence of irregular maturities. The irregular structure also complicates the construction of forecast measures, since the forecasted maturities therein cannot be directly observed. Furthermore, the use of interpolated maturities can lead to problems of measurement error, contaminating the usual measures of forecast accuracy, such as the mean squared or mean absolute errors.

Figure 5.1 Evolution of Interbank Deposits (DI) Term Structure.



Note that the irregular structure can be treated directly in the Bayesian estimation method used in this article, calculating the contribution to the likelihood function for the observed data only. In Appendix Table A.5 we show how to perform this procedure using an additional variable (counters) that indicates which maturities were actually observed on each day, and so we use the contribution to the likelihood function only for these observations, looping over observations and days.

This procedure avoids the use of interpolation procedures, which might generate arbitrage conditions (Laurini and Moura, 2010) and could influence the results of our analysis. An additional advantage of this methodology is that it is possible to use the model itself to interpolate the missing maturities, since we can recover the complete interest rate curve in each moment with the estimation of the model parameters, simply assuming the maturity τ of interest in equation (2.3).

6. ESTIMATION RESULTS

The results of Bayesian estimation for the parameters and the five latent factors are shown in the following figures and tables. The specification is based on a diagonal structure for the autoregressive process of latent factors, assuming that each latent factor is only a function of its own past. We also estimate a full vector autoregression, but the results are basically equivalent, and we decide to use the more parsimonious specification. Table 6.2 shows the posterior mean and 95% credibility interval for the estimated parameters of the autoregressive structure of latent factors and also the decay parameters λ_1 and λ_2, with μ denoting the intercepts and ϕ the autoregressive parameter for each latent factor.

	2.5%	Mean	97.5%
μ_1	0.00020	0.00129	0.00241
ϕ_1	0.98740	0.99266	0.99770
μ_2	-0.00083	-0.00033	0.00017
ϕ_2	0.98890	0.99439	0.99990
μ_3	-0.00035	0.00029	0.00094
ϕ_3	0.98590	0.99215	0.99790
μ_4	-0.00067	-0.00009	0.00050
ϕ_4	0.98700	0.99273	0.99800
μ_5	-0.00078	-0.00024	0.00029
ϕ_5	0.98540	0.99182	0.99800
λ_1	0.09643	0.09646	0.09647
λ_2	0.84850	0.84852	0.84860

Table 6.2 Estimated Parameters – AFDNS Model.

Figure 6.2, demonstrates the evolution of the level factor β_{1t}, representing the average values of the term structure of interest rates in Brazil. Note that the figure is a negative rotation of the literal level factor. This rotation occurs, because in the model with five factors the model is not uniquely identified. Thus it is possible to rotate the factors to achieve the same model fit.

Figure 6.2. Evolution of Level Factor.

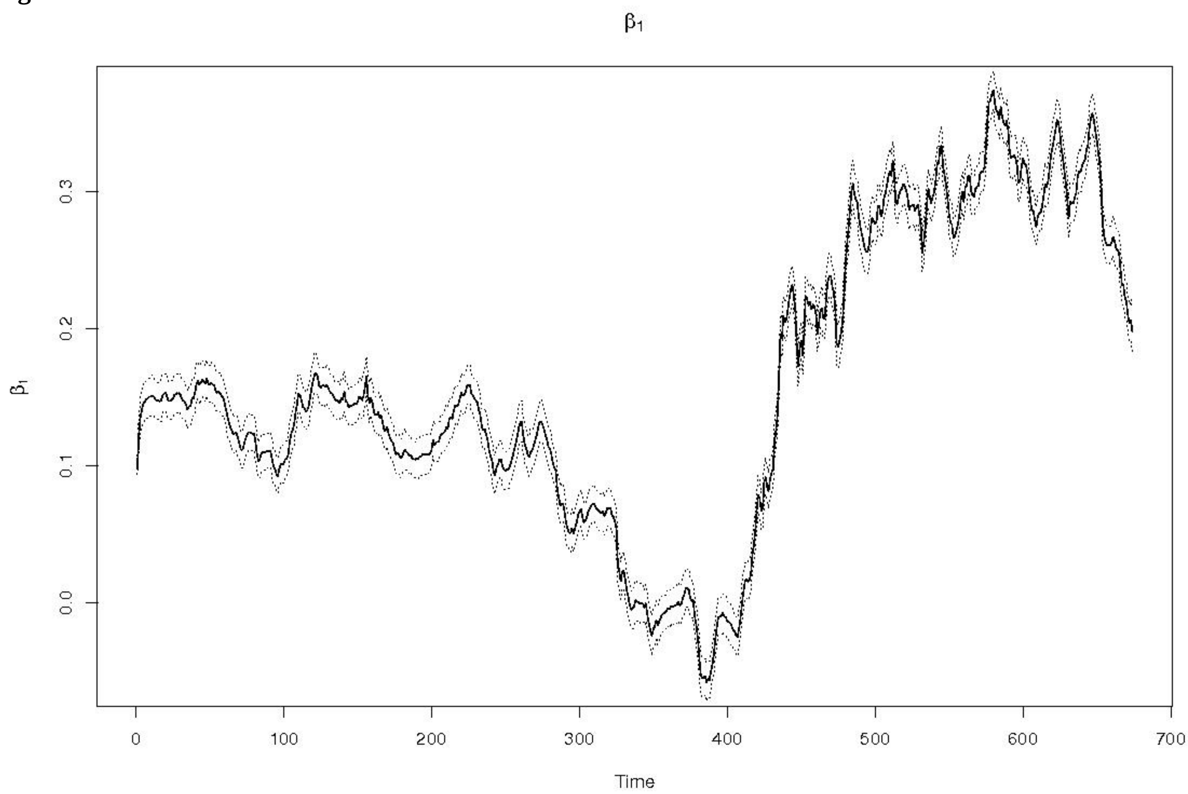
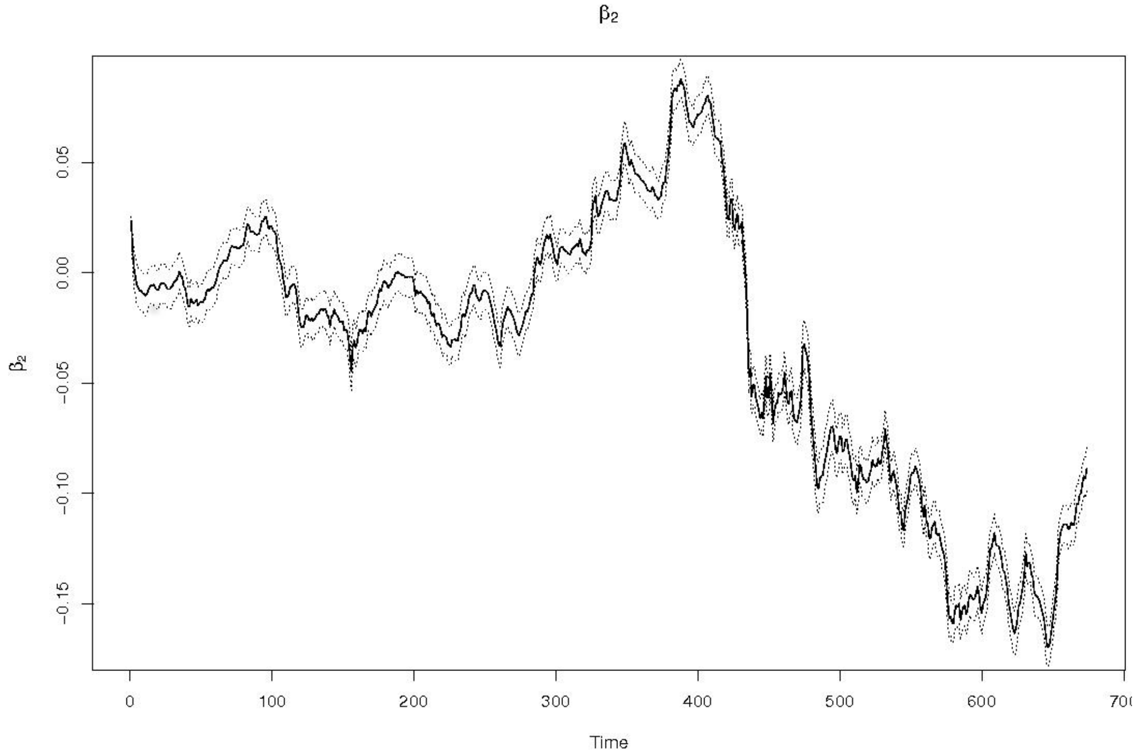
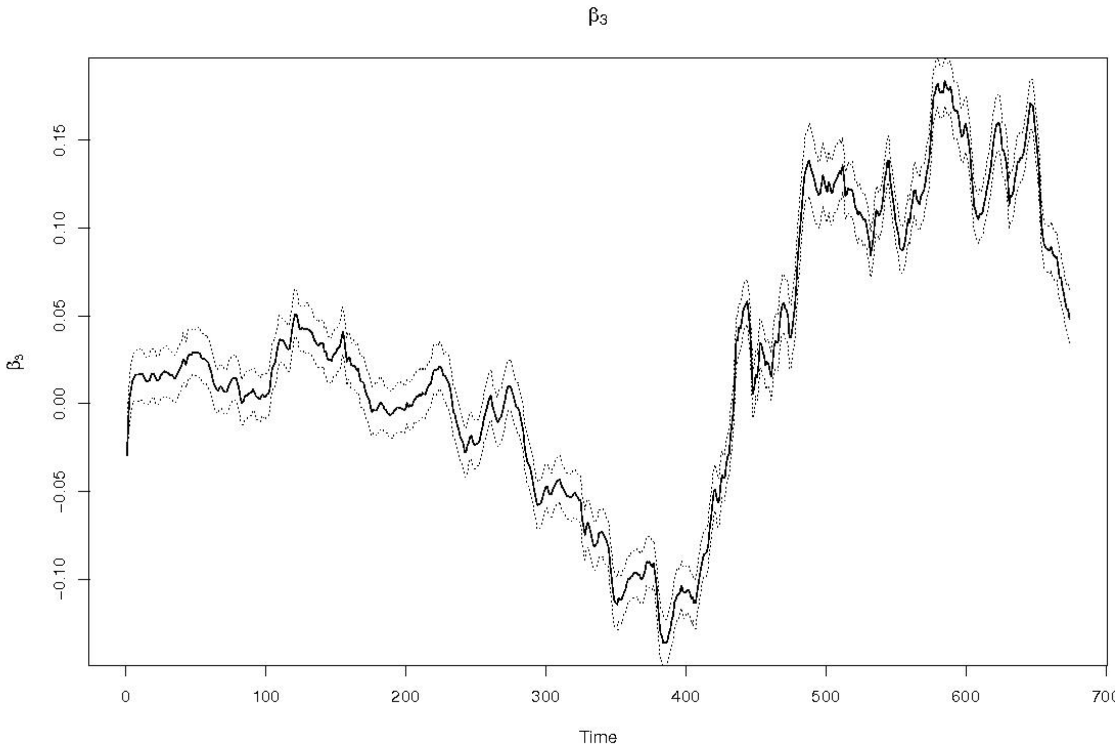


Figure 6.3 contains two graphs showing the evolution of the two slope factors estimated, according to the model represented in equation (3.5). We can note that the two factors capture the slope of the term structure differently from each other. This complimentarily of the factors contributes to the quality of the model fit.

Figure 6.3 Evolution of Slope Factors.



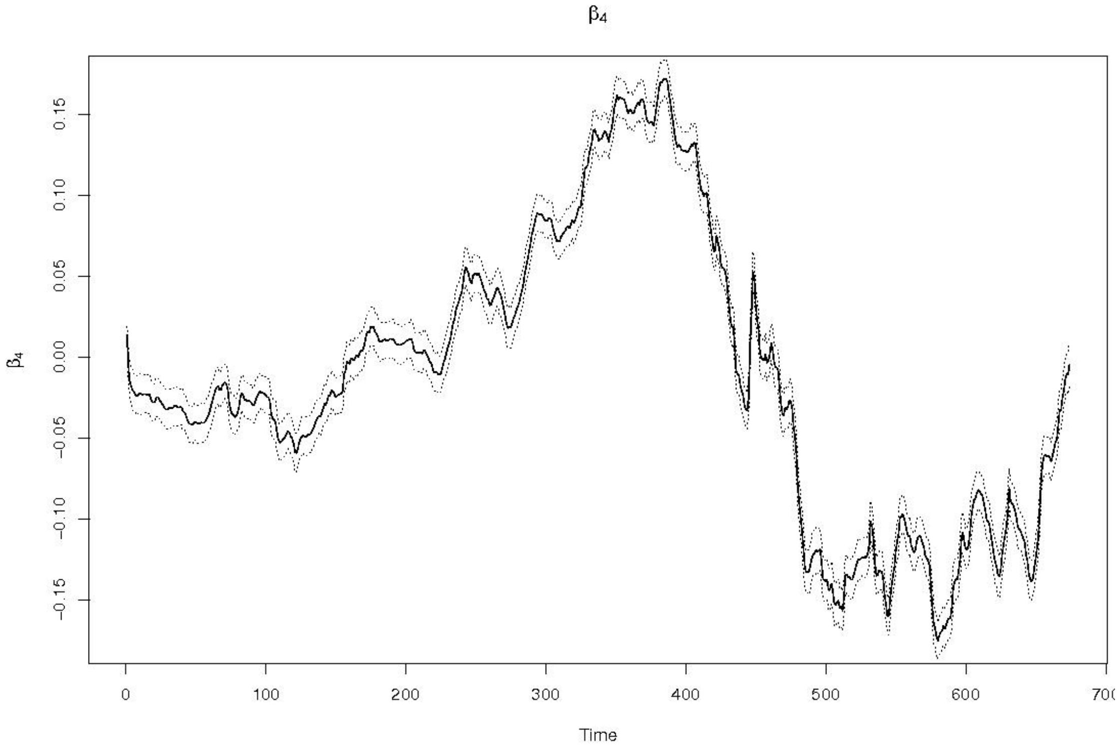
(a) 1º Slope Factor



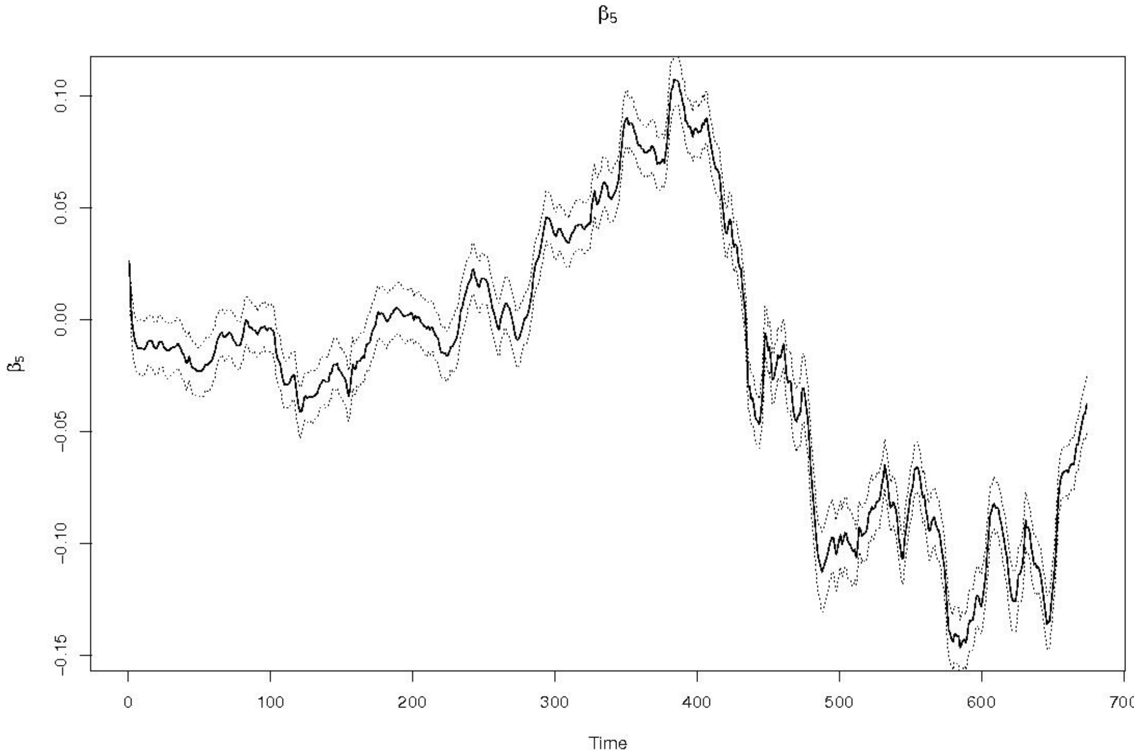
(b) 2º Slope Factor.

The factors of curvature illustrated in Figure 6.4 show a very similar dynamic evolution in the sample under analysis. Although the two factors to have a very similar behavior, the double curvature structure allows for a marginally better fit to the observed curve. Due to the requirement of matching each slope factor with a correspondent curvature factor to impose the no-arbitrage restriction, in the Svensson formulation it is not possible to eliminate the second curvature factor for a more parsimonious specification.

Figure 6.4 Evolution of Curvature Factors.



(a) 1º Curvature Factor



(b) 2º Curvature Factor

After observing the evolution of the factors that make up the structure of the dynamic model, the next step is to analyze the fitting provided by the proposed models. We present in Figures

6.5-6.6 the fitted curves for several days with different patterns of yield, showing the good fit of the model in all these cases. In these figures all maturities are measured in years.

Figure 6.5 Model Fit – DI Curve **-** 27/11/2008.

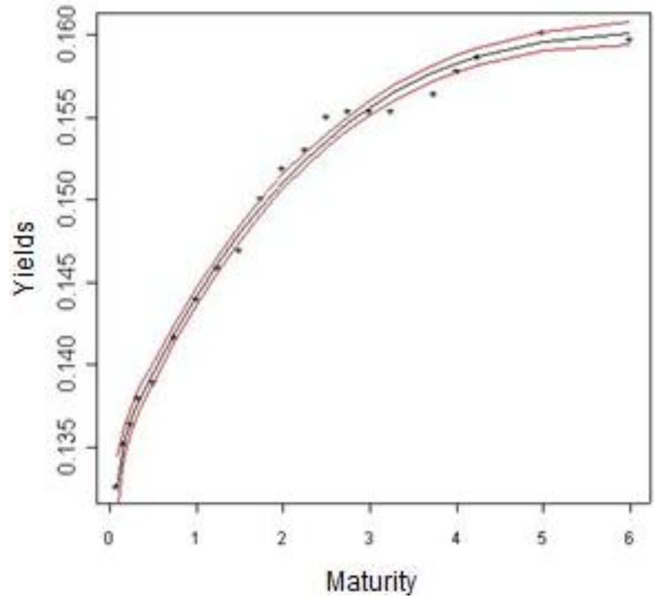
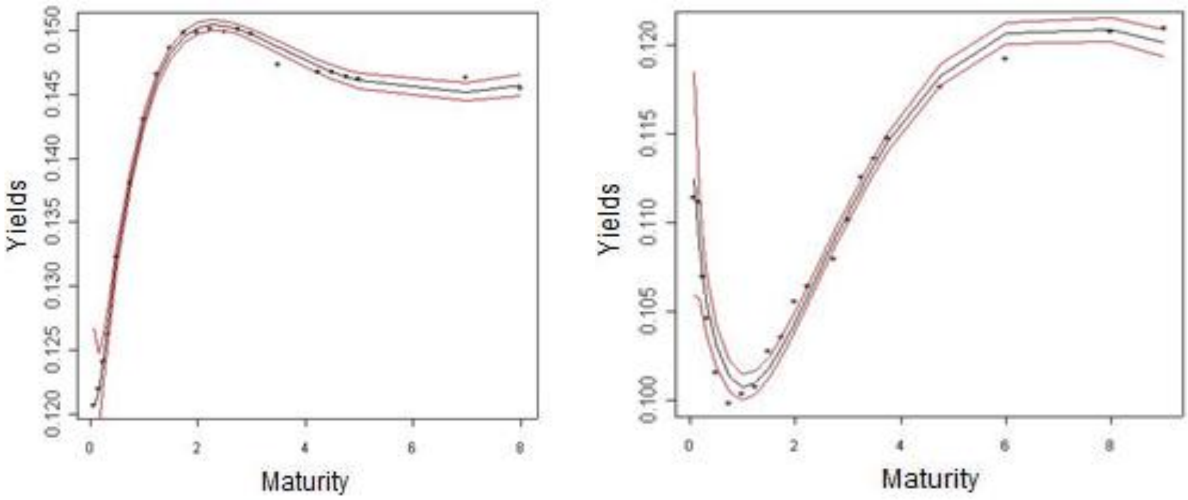


Figure 6.6 Model Fit – DI Curves – Days 20/06/2008 and 16/03/2009.



In a preliminary analysis of the importance of the no-arbitrage corrections, Figures 6.7 and 6.8 present graphs containing the fit of the model estimated with and without the no-arbitrage restriction. It is important to note that the fitting for the observed yields is almost identical. Both lines appear to be superimposed in Figures 6.7 and 6.8 respectively, especially for the medium and longer maturities. However, although the difference is marginal, for short-term rates, the adjustments provided by the model without the correction of no-arbitrage performed best, capturing both the shape and the slope of the curve in these parts.

Although the use of interest rates data without interpolation hinders the construction of adjustment measures, we performed a similar analysis by interpolating the results obtained by estimating the models with (AFDNS) and without (DNS) correction for no-arbitrage for the most liquid maturities of the yield curve, compared to yields interpolated for the same maturities. Table 6.3 shows the one-step-ahead predictions for smoothed observations, assuming fixed parameters for maturities of 1-48 months of interpolated DI contracts during this period. We show the mean error (ME), root mean squared error (RMSE), and mean absolute error (MAE) for these predictions. The obtained results show that the imposition of no-arbitrage improves point predictions for some maturities, but is not possible to infer a clear result of improved forecasts using these corrections.

Figure 6.7 Adjustment of models with (solid line) and without (dotted line) no-arbitrage restrictions. Day - 20/06/2008.

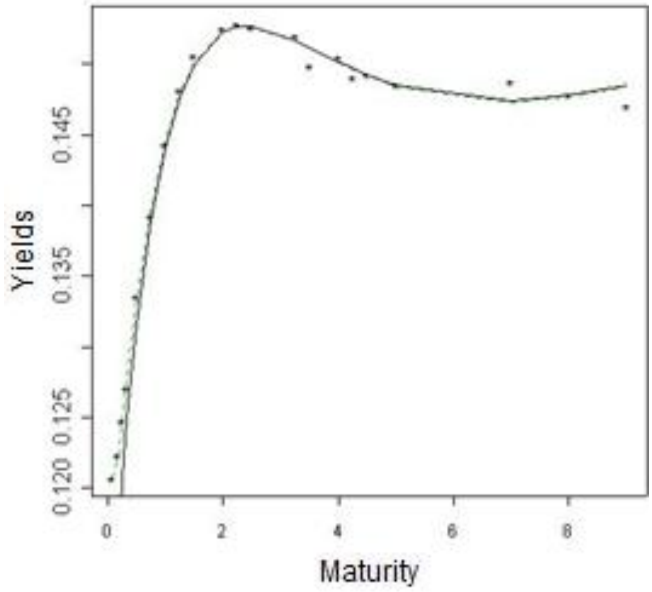
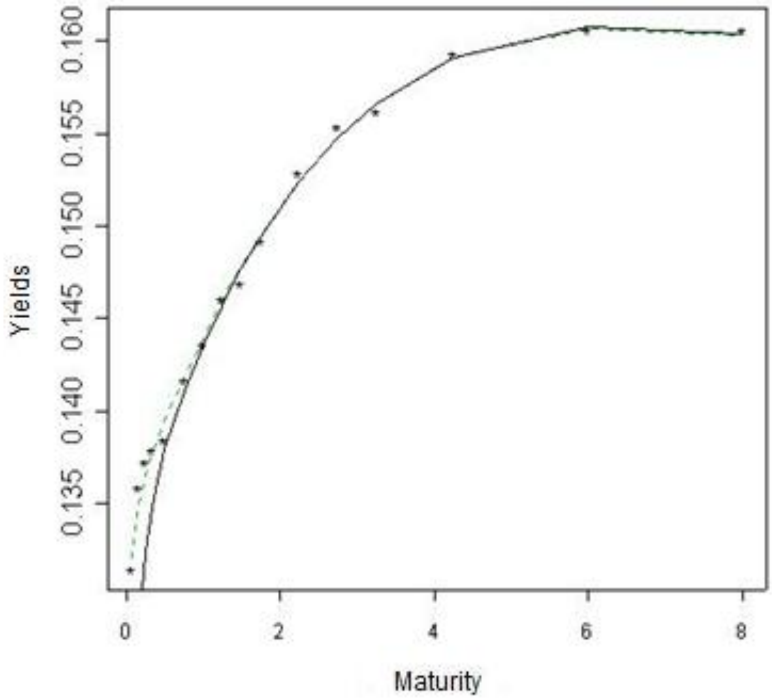


Figure 6.8 Adjustment of models with (solid line) and without (dotted line) no-arbitrage restrictions. Day. 26/11/2008.



	ME	RMSE	MAE
Mat 1 DNS	-2.8122	5.7841	3.7621
Mat 1 AFDNS	16.8962	17.6925	17.2028
Mat 6 DNS	-2.9519	7.4186	5.6072
Mat 6 AFDNS	0.2079	6.8099	5.0805
Mat 12 DNS	-4.9215	8.8237	6.7927
Mat 12 AFDNS	-3.4148	8.0828	6.0401
Mat 18 DNS	0.6394	5.2270	3.9355
Mat 18 AFDNS	1.5965	5.4275	4.1027
Mat 24 DNS	4.0240	7.6530	5.8079
Mat 24 AFDNS	4.7088	8.0347	6.1933
Mat 30 DNS	2.7249	8.2024	5.2921
Mat 30 AFDNS	3.2470	8.3902	5.5166
Mat 36 DNS	0.2637	7.9544	4.5652
Mat 36 AFDNS	0.6782	7.9789	4.6470
Mat 42 DNS	-1.7102	6.5887	4.6220
Mat 42 AFDNS	-1.3725	6.5087	4.5791
Mat 48 DNS	-2.3927	6.4452	4.5272
Mat 48 AFDNS	-2.1109	6.3449	4.4259

Table 6.3 One Step-Ahead Smoothed Forecasts – DNS and AFDNS Models**.**

As previously discussed, the use of an irregular maturity structure makes it difficult to construct appropriate measures of predictive performance, and thus it is difficult to construct and summarize these steps for all different maturities observed in the sample. A possible alternative would be comparing the forecasts made with interpolated maturities for distinct forecast horizons. A measurement error that contaminates the measurements of forecast error may result with this method however. Thus, the analysis of a smoothed one-step-ahead prediction was just an illustration to show the good fit of the model to the data and was not the main point of the article.

The key point of this analysis is to give support to the argument of Duffee (2011) that no-arbitrage conditions are only important in the presence of misspecification problems, which would be evident if the model without no-arbitrage corrections had a general lower predictive performance compared to the model with this correction. This analysis shows that the model with and without no-arbitrage correction have similar predictive performance. Furthermore, both models apparently do not suffer from incorrect specification, thus giving support to the initial hypothesis that no-arbitrage conditions are not important in this context.

7. VALIDITY OF NO-ARBITRAGE CORRECTIONS

To statistically verify the validity of the no-arbitrage conditions we use a Bayesian hypothesis testing procedure. Note that the null hypothesis that no-arbitrage corrections are not relevant is equivalent to testing if the no-arbitrage corrections are equal to zero, compared to the alternative hypothesis that these corrections are nonzero.

Under the Bayesian perspective, however, this problem is not well formulated since, in this perspective, the goal is to recover posterior distributions of the parameters, which corresponds to interval estimation. Thus this null hypothesis has measure zero. However, this problem can be formulated as a model selection procedure, and thus a decision problem. In this problem the objective is to find the model with the largest posterior probability among the possible models. The mechanism of Bayesian hypothesis testing/model selection in this context is based on the use of so-called Bayes factors. The posterior probability of the model *M* is given by:

$$p(M_i \mid y_t) = \frac{p(y_t \mid M_i)\, p(M_i)}{p(y_t)}$$

where $p(y_t|M_i)$ denotes the marginal likelihood of model *i*, and the Bayes factor is given by:

$$K = \frac{p(M_i \mid y_t)}{p(M_j \mid y_t)} = \frac{\int p(\Theta \mid M_i)\, p(M_i)\, p(y_t \mid M_i, \Theta)\, d\Theta}{\int p(\Theta \mid M_j)\, p(M_j)\, p(y_t \mid M_j, \Theta)\, d\Theta}$$

In this expression we note that the calculation of the Bayes factor involves obtaining the marginal likelihoods, which requires integration of the likelihood over the parameter vector. Except in simple models, this calculation cannot be performed with analytical forms, and requires numerical integration methods. The method used for calculating the marginal likelihood (also known as integrated likelihood) used in our paper is the methodology proposed by Raftery et al. (2007), which uses the modified harmonic mean of the likelihoods obtained during the MCMC simulation as the integrated likelihood estimator.

Values greater than one indicates evidence in favor of model *i*, while values less than one indicate evidence in favor of model *j* in the Bayes factor. In our analysis model *i* is given by the model without corrections of no-arbitrage, and model j corresponds to the model with the imposition of no-arbitrage restrictions.

The calculated value for the Bayes factor between the unconstrained model and the model with no-arbitrage restrictions was 1.0443. In the interpretation proposed by Jeffreys (1961) for the results of the Bayes factor, only values above 3 indicate some evidence of the model without no-arbitrage corrections against the model with no-arbitrage imposed. Thus according to the Bayes factor, the models are equivalent, indicating that the correction for no-arbitrage restriction is not required for the term structure of interest rates for DI instruments observed in this sample.

8. CONCLUSIONS

The purpose of this study was to evaluate the importance of no-arbitrage corrections within a parametric model based on the formulation of Christensen et al. (2009), using the modeling of the term structure of interest rates in Brazil,. We used five latent factors to capture the movements of the yield curve of the interbank DI contracts. Additionally, to ensure no-arbitrage consistency in the model, we added the convexity correction term in the yield curve equation, in accordance with the asset pricing theory.

The results indicate that the no-arbitrage corrections are not statistically significant. Note that we can interpret this result in two ways. The first is that there are no systematic arbitrage opportunities in the DI contracts, which could be interpreted as the result of a liquid market in which all arbitrage opportunities are exhausted by rational agents; thus equilibrium prices are free of arbitrage opportunities.

The alternative interpretation based on Duffee (2011) seems more consistent with the results and the methodology used however. Duffee (2011) shows that within the framework of affine processes, in models correctly specified, no-arbitrage corrections are not relevant. These corrections only mean better predictions and adjustments for models with specification problems, such as an inadequate number of factors or omitted variables. A similar interpretation was obtained in Joslin et al. (2011). who show that in affine models the imposition of no-arbitrage restrictions in the cross-section structure of the term structure of interest rates cannot improve the predictive results of the model.

In general our results show that the generalized Nelson-Siegel models of Diebold and Li (2006) and Christensen et al. (2009) provide an adequate modeling of the complex dynamic patterns that exist in the interbank DI term structure. Our implementation does not need interpolation procedures, and the Bayesian method does not suffer from the usual problems of multiple local maxima that exist in the classical methods based on numerical optimization. Hence, our chosen methods allowed us to construct exact credibility intervals for parameters, latent factors, and forecasts derived from the model.

Our study contributes to the discussion on the estimation of dynamic Nelson-Siegel models using only traded contracts, without using interpolations and aggregations, which can generate artificial no-arbitrage conditions. This methodology is especially relevant for emerging markets, where the liquidity of emerging markets varies substantially over time, especially in periods of crises. Our methodology adapts to the contractions and expansions of observed maturities, and, thus, can be used without the need for interpolations or extrapolations used especially to build long maturities that have less liquidity in these emerging markets. Note that this same methodology can be used in many generalizations of the Nelson-Siegel model, such as the models with macroeconomic variables and stochastic volatilities discussed in Diebold and Rudebusch (2013).

A possible extension of this estimation methodology is to simultaneously model the yields and the liquidity process related to the maturities observed in the interest rate market. In this case we can model the number of observed maturities as a counting process, or then model the observed maximum maturities as a process of conditional duration, using the latent factors of the model as explanatory variables and possibly other variables related to liquidity in this market. This simultaneous estimation can be easily implemented through additional latent factors in the Bayesian estimation methodology used in this work.

APPENDIX

	Mean	Median	Max.	Min.	Std. Dev.	Symmetry	Kurtosis	Obs.
M22	0.116453	0.1157	0.1386	0.085	0.0139	-0.6396	2.845219	670
M44	0.116146	0.1162	0.1418	0.0858	0.014342	-0.55961	2.710521	667
M66	0.116062	0.1162	0.1445	0.0859	0.014778	-0.46309	2.621523	658
M88	0.115981	0.11525	0.1455	0.0858	0.015325	-0.35128	2.514608	646
M132	0.116208	0.1156	0.1507	0.0858	0.016009	-0.20169	2.460761	673
M198	0.117056	0.1158	0.16	0.0865	0.016981	0.046699	2.344291	674
M264	0.118272	0.1169	0.164	0.0885	0.017485	0.252551	2.265945	674
M330	0.119856	0.1172	0.1702	0.0917	0.01775	0.420079	2.212559	653
M396	0.121223	0.1181	0.1685	0.095	0.017494	0.479067	2.138884	647
M462	0.12206	0.118	0.171	0.0973	0.017123	0.596646	2.223301	639
M528	0.122663	0.11835	0.175	0.0999	0.016378	0.719916	2.460912	642
M594	0.12431	0.1188	0.177	0.1025	0.016787	0.696294	2.366125	592
M660	0.12395	0.1188	0.18	0.102	0.016239	0.778424	2.662608	569
M726	0.123896	0.1188	0.1785	0.1015	0.01596	0.96777	3.247783	544
M792	0.123887	0.1191	0.184	0.1008	0.01526	1.021025	3.551737	566
M858	0.124696	0.1186	0.1805	0.1007	0.016281	0.988503	3.221331	490
M924	0.124323	0.1187	0.1793	0.1005	0.014759	0.901282	3.293741	454
M990	0.124702	0.1207	0.1745	0.1	0.014244	0.852295	3.549224	387
M1056	0.125993	0.1225	0.1797	0.0998	0.014583	1.010142	4.102564	403
M1122	0.127394	0.1209	0.1812	0.0993	0.017569	0.818134	2.972903	367
M1188	0.125297	0.122	0.1745	0.0991	0.014547	0.576614	3.248487	379
M1254	0.124886	0.1221	0.174	0.0988	0.013797	0.753758	3.921368	313
M1320	0.126859	0.1244	0.1755	0.0985	0.012838	1.158499	5.252697	271
M1584	0.127597	0.123	0.1815	0.0983	0.017239	0.957634	3.45489	294
M1848	0.126526	0.1241	0.1815	0.0981	0.014641	0.712968	3.928971	443
M2112	0.124258	0.1233	0.1728	0.0973	0.01445	0.573032	3.387993	366
M2376	0.126066	0.12465	0.182	0.0976	0.015526	0.887602	4.392208	398

Table A.4 Descriptive Statistics of most liquid DI Contracts.

```
model{
for (kk in 1:TM){
  for (i in 1:counters[kk])
  {yields[kk,i]~dnorm(m[kk,i],yisigma2)
        m[kk,i]<- beta[kk,1]+
        beta[kk,2]*(1-exp(-lambda1*(Maturity[kk,i]))/(lambda1*(Maturity[kk,i])))+
        beta[kk,3]*(1-exp(-lambda2*(Maturity[kk,i]))/(lambda2*(Maturity[kk,i])))+
beta[kk,4]*(1-exp(-lambda1*(Maturity[kk,i]))/(lambda1*(Maturity[kk,i]))-exp(-lambda1*(Maturity[kk,i])))+
beta[kk,5]*(1-exp(-lambda2*(Maturity[kk,i]))/(lambda2*(Maturity[kk,i]))-exp(-lambda2*(Maturity[kk,i])))
        }
}

for (kk in 2:TM){
        beta[kk,1:5]~dmnorm(betae[kk,1:5],P.F1[1:5,1:5])
        betae[kk,1]<-phi11+phi12*beta[kk-1,1]
        betae[kk,2]<-phi21+phi22*beta[kk-1,2]
        betae[kk,3]<-phi31+phi32*beta[kk-1,3]
        betae[kk,4]<-phi41+phi42*beta[kk-1,4]
        betae[kk,5]<-phi51+phi52*beta[kk-1,5]

}

beta[1,1]~dnorm(.09,.09)
beta[1,2]~dnorm(-.24,.09)
beta[1,3]~dnorm(.1,.1)
beta[1,4]~dnorm(0.1,.01)
beta[1,5]~dnorm(.3,.01)
yisigma2 ~ dgamma(2.5,0.25);
itau2 ~ dgamma(2.5,0.25);
lambda1<-exp(plambda1)
lambda2<-exp(plambda2)
plambda1~dnorm(-.75,.5)
plambda2~dnorm(-.28,.5)
phi11~dnorm(.5,.1)
phi21~dnorm(.5,.1)
phi31~dnorm(.5,.1)
phi41~dnorm(.5,.1)
phi51~dnorm(.5,.1)
phi12~dnorm(.5,.1)
phi22~dnorm(.5,.1)
phi32~dnorm(.5,.1)
phi42~dnorm(.5,.1)
phi52~dnorm(.5,.1)
P.F1[1:5,1:5] ~ dwish(Omega[,],5)
Sigma[1:5,1:5]<-inverse(P.F1[,])
}
```

Table A.5 Prototype of a dynamic Nelson-Siegel specification in WinBUGS/JAGS model language.

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