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Flux Surfaces According to Killing Magnetic Vectors in Riemannian Space $\mathbb{S}ol3$

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Abstract

In this paper, we define flux surface as surfaces in which its normal vector is orthogonal to the vector corresponding to a flux with its associate scalar flux functions in ambient manifold M. Next, we determine, in 3-dimensional homogenous Riemannian manifold $\mathbb{S}ol3$, the parametric flux surfaces according to the flux corresponding to the Killing magnetic vectors and we calculate its associate scalar flux functions. Finally, examples of such surfaces are presented with their graphical representation in Euclidean space.

1. Introduction

An effect, which passes or moves through a surface or substance, is called a flux or flow. It has many applications that we can cited a fluid mechanics, thermodynamics, electromagnetism, radiation, energy and in particular particle flux. Surfaces that do not disturb the flux are called flux surface, it plays an important role in physics, particulary the magnetism, and geometry (see [1–5] and [6–10]).
Geometrically, let $\mathbb{M}$ be a smooth surface in Riemannian manifold $(\mathbb{N},g)$, $\overrightarrow{\mathbf{n}}$ is the normal vector and $\overrightarrow{V}$ is a smooth vector field on $\mathbb{N}$. The flux $\mathscr{F}$ corresponding to the smooth vector $\overrightarrow{V}$, (to simplify, we denote the vectors $\overrightarrow{V}, \overrightarrow{\mathbf{n}}$ by $V, \mathbf{n}$) passing through the surface $\mathbb{M}$ is given by

$$\mathscr{F} = \int_{\mathbb{M}} g(V,\mathbf{n})ds.$$

The smooth surface $\mathbb{M}$ is called a *flux surface* of a smooth vector field V if

$$g(V,\mathbf{n}) = 0$$

everywhere on $\mathbb{M}$. We denoted $\mathbb{M}$ by V*-flux surface* (see Figure 1.1).

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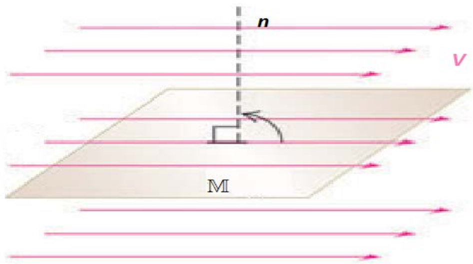

Figure 1.1: Flux surface $\mathbb{M}$ for linear flux in $(\mathbb{R}^3, g_{euc})$

When V is magnetic vector fields which is zero divergence according to Biot and Savart's law ([11]), V does not cross the surface $\mathbb{M}$ anywhere, i.e. the magnetic flux traversing $\mathbb{M}$ is zero, then $\mathbb{M}$ is called flux surface corresponding to the magnetic vector V. In this case $\mathbb{M}$ is denoted by *Magnetic V-flux surface.*
Hence, we can define a scalar flux function f according to the *magnetic vector* V, such that its value is constant on the surface $\mathbb{M}$, and

$$g(V, \nabla f) = 0$$

where ∇f is Riemannian gradient on $\mathbb{M}$.
Moreover, If V is a Killing i.e. magnetic vector fields satisfying the Killing equation

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0 \tag{1.1}$$

then $\mathbb{M}$ is called *Killing magnetic V-flux surface, where* ∇ is a connection and X, Y are a vector fields on $\mathbb{M}$.
The plasma is an example of flux surfaces. Considered as fourth state of matter, it is a hot ionized gas made up of approximately equal numbers of positively charged ions and negatively charged electrons which makes it a good electrical conductor. The electrical conductivity creates currents flowing in a plasma that interact with magnetic fields to produce the forces necessary for containment. Ordinary matter ionizes and forms a plasma at temperatures above about 5000 K, and most of the visible matter in the universe is in the plasma state (see for more detail [1, 5, 10–12]).
In magnetic confinement fusion, a flux surface is a surface on which magnetic field lines lie. Poincare-Hopf prove that such surfaces must be either a torus, or a knot (see [13]). Another applications of a flux surfaces, in Minkowski context, on the dynamics of solitons and dispersive effects can be found in ([6–8]).
In [2] and [9], the Killing V-magnetic flux surfaces was determined in Heisenberg three group and in Euclidean space, respectively. In our study, we determine all Killing V-magnetic flux surfaces and its associate Killing scalar flux functions in three-dimensional Riemannian manifold $\mathbb{S}ol3$ which is among the eight models of the geometry of Thurston ([14]).
The paper is organized as follow. In Section 2, we present the geometry of $\mathbb{S}ol3$ and its three Killing vectors representations. We determine, in the Section 3, all parameterizations of Killing V-flux surfaces and its associate Killing V-magnetic scalar flux functions with examples.
We use the computer software "Wolfram Mathematica" to present the computer graphics in Euclidean 3-space.

2. Geometry of Riemannian Space $\mathbb{S}ol3$

The $\mathbb{S}ol3$ space is seen as $\mathbb{R}^3$ with the standard representation in $SL(3, \mathbb{R})$ as

$$\mathbb{S}ol3 = \left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y, z) \in \mathbb{R}^3 \right\}$$

endowed with the multiplication

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + e^{-z} x_2, y_1 + e^{z} y_2, z_1 + z_2).$$

The Riemannian metrics on the $\mathbb{S}ol3$ is given by

$$g_{\mathbb{S}ol_3} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2. \tag{2.1}$$

We define an orthonormal basis $(e_i)_{i=\overline{1,3}}$ as

$$e_1 = e^{-z} \partial x, \ \ e_2 = e^{z} \partial y, \ \ e_3 = \partial z, \tag{2.2}$$

and its dual basis $\left(\omega^i\right)_{i=\overline{1,3}}$ by

$$\omega^1 = e^{z} dx; \ \ \omega^2 = e^{-z} dy; \ \ \omega^3 = dz.$$

The Lie bracket of the basis $(e_i)_{i=\overline{1,3}}$ are given by the following identities

$$[e_1,e_2]=0;\quad [e_2,e_3]=-e_2,\ \ [e_1,e_3]=e_1. \tag{2.3}$$

The Levi-Civita connection ∇ of the metric $g_{\mathbb{S}ol_3}$ with respect to the orthonormal basis $(e_i)_{i=\overline{1,3}}$ is

$$\left\{\begin{array}{lllll} \nabla_{e_1}e_1=-e_3 & & \nabla_{e_2}e_1=0 & & \nabla_{e_3}e_1=0 \\ \nabla_{e_1}e_2=0 & ; & \nabla_{e_2}e_2=e_3 & ; & \nabla_{e_3}e_2=0 \\ \nabla_{e_1}e_3=e_1 & & \nabla_{e_2}e_3=-e_2 & & \nabla_{e_3}e_3=0 \end{array}\right. . \tag{2.4}$$

The algebra of Killing vector field of $\mathbb{S}ol3$ it is generated by the basis $\mathbb{K}=(\mathbf{K}_i)_{i=\overline{1,3}}$, which are solutions of Eq. (1.1), where the Killing vectors $(\mathbf{K}_i)_{i=\overline{1,3}}$ are presented in the following

$$\mathbf{K}_1=\partial x,\ \ \mathbf{K}_2=\partial y \text{ and } \mathbf{K}_3=x\partial x-y\partial y-\partial z.$$

The Killing vectors, in the base $(e_i)_{i=\overline{1,3}}$ from the Eq. (2.2), are

$$\mathbf{K}_1=e^z e_1,\ \ \mathbf{K}_2=e^{-z}e_2 \text{ and } \ \mathbf{K}_3=xe^z e_1-ye^{-z}e_2-e_3, \tag{2.5}$$

(for more detail see [3,4,15]).

3. Killing V-flux Surfaces in $\mathbb{S}ol3$

Definition 3.1. *Let M be a smooth surface in a Riemannian manifold (N,g) and $\mathbf{n}$ be its normal vector field. We call M a flux surface of a smooth vector field V (denoted by V-flux surface) on (N,g) if*

$$g(V,\mathbf{n})=0$$

everywhere on M.
Moreover, if V is a Killing field then we call M flux surface according to the Killing vector V denoted by Killing V-flux surface.

Lemma 3.2. *Let f be a scalar function in (N,g), then the Riemannian gradient of f is*

$$\nabla f=f_x\partial x+f_y\partial y+f_z\partial z=e^z f_x e_1+e^{-z}f_y e_2+f_z e_3.$$

The determination of flux surfaces needs the resolution of partial differential equations denoted by PDE, therefore we use the resolution method in the following proposition.

Proposition 3.3. *Let P and Q two functions in real parameters s and t. The general solutions of the PDE*

$$P(s,t)h_s+Q(s,t)\partial_t h_t=R(s,t) \tag{3.1}$$

are in the following form:

1. *When the PDE (3.1) is homogeneous (i.e. $R\equiv 0$)*

i. If $P\equiv 0$ (resp. $Q\equiv 0$) then $h(s,t)=h(s)$ (resp. $h(s,t)=h(t)$)

ii. If P and Q are non null functions, then

$$h(s,t)=\varphi(\psi(s,t))$$

where $\psi(s,t)=c$ (c is a constant) is the solution of ODE

$$\frac{ds}{P}=\frac{dt}{Q} \tag{3.2}$$

and φ is arbitrary real function.

2. *When the PDE (3.1) is nonhomogeneous (i.e. $R\neq 0$)*

i. If $P\equiv 0$ (resp. $Q\equiv 0$) then $h(s,t)=\int\frac{R}{Q}dt$ (resp. $h(s,t)=\int\frac{R}{P}ds$)

ii. If P and Q are non null functions, then the solution h is given implicitly from

$$\overline{\psi}_1(s,t,h)=\varphi(\overline{\psi}_2(s,t,h))$$

where $\overline{\psi}_{1,2}(s,t)=c_{1,2}$ ($c_{1,2}$ are a constants) are the choice of two functions among three functions solutions of three ODEs

$$\frac{ds}{P}=\frac{dt}{Q}=\frac{dh}{R}$$

and φ is arbitrary function in $\mathbb{R}$. (See the method to solve linear PDE in [16]).

3.1. Killing K_1-flux surfaces in $\mathbb{S}ol3$

Let $\mathbb{M}$ be a surface in $\mathbb{S}ol3$ and $X(s,t) = (x(s,t), y(s,t), z(s,t))$ its parametrization. The tangent vectors X_s and X_t are described by

$$\begin{cases} X_s = x_s\partial x + y_s\partial y + z_s\partial z = e^z x_s e_1 + e^{-z} y_s e_2 + z_s e_3 \\ X_t = x_t\partial x + y_t\partial y + z_t\partial z = e^z x_t e_1 + e^{-z} y_t e_2 + z_t e_3. \end{cases}$$

Its normal vector $\mathbf{n}$, in the base $(e_i)_{i=\overline{1,3}}$, is

$$\mathbf{n} = \frac{X_s \times X_t}{\|X_s \times X_t\|} = \frac{1}{\|X_s \times X_t\|}\begin{pmatrix} (y_s z_t - y_t z_s)\, e^{-z} \\ (x_t z_s - x_s z_t)\, e^{z} \\ x_s y_t - x_t y_s \end{pmatrix}. \tag{3.3}$$

Now, we have the theorem.

Theorem 3.4. *Let $\mathbb{M}$ be a surface in $\mathbb{S}ol3$ and $X(s,t) = (x(s,t), y(s,t), z(s,t))$ its parametrization. Then $\mathbb{M}$ is a Killing K_1-flux surface if and only if*

$$y_s z_t - y_t z_s = 0 \tag{3.4}$$

Proof. It's a direct consequence by using the inner product given in Eq. (2.1), in the orthonormal base $(e_i)_{i=\overline{1,3}}$ defined in the Definition 3.1, of the normal vector $\mathbf{n}$ given from the Eq. (3.3) and the Killing vector K_1. □

Proposition 3.5. *All Killing K_1-flux surfaces in $\mathbb{S}ol3$ are parameterized by*

$$\begin{aligned} \mathbf{1}.\ X(s,t) &= (x(s,t), y(s,t), \varphi(\psi_2(s,t))), \\ \mathbf{2}.\ X(s,t) &= (x(s,t), \varphi(\psi_3(s,t)), z(s,t)), \\ \mathbf{3}.\ X(s,t) &= (x(s,t), \varphi_1(s), \varphi_2(s)), \\ \mathbf{4}.\ X(s,t) &= (x(s,t), \varphi_1(t), \varphi_2(t)), \end{aligned}$$

where x, y, z and φ ,$\varphi_{1,2}$ are arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

Proof. Using the Proposition 3.3(1-ii), the parameterizations $X(s,t)$ are a general solution of the first order linear PDE given in the Theorem 3.4 for arbitrary functions x, y and x, z for the assertions 1 and 2, respectively. For the assertions 3 and 4, it's a direct consequence from the Proposition 3.3(1-i). □

Example 3.6. ***1.*** *Let $y(s,t) = \cos st$, from the Proposition 3.5-1(ii) and substituting the value of y in the Eq. (3.4), we have $P = y_s$ and $Q = y_t$ and*

$$\frac{ds}{s\sin st} = \frac{-dt}{t\sin st}$$

its solution is

$$\psi_2(s,t) = st = c,\ c \text{ is a constant}$$

then the surface $\mathbb{M}_{1.1}$ parameterized by

$$X(s,t) = (x(s,t), \cos st, \varphi(st))$$

is Killing K_1-flux surface in $\mathbb{S}ol3$, where φ and x are arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$ respectively. We present the surface $\mathbb{M}_{1.1}$ in the Figure 3.1 for $(s,t) \in [-\pi,\pi]^2$ in Euclidean space $(\mathbb{R}^3, g_{euc})$.

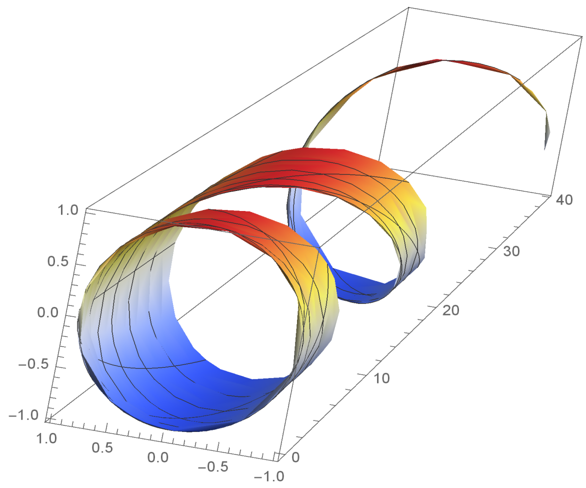

Figure 3.1: Killing K_1-flux surface $\mathbb{M}_{1.1} = \left((s+t)^2, \cos st, \sin st\right)$

2. *Let $z(s,t) = e^{st}$, similarly the Eq. (3.4) turns to*

$$sy_s - ty_t = 0$$

we have $P = s, Q = t$ and

$$\frac{ds}{s} = \frac{-dt}{t}$$

its solution is

$$\psi_3(s,t) = st = c, \ c \text{ is a constant}$$

Using the assertion 2 of Proposition 3.5, then the surface $\mathbb{M}_{1.2}$ parameterized by

$$X(s,t) = \left(x(s,t), \varphi(st), e^{st}\right)$$

is Killing K_1-flux surface in $\mathbb{S}ol3$, where φ and x are arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$ respectively.
3. *Let $z(s,t) = e^{s+t}$, we have*

$$\frac{ds}{e^{s+t}} = \frac{-dt}{e^{s+t}}$$

its solution is

$$\psi_3(s,t) = s+t = c, \ c \text{ is a constant}$$

By the assertion 2 of Proposition 3.5, the surface $\mathbb{M}_{1.3}$ parameterized by

$$X(s,t) = \left(x(s,t), \varphi(s+t), e^{s+t}\right)$$

is Killing K_1-flux surface in $\mathbb{S}ol3$, where φ and x are arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

3.1.1. Scalar flux functions

Definition 3.7. *Let f be a function on (N,g). Then f is called a scalar flux function corresponding to the magnetic vector field V if its value is constant on the surface M, and*

$$g(V, \nabla f) = 0$$

we denoted here f, to simplify, a V-magnetic scalar flux function. Moreover, if V is Killing, f is denoted Killing V-magnetic scalar flux function.

Now, we can present the following theorem.

Theorem 3.8. *Let $\mathbb{M}$ be a Killing magnetic K_1-flux surface in $\mathbb{S}ol3$. Then the function f given by*

$$f(x,y,z) = f(y,z) \ (f \text{ depend only on parameters } y,z)$$

and constant on $\mathbb{M}$ is Killing K_1-magnetic scalar flux function to $\mathbb{M}$, where ψ is arbitrary smooth function in $\mathbb{R}$.

Proof. Using the Definition 3.7 and the Lemma 3.2, we have

$$g_{\mathbb{S}ol_3}(K_1,\nabla f)=e^{2z}f_x=0$$

by solving the above first order PDE we get

$$f(x,y,z)=f(y,z)$$

and f must be also constant on $\mathbb{M}$ to be Killing K_1-magnetic scalar flux function on $\mathbb{M}$ □

Example 3.9. *Using the Example 3.6(3), the surface $\mathbb{M}_{1.3}$ parameterized by*

$$X(s,t)=\left(\cos st,\sqrt{e^{s+t}},e^{s+t}\right)$$

is Killing magnetic K_1-flux surface. The Killing K_1-magnetic scalar flux function f to $\mathbb{M}_{1.3}$, from the Theorem 3.7, is in the form $f(x,y,z)=f(y,z)$ and it must be constant on $\mathbb{M}$, (i.e. $f(X(s,t))\equiv C$, C is a constant). Let

$$f(x,y,z)=y^2-z+a,\ a\in\mathbb{R}$$

We have

$$f(X(s,t))=a$$

then $f(x,y,z)=y^2-z+a$ is Killing K_1-magnetic scalar flux function to the Killing magnetic K_1-flux surface $\mathbb{M}_{1.3}$ parameterized by $X(s,t)=\left(\cos st,\sqrt{e^{s+t}},e^{s+t}\right)$.

3.2. Killing K_2-flux surfaces in $\mathbb{S}ol3$

Similar as Section 3.1, we characterise and present all Killing K_2-flux surfaces given in the Eq. (2.5_2).

Theorem 3.10. *Let $\mathbb{M}$ be a surface in $\mathbb{S}ol3$ and $X(s,t)=(x(s,t),y(s,t),z(s,t))$ its parametrization. Then $\mathbb{M}$ is a Killing K_2-flux surface if and only if*

$$x_u z_v - x_v z_u = 0.$$

Proof. The proof is similar as the proof of the Theorem 3.4 using the Killing vector K_2 given in Eq. (2.5_2). □

Proposition 3.11. *All Killing K_1-flux surfaces in $\mathbb{S}ol3$ are parameterized by*

$$\begin{aligned}
\mathbf{1}.\ X(s,t) &= (x(s,t),y(s,t),\varphi(\psi_1(u,v))),\\
\mathbf{2}.\ X(s,t) &= (\varphi(\psi_3(s,t)),y(s,t),z(s,t)),\\
\mathbf{3}.\ X(s,t) &= (\varphi_1(s),y(s,t),\varphi_2(s)),\\
\mathbf{4}.\ X(s,t) &= (\varphi_1(t),y(s,t),\varphi_2(t)),
\end{aligned}$$

where x,y,z and φ, $\varphi_{1,2}$ are arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

Proof. The proof is similar as the proof of the Proposition 3.5. □

Example 3.12. ***1.*** *Let $x(s,t)=\sin(s+t)$, from the assertion 1 of Proposition 3.5-1(ii) and same computations as the Example 3.6, we have*

$$\psi_1(s,t)=s+t=c,\ c \text{ is a constant,}$$

then the surface $\mathbb{M}_{2.1}$ parameterized by

$$X(s,t)=(\sin(s+t),y(s,t),\varphi(s+t))$$

is Killing K_2-flux surface in $\mathbb{S}ol3$.

2. *Similarly, from the assertion 2 of Proposition 3.5, let $z(s,t)=(1+\cos t)\sin s$, we have*

$$\psi_3(s,t)=(1+\cos t)\sin s=c,\ c \text{ is a constant,}$$

then the Killing K_2-flux surface $\mathbb{M}_{2.2}$ in $\mathbb{S}ol3$ have the parametrization

$$X(s,t)=(\varphi((1+\cos t)\sin s),y(s,t),(1+\cos t)\sin s),$$

where φ and y are arbitrary smooth functions in $\mathbb{R}$ and $\mathbb{R}^2$ respectively. The following Figure 3.2 presents the surface $\mathbb{M}_{2.2}$ in $(\mathbb{R}^3,g_{euc})$ for parameters $(s,t)\in[-\pi,2\pi]\times[-\pi,\pi]$.

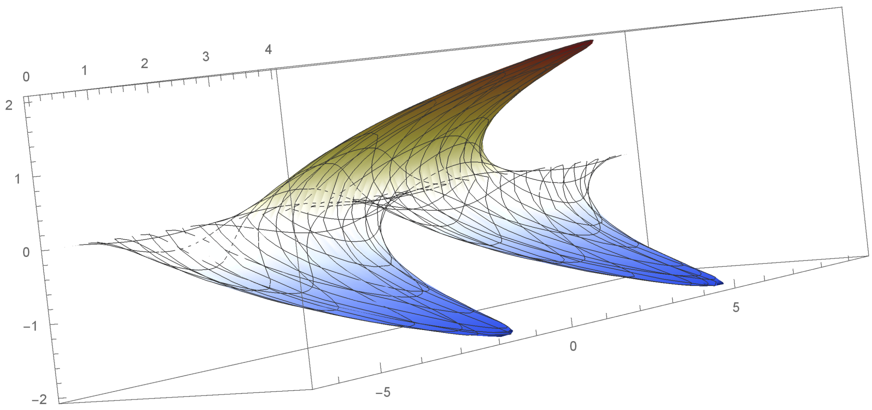

Figure 3.2: Killing K_2-flux surface $\mathbb{M}_2 = (((1+\cos t)\sin s)^2, s+t, (1+\cos t)\sin s)$

3. *Similarly, from the assertion* 4 *of Proposition 3.5, the surface* $\mathbb{M}_{2.3}$ *parameterized by* $X(s,t) = (t\cos t, s^3 + t, t\sin t)$ *is Killing* K_2*-flux surface in* $\mathbb{S}ol3$*. The Figure 3.3 presents* $\mathbb{M}_{2.3}$ *for* $(s,t) \in [-3,3] \times [-2\pi, 2\pi]$ *..*

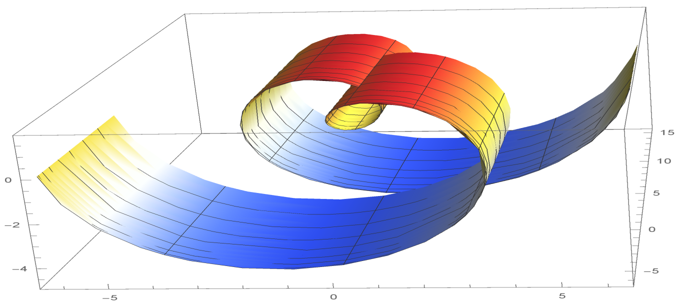

Figure 3.3: Killing K_2-flux Surface $\mathbb{M}_{2.3}$

3.2.1. Killing K_2-magnetic scalar flux functions

We present the Killing K_2-magnetic scalar flux functions in the following theorem.

Theorem 3.13. *Let* $\mathbb{M}$ *be a Killing magnetic* K_2*-flux surface in* $\mathbb{S}ol3$*. Then the function* f *given by*

$$f(x,y,z) = f(x,z)$$

and constant on $\mathbb{M}$ *is Killing* K_2*-magnetic scalar flux function to* $\mathbb{M}$*.*

Proof. Using the Definition 3.7 and the Lemma 3.2, we have

$$g(K_2, \nabla f) = e^{-2z} f_y = 0$$

we get Killing K_2-magnetic scalar flux function f by solving the above linear first order PDE and f must be constant on $\mathbb{M}$. □

Example 3.14. *From the Theorem 3.13, the Killing* K_2*-magnetic scalar flux function corresponding to the* $\mathbb{M}$ *parameterized by*

$$X(s,t) = \left(\sin(s+t), \cos(s^2+t^2), \arcsin(s+t)\right)$$

given in Example 3.12 is in the form $f(x,y,z) = f(x,z) = \arcsin x - \sin z + a$; $a \in \mathbb{R}$ *and* $f(X(s,t)) = a$*, i.e.* f *is constant on* $\mathbb{M}$*. See Figure 3.4 (here* $(s,t) \in [-\pi,\pi]^2$*).*

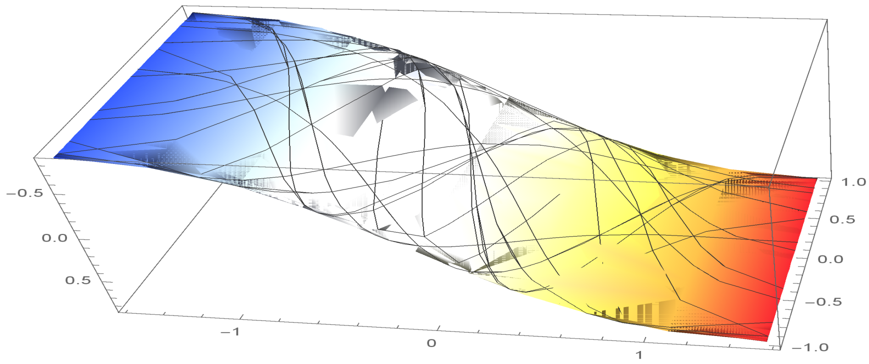

Figure 3.4: Flux surface $\mathbb{M}$ of Killing K_2-magnetic scalar flux function f

3.3. Killing K_3-flux surfaces in $\mathbb{S}ol3$

Following the above subsections, we have the following theorem.

Theorem 3.15. *Let $\mathbb{M}$ be a surface in $\mathbb{S}ol3$ and $X(s,t)=(x(s,t),y(s,t),z(s,t))$ its parametrization. Then $\mathbb{M}$ is a Killing K_3-flux surface if and only if*

$$x_t y_s - x_s y_t + x y_s z_t - x y_t z_s + y x_s z_t - y x_t z_s = 0. \tag{3.5}$$

Proof. The proof is similar as the Theorem 3.4 using the Killing vector K_3 given in Eq. (2.5$_3$). □

The surface $\mathbb{M}$ in $\mathbb{S}ol3$ parameterized by $X(s,t)=(x(s,t),y(s,t),z(s,t))$ is Killing K_3-flux surface if the Eq. (3.5) holds. However, the Eq. (3.5) have three cases to solve. The first case is where we assume that $x(s,t)$ and $y(s,t)$ are arbitrary functions and find the solution $z(s,t)$ of the Eq. (3.5) in the form

$$[yx_t + xy_t]\, z_s - [xy_s + yx_s]\, z_t + [x_s y_t - x_t y_s] = 0.$$

The second and the third cases are when we assume $x(s,t)$, $z(s,t)$ and $y(s,t)$, $z(s,t)$ to be arbitrary functions and find the solutions $y(s,t)$ and $x(s,t)$, respectively. We present in the following proposition only the first case. A similar result, with a similar method, can be obtained for the second and third cases which will not be presented in this paper.

Proposition 3.16. *The parametric surfaces in $\mathbb{S}ol3$ with the parametrization $X(s,t)$ given by*

$$\begin{cases} \mathbf{1}.\ X(s,t) = (x(s,t), \varphi_1(x(s,t)), \varphi_4(\psi(s,t))), \\ \mathbf{2}.\ X(s,t) = \left(x(s,t), \frac{\varphi_2(s)}{x(s,t)}, z(s)\right), \\ \mathbf{3}.\ X(s,t) = \left(x(s,t), \frac{\varphi_3(t)}{x(s,t)}, z(t)\right), \\ \mathbf{4}.\ X(s,t) = (x(s,t), y(s,t), \overline{\psi}(s,t)), \end{cases}$$

are Killing K_3-flux surfaces, where x,y,z and $\varphi_{\overline{1,4}}$ are arbitrary smooth functions in $\mathbb{R}^2$ and $\mathbb{R}$, the functions $\psi,\overline{\psi}$ are given in the Eqs. (3.7) and (3.8), respectively, and we assume that $\frac{\partial(x,y)}{\partial(s,t)} \neq 0$.

Proof. (case 1). Let $x(s,t)$ and $y(s,t)$ be an arbitrary smooth functions then the Eq. (3.5) turns to the PDE

$$\Big(\underbrace{xy_t + yx_t}_{P}\Big) z_s + \Big(\underbrace{-xy_s - yx_s}_{Q}\Big) z_t + \Big(\underbrace{x_s y_t - x_t y_s}_{R}\Big) = 0 \tag{3.6}$$

in the form of the PDE in Eq. (3.1), with respect to z.

i. If $R(s,t) = x_t y_s - x_s y_t = 0$ (i.e. the Eq. (3.6) is homogeneous PDE), using the assertion 1 of the Proposition 3.3, then y must be

$$y(s,t) = \varphi_1(x(s,t))$$

With cases when $y(s,t) = \frac{\varphi_2(s)}{x(s,t)}$ (i.e. $P \equiv 0$) (resp. $y(s,t) = \frac{\varphi_3(t)}{x(s,t)}$ (i.e. $Q \equiv 0$), by using again the assertion 1(i) of the Proposition 3.3, we obtain $z = z(t)$ (resp. $z = z(s)$), which prove the assertions 2 and 3. If $P,Q \neq 0$ then

$$z(s,t) = \varphi_4(\psi(s,t)) \tag{3.7}$$

where ψ is solution of the OED

$$\frac{dt}{xy_s+yx_s}=-\frac{ds}{xy_t+yx_t}$$

and $\varphi_{\overline{1,4}}$ are arbitrary real smooth functions, then the assertion 1 is proved.

ii. If $P=0$ $\left(\text{i.e. } y(s,t)=\frac{\varphi_2(s)}{x(s,t)}\right)$ we obtain $R=0$ as (i), and similarly when $Q=0$.

iii. If $P,R,Q\neq 0$ then the solution $z(s,t)=\overline{\psi}(s,t)$ of the nonhomogeneous PDE (3.6), using the assertion 2(ii) of the Proposition 3.3, is given implicitly from the equation

$$\overline{\psi}_1(s,t,z)=\varphi\left(\overline{\psi}_2(s,t,z)\right), \tag{3.8}$$

which prove the assertion 4.

□

Example 3.17. *We construct an example using the assertion* 1 *of the Proposition 3.16. Let* $x(s,t)=st$ *and* $y(s,t)=(st)^2$, *using the Proposition 3.3, we have*

$$P=3s^3t^2,\ Q=-3s^2t^3 \text{ and } R=0$$

and the ODE

$$\frac{ds}{P}=\frac{dt}{Q}$$

with solution

$$\psi_1(s,t)=s^3t^3=c,\ c \text{ is a constant}$$

then the surface $\mathbb{M}_{3.1}$ *parameterized by* $X(s,t)=\left(st,(st)^2,\varphi(s^3t^3)\right)$ *is Killing* K_3*-flux surface in* $\mathbb{S}ol3$, *where* φ *is arbitrary real smooth function. We present, in* $(\mathbb{R}^3,g_{euc})$, *the Killing* K_3*-flux surface* $X(s,t)=\left(st,(st)^2,\sin(s^3t^3+1)\right)$ *in* $\mathbb{S}ol3$ *in Figure 3.5, where* $(s,t)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[-\pi,\pi]$

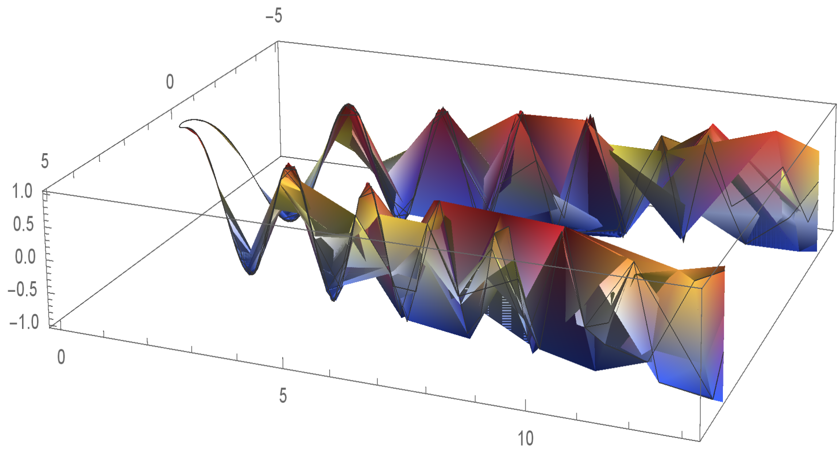

Figure 3.5: Killing K_3-flux surface $\mathbb{M}_{3.1}$

Example 3.18. *We present an example, using the assertion* 2 *of the Proposition 3.16, of Killing* K_3*-flux surface* $\mathbb{M}_{3.2}$ *in* $\mathbb{S}ol3$ *parameterized by* $X(s,t)=\left(st,\frac{s}{t},\cos s\right)$ *see Figure 3.6* $(s,t)\in[-2\pi,2\pi]^2$.

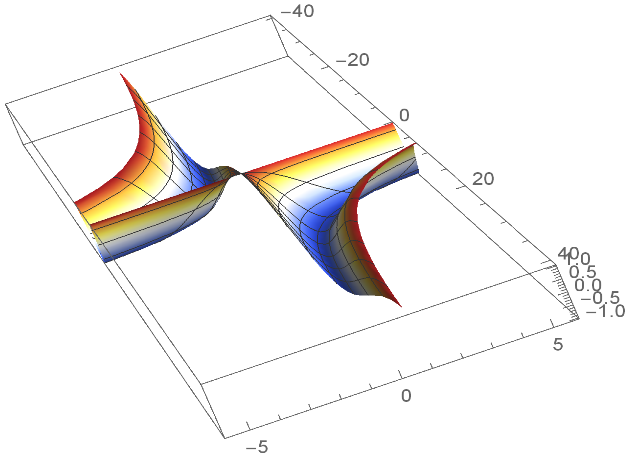

Figure 3.6: Killing K_3-Flux surface $\mathbb{M}_{3.2}$

Example 3.19. *Now, we consider the assertion* 4 *of the Proposition 3.16. Let* $x(s,t) = \cos(s+t)$ *and* $y(s,t) = st^2$, *then the Eq. (3.6) turns to an nonhomogeneous PDE given by*

$$s[\cos(s+t) - t\sin(s+t)]z_s + t[s\sin(s+t) - \cos(s+t)]z_t - [s\sin(s+t) + t\sin(s+t)] = 0 \tag{3.9}$$

using the Proposition 3.3-2(ii), we must make a choice of two functions among three functions which are solutions of three following ODEs

$$\underbrace{\frac{ds}{s[\cos(s+t) - t\sin(s+t)]}}_{(1)} = \underbrace{\frac{dt}{t[s\sin(s+t) - \cos(s+t)]}}_{(2)} = \underbrace{\frac{-dz}{[s\sin(s+t) + t\sin(s+t)]}}_{(3)}$$

The ODE $(1=2)$ *has a solution* $\psi_1(s,t,z) = -2st\cos(s+t) = c_1$ *and the solution of the ODE* $(1=3)$ *has a solution* $\psi_2(s,t,z) = (s+t+tz)\cos(s+t) - (1+stz)\sin(s+t) = c_2$, *where* $c_{1,2}$ *are a real constants.*
Hence, the solution z *of the Eq. (3.9) is given implicitly from the equation*

$$\psi_1(s,t,h) = \varphi(\psi_2(s,t,h))$$

which turns, after substitution the values of $\psi_{1,2}$, *to*

$$-2st\cos(s+t) = \varphi((s+t+tz)\cos(s+t) - (1+stz)\sin(s+t))$$

where φ *is arbitrary function in* $\mathbb{R}$. *By taking* $\varphi = Id_{\mathbb{R}}$, *we get*

$$z(s,t) = \frac{\sin(s+t) - (s+t+2st)\cos(s+t)}{t\cos(s+t) + st\sin(s+t)}$$

and the surface $\mathbb{M}_{3.3}$ *parameterized by*

$$X(s,t) = \left(\cos(s+t), st^2, \frac{\sin(s+t) - (s+t+2st)\cos(s+t)}{t\cos(s+t) + st\sin(s+t)}\right)$$

is Killing K_3*-flux surface in* $\mathbb{S}ol3$. *We present, in* $(\mathbb{R}^3, g_{euc})$, *the Killing* K_3*-flux surface* $\mathbb{M}_{3.3}$, *in* $\mathbb{S}ol3$, *in Figure 3.7 where* $(s,t) \in \left[-\frac{\pi}{2}, -\frac{\pi}{2}\right]^2$.

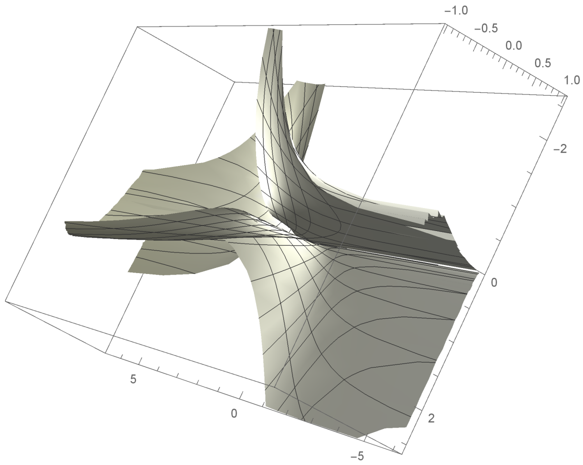

Figure 3.7: Killing K_3-flux surface $\mathbb{M}_{3.3}$

3.3.1. Killing K_3-magnetic scalar flux functions

For the Killing K_2-magnetic Scalar flux functions, we have the following theorem.

Theorem 3.20. *Let $\mathbb{M}$ be a Killing magnetic K_3-flux surface in $\mathbb{S}ol3$. Then the function f given by*

$$f(x,y,z) = \Psi\left(2\ln x + e^{2z}, 2\ln y - e^{-2z}\right)$$

and constant on $\mathbb{M}$ is Killing K_3-magnetic scalar flux function to $\mathbb{M}$.

Proof. Using the Definition 3.7 and the Lemma 3.2, we have

$$g_{\mathbb{S}ol3}(K_3, \nabla f) = xe^{2z} f_x - ye^{-2z} f_y - f_z = 0$$

solving the above linear first order PDE, we get

$$f(x,y,z) = \Psi\left(2\ln x + e^{2z}, 2\ln y - e^{-2z}\right)$$

Killing magnetic K_3-scalar flux function f and constant on $\mathbb{M}$, where Ψ is arbitrary function. □

Example 3.21. *Using the the assertion 2 of the Proposition 3.16, let $\mathbb{M}$ be a Killing magnetic K_3-flux surface in $\mathbb{S}ol3$ parameterized by*

$$X(s,t) = \left(st, \frac{\varphi(s)}{st}, z(s)\right)$$

Next, the Killing K_3-magnetic scalar flux function f is in the form

$$\begin{aligned} f(x,y,z) &= \Psi\left(2\ln x + e^{2z}, 2\ln y - e^{-2z}\right) \\ &= \Psi\left(2\ln(st) + e^{2z(s)}, 2\ln\frac{\varphi(s)}{st} - e^{-2z(s)}\right) \end{aligned}$$

and constant on $\mathbb{M}$. By choosing

$$\Psi(u,v) = u + v;\ \varphi(s) = -\sinh 2z(s)$$

we get

$$f(x,y,z) = 2(\ln xy + \sinh 2z) \text{ and } f(X(s,t)) = \Psi(0)$$

then f is constant on $\mathbb{M}$ parameterized by $X(s,t) = \left(st, \frac{-\sinh 2z(s)}{st}, z(s)\right)$.

Conclusion

The flux surface are surfaces that appear in many phenomena that we can cited, in Euclidean context, magnetic confinement fusion, dynamics of solitons and dispersive effects and plasma state. These surfaces are characterized that its normal vector is orthogonal to the vector corresponding to a flux in ambient spaces. Moreover, if the flux is magnetic (i.e. the associate vector to the flux is magnetic vector), we can define flux functions is which its gradient is orthogonal to the magnetic vector and constant on the associate flux surface. Inspired by the determination of the flux surfaces and associate flux function in Euclidean and Heisenberg group as three-dimensional manifolds. We have extended this determination, in this paper, to three-dimensional Riemannian space Sol3.

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