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ON D_4 INVARIANTS OF POLYNOMIAL ALGEBRAS

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Abstract. Let D_4 be the dihedral group of order 8. In the present study, we give generators of the algebra of D_4 invariants in the polynomial algebra with four generators over a field of characteristic zero.

1. Introduction

Let $K[X_n]$ be the polynomial algebra of rank n over a field K of characteristic zero, and let $GL_n(K)$ be the general linear group. Hilbert's fourteen problem (see [1, 2, 3, 4]) asks whether algebra $K[X_n]^G$ of constants of any subgroup G of $GL_n(K)$ is finitely generated. Although it was negated by Nagata [5] in general, Noether [6] showed that $K[X_n]^G$ is finitely generated for finite groups G. Our work was motivated by the approach above: What are the finite generators of $K[X_n]^G$ for some concrete groups G, in particular when G is a subgroup of the symmetric group S_n. The dihedral group

$$D_4 = \langle r, s \mid r^2 = s^4,\ rsr = s^3 \rangle = \{1, r, s, rs, s^2, rs^2, s^3, rs^3\}$$

of order 8 can be realized as a subgroup of the symmertic group S_4 by defining $r = (12)(34)$, $s = (1234)$. In this case we have that

$$D_4 = \{(1), (12)(34), (1234), (24), (13)(24), (14)(23), (1432), (13)\}.$$

Let $K[X_4] = K[x_1, x_2, x_3, x_4]$ be the commmutative unitary polynomial algebra of rank 4 over K. We define the invariant subalgebra

$$K[X_4]^{S_4} = \{f \in K[X_4] \mid f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = f(x_1, x_2, x_3, x_4), \forall \pi \in S_4\}$$

induced by the group S_4. The algebra $K[X_4]^{S_4}$ is called the algebra of symmetric polynomials, and each polynomial in $K[X_4]^{S_4}$ is called a symmetric polynomial. It is well known that $K[X_4]^{S_4}$ is generated by elementary symmetric polynomials (see e.g. [7])

$$\alpha_1 = x_1 + x_2 + x_3 + x_4\ ,\quad \alpha_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\ ,$$

$$\alpha_3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4\ ,\quad \alpha_4 = x_1x_2x_3x_4.$$

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Elementary symmetric polynomials are algebraically independent. There exists another set $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ of generators for $K[X_4]^{S_4}$, where

$$\beta_1 = x_1 + x_2 + x_3 + x_4 \ , \quad \beta_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \ ,$$
$$\beta_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 \ , \quad \beta_4 = x_1^4 + x_2^4 + x_3^4 + x_4^4,$$

that is not algebraically independent. We extend this notation to

$$\beta_n = x_1^n + x_2^n + x_3^n + x_4^n$$

for all $1 \leq n$. Similar to $K[X_4]^{S_4}$, we have that

$$K[X_4]^{D_4} = \{f \in K[X_4] \mid f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = f(x_1, x_2, x_3, x_4), \forall \pi \in D_4\}$$

Clearly, $K[X_4]^{S_4} \subsetneq K[X_4]^{D_4}$, since $p(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4 \in K[X_4]^{D_4}$, while $p \notin K[X_4]^{S_4}$. In this paper, we aim to show that $K[X_4]^{D_4}$ is generated by polynomials $\alpha_1, \alpha_2, \alpha_3, \alpha_4, p$.

2. Preliminaries

We give some preliminary results, in this section. Let us define

$$p_{a,b} = x_1^a x_3^b + x_1^b x_3^a + x_2^a x_4^b + x_2^b x_4^a \ , \quad 0 \leq a, b.$$

One may easily check that $p_{a,b} \in K[X_4]^{D_4}$, and that $p = \frac{1}{2}p_{11}$. We give the next list of equations without proof which is straightforward.

$$p_{a,b} = p_{b,a} \ , \quad 0 \leq a, b \tag{2.1}$$

$$p_{a,b} = \frac{1}{2}p_{11}p_{a-1,b-1} - \alpha_4 p_{a-2,b-2} \ , \quad 2 \leq a, b \tag{2.2}$$

$$p_{2,2} = \frac{1}{2}p_{11}^2 - 4\alpha_4 \tag{2.3}$$

$$p_{1,b+3} = \frac{1}{2}\alpha_1 p_{1,b+2} - \frac{1}{2}p_{2,b+2} - \frac{1}{2}\alpha_3\beta_{b+1} + \frac{1}{2}\alpha_4\beta_b + \frac{1}{4}p_{1,1}\beta_{b+2} \ , \quad 0 \leq b \tag{2.4}$$

$$p_{1,3} = \frac{1}{2}\alpha_1 p_{1,2} - \frac{1}{2}p_{2,2} - \frac{1}{2}\alpha_3\beta_1 + 2\alpha_4 + \frac{1}{4}p_{1,1}\beta_2 \tag{2.5}$$

$$p_{1,2} = \frac{1}{2}\alpha_1 p_{1,1} - \alpha_3 \tag{2.6}$$

The next lemma is necessary in the proof of the main result.

Lemma 2.1. $K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{a,b} \mid 0 \leq a, b] = K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p]$.

Proof. Clearly $K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p] \subset K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{a,b} \mid 0 \leq a, b]$ because

$$p = \frac{1}{2}p_{1,1} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{a,b} \mid 0 \leq a, b].$$

In order to complete the proof, it is sufficient to show that $p_{a,b}$ is included in the algebra generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, p$ for every $0 \leq a, b$. Initially, using the equations (2.1), (2.2), (2.3) inductively, we obtain that every polynomial

$$p_{a,b} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{1,1}]$$

for every $2 \leq a, b$. Now by (2.1), (2.4), (2.5), we have that

$$p_{1,b} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{1,1}, p_{1,2}]$$

for all $3 \leq b$. Finally, we terminate the proof by (2.6) implying that

$$p_{1,2} \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{1,1}].$$

□

3. Main Results

The next theorem is the main result of the paper.

Theorem 3.1. *The algebra $K[X_4]^{D_4}$ is generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, p$.*

Proof. It is sufficient to show that $K[X_4]^{D_4} \subset K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \leq a, b]$ by Lemma 2.1. Let

$$g = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d$$

be an arbitrary element of $K[X_4]^{D_4}$ of the form $g = g_1 + g_2 + g_3 + g_4$, where

$$g_1 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 1,$$

$$g_2 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 2,$$

$$g_3 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 3,$$

$$g_4 = \sum_{0 \leq a,b,c,d} \varepsilon_{abcd} x_1^a x_2^b x_3^c x_4^d \ , \quad \mid \{a, b, c, d\} \mid = 4.$$

It is clear that each g_i, $i = 1, 2, 3, 4$, is D_4 invariant; i.e.,

$$g_i(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = g_i(x_1, x_2, x_3, x_4), \forall \pi \in D_4.$$

Initially,

$$g_1 = \sum_{0 \leq a} \varepsilon_{aaaa} (x_1 x_2 x_3 x_4)^a = \sum_{0 \leq a} \varepsilon_{aaaa} \alpha_4^a \ \in K[\alpha_4].$$

On the other hand g_2 is of the form $g_2 = g_{21} + g_{22} + g_{23} + g_{24}$, where

$$g_{21} = \sum_{a<b} \varepsilon_{bbaa} x_1^b x_2^b x_3^a x_4^a + \varepsilon_{aabb} x_1^a x_2^a x_3^b x_4^b + \varepsilon_{abba} x_1^a x_2^b x_3^b x_4^a + \varepsilon_{baab} x_1^b x_2^a x_3^a x_4^b$$

$$g_{22} = \sum_{a<b} \varepsilon_{bbab} x_1^b x_2^b x_3^a x_4^b + \varepsilon_{bbba} x_1^b x_2^b x_3^b x_4^a + \varepsilon_{abbb} x_1^a x_2^b x_3^b x_4^b + \varepsilon_{babb} x_1^b x_2^a x_3^b x_4^b$$

$$g_{23} = \sum_{a<b} \varepsilon_{aaab} x_1^a x_2^a x_3^a x_4^b + \varepsilon_{aaba} x_1^a x_2^a x_3^b x_4^a + \varepsilon_{abaa} x_1^a x_2^b x_3^a x_4^a + \varepsilon_{baaa} x_1^b x_2^a x_3^a x_4^a$$

$$g_{24} = \sum_{a<b} \varepsilon_{abab} x_1^a x_2^b x_3^a x_4^b + \varepsilon_{baba} x_1^b x_2^a x_3^b x_4^a$$

One may easily show that no summand in a fixed g_{2i} turns into a summand in g_{2j}, $i \neq j$, under the action of D_4. Thus by

$$g_2(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = g_2(x_1, x_2, x_3, x_4), \forall \pi \in D_4,$$

we get that

$$g_{2i}(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) = g_{2i}(x_1, x_2, x_3, x_4), \forall \pi \in D_4, \quad i = 1, 2, 3, 4.$$

Easy calculations computing the actions of permutations from D_4 gives that all coefficients in each g_{2i} for a fixed (a,b) are equal; i.e.,

$$g_{21} = \sum_{a<b} \varepsilon_{bbaa}(x_1^b x_2^b x_3^a x_4^a + x_1^a x_2^a x_3^b x_4^b + x_1^a x_2^b x_3^b x_4^a + x_1^b x_2^a x_3^a x_4^b)$$

$$g_{22} = \sum_{a<b} \varepsilon_{bbab}(x_1^b x_2^b x_3^a x_4^b + x_1^b x_2^b x_3^b x_4^a + x_1^a x_2^b x_3^b x_4^b + x_1^b x_2^a x_3^b x_4^b)$$

$$g_{23} = \sum_{a<b} \varepsilon_{aaab}(x_1^a x_2^a x_3^a x_4^b + x_1^a x_2^a x_3^b x_4^a + x_1^a x_2^b x_3^a x_4^a + x_1^b x_2^a x_3^a x_4^a)$$

$$g_{24} = \sum_{a<b} \varepsilon_{abab}(x_1^a x_2^b x_3^a x_4^b + x_1^b x_2^a x_3^b x_4^a)$$

which are in the form

$$g_{21} = \frac{1}{2} \sum_{a<b} \varepsilon_{bbaa}(p_{a,b}^2 - p_{2a,2b} - p_{a+b,a+b})$$

$$g_{22} = \sum_{a<b} \varepsilon_{bbab} \alpha_4^a \left(\frac{1}{2} p_{b-a,b-a} \beta_{b-a} - p_{2(b-a),b-a} \right)$$

$$g_{23} = \sum_{a<b} \varepsilon_{aaab} \alpha_4^a \beta_{b-a}$$

$$g_{24} = \frac{1}{2} \sum_{a<b} \varepsilon_{abab} \alpha_4^a p_{b-a,b-a}$$

Therefore $g_2 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \leq a, b]$.

Next, g_3 is of the form $g_3 = g_{31} + g_{32} + g_{33} + g_{34} + g_{35} + g_{36}$, where

$$g_{31} = \sum_{a<b<c} \begin{pmatrix} \varepsilon_{accb} x_1^a x_2^c x_3^c x_4^b + \varepsilon_{ccba} x_1^c x_2^c x_3^b x_4^a + \varepsilon_{cbac} x_1^c x_2^b x_3^a x_4^c + \varepsilon_{bacc} x_1^b x_2^a x_3^c x_4^c \\ \varepsilon_{cabc} x_1^c x_2^a x_3^b x_4^c + \varepsilon_{ccab} x_1^c x_2^c x_3^a x_4^b + \varepsilon_{bcca} x_1^b x_2^c x_3^c x_4^a + \varepsilon_{abcc} x_1^a x_2^b x_3^c x_4^c \end{pmatrix}$$

$$g_{32} = \sum_{a<b<c} \begin{pmatrix} \varepsilon_{acbb} x_1^a x_2^c x_3^b x_4^b + \varepsilon_{cbba} x_1^c x_2^b x_3^b x_4^a + \varepsilon_{bbac} x_1^b x_2^b x_3^a x_4^c + \varepsilon_{bacb} x_1^b x_2^a x_3^c x_4^b \\ \varepsilon_{cabb} x_1^c x_2^a x_3^b x_4^b + \varepsilon_{bcab} x_1^b x_2^c x_3^a x_4^b + \varepsilon_{bbca} x_1^b x_2^b x_3^c x_4^a + \varepsilon_{abbc} x_1^a x_2^b x_3^b x_4^c \end{pmatrix}$$

$$g_{33} = \sum_{a<b<c} \begin{pmatrix} \varepsilon_{aabc} x_1^a x_2^a x_3^b x_4^c + \varepsilon_{abca} x_1^a x_2^b x_3^c x_4^a + \varepsilon_{bcaa} x_1^b x_2^c x_3^a x_4^a + \varepsilon_{caab} x_1^c x_2^a x_3^a x_4^b \\ \varepsilon_{aacb} x_1^a x_2^a x_3^c x_4^b + \varepsilon_{baac} x_1^b x_2^a x_3^a x_4^c + \varepsilon_{cbaa} x_1^c x_2^b x_3^a x_4^a + \varepsilon_{acba} x_1^a x_2^c x_3^b x_4^a \end{pmatrix}$$

$$g_{34} = \sum_{a<b<c} \varepsilon_{acbc} x_1^a x_2^c x_3^b x_4^c + \varepsilon_{cbca} x_1^c x_2^b x_3^c x_4^a + \varepsilon_{bcac} x_1^b x_2^c x_3^a x_4^c + \varepsilon_{cacb} x_1^c x_2^a x_3^c x_4^b$$

$$g_{35} = \sum_{a<b<c} \varepsilon_{abcb} x_1^a x_2^b x_3^c x_4^b + \varepsilon_{bcba} x_1^b x_2^c x_3^b x_4^a + \varepsilon_{cbab} x_1^c x_2^b x_3^a x_4^b + \varepsilon_{babc} x_1^b x_2^a x_3^b x_4^c$$

$$g_{36} = \sum_{a<b<c} \varepsilon_{abac} x_1^a x_2^b x_3^a x_4^c + \varepsilon_{baca} x_1^b x_2^a x_3^c x_4^a + \varepsilon_{acab} x_1^a x_2^c x_3^a x_4^b + \varepsilon_{caba} x_1^c x_2^a x_3^b x_4^a$$

Similarly, g_{3i}, $i = 1, 2, 3, 4, 5, 6$, is D_4-invariant, and hence we have that

$$g_{31} = \sum_{a<b<c} \varepsilon_{accb}(p_{a,c} p_{b,c} - p_{a+b,2c} - p_{a+c,b+c})$$

$$g_{32} = \sum_{a<b<c} \varepsilon_{acbb}(p_{a,b} p_{b,c} - p_{a+c,2b} - p_{a+b,b+c})$$

$$g_{33} = \sum_{a<b<c} \varepsilon_{aabc}(p_{a,b} p_{a,c} - p_{2a,b+c} - p_{a+c,a+b})$$

$$g_{34} = \sum_{a<b<c} \varepsilon_{acbc}\alpha_4^a \left(\frac{1}{2}p_{c-a,c-a}\beta_{b-a} - p_{b+c-2a,c-a}\right)$$

$$g_{35} = \sum_{a<b<c} \varepsilon_{abcb}\alpha_4^a \left(\frac{1}{2}p_{b-a,b-a}\beta_{c-a} - p_{b+c-2a,b-a}\right)$$

$$g_{36} = \frac{1}{2}\sum_{a<b<c} \varepsilon_{abac}(p_{a,a}p_{b,c} - 2p_{a+b,a+c})$$

and that $g_3 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \le a, b]$.

Finally, g_3 is of the form $g_4 = g_{41} + g_{42} + g_{43}$, where

$$g_{41} = \sum_{a<b<c<d} \begin{pmatrix} \varepsilon_{abcd}x_1^ax_2^bx_3^cx_4^d + \varepsilon_{bcda}x_1^bx_2^cx_3^dx_4^a + \varepsilon_{cdab}x_1^cx_2^dx_3^ax_4^b + \varepsilon_{dabc}x_1^dx_2^ax_3^bx_4^c \\ \varepsilon_{badc}x_1^bx_2^ax_3^dx_4^c + \varepsilon_{cbad}x_1^cx_2^bx_3^ax_4^d + \varepsilon_{dcba}x_1^dx_2^cx_3^bx_4^a + \varepsilon_{adcb}x_1^ax_2^dx_3^cx_4^b \end{pmatrix}$$

$$g_{42} = \sum_{a<b<c<d} \begin{pmatrix} \varepsilon_{acbd}x_1^ax_2^cx_3^bx_4^d + \varepsilon_{cbda}x_1^cx_2^bx_3^dx_4^a + \varepsilon_{bdac}x_1^bx_2^dx_3^ax_4^c + \varepsilon_{dacb}x_1^dx_2^ax_3^cx_4^b \\ \varepsilon_{cadb}x_1^cx_2^ax_3^dx_4^b + \varepsilon_{bcad}x_1^bx_2^cx_3^ax_4^d + \varepsilon_{dbca}x_1^dx_2^bx_3^cx_4^a + \varepsilon_{adbc}x_1^ax_2^dx_3^bx_4^c \end{pmatrix}$$

$$g_{43} = \sum_{a<b<c<d} \begin{pmatrix} \varepsilon_{abdc}x_1^ax_2^bx_3^dx_4^c + \varepsilon_{bdca}x_1^bx_2^dx_3^cx_4^a + \varepsilon_{dcab}x_1^dx_2^cx_3^ax_4^b + \varepsilon_{cabd}x_1^cx_2^ax_3^bx_4^d \\ \varepsilon_{bacd}x_1^bx_2^ax_3^cx_4^d + \varepsilon_{dbac}x_1^dx_2^bx_3^ax_4^c + \varepsilon_{cdba}x_1^cx_2^dx_3^bx_4^a + \varepsilon_{acdb}x_1^ax_2^cx_3^dx_4^b \end{pmatrix}$$

Similar computations give that

$$g_{41} = \sum_{a<b<c<d} \varepsilon_{abcd}\alpha_4^a(\beta_{c-a}p_{d-a,b-a} - p_{d-a,b+c-2a} - p_{b-a,d+c-2a})$$

$$g_{42} = \sum_{a<b<c<d} \varepsilon_{acbd}\alpha_4^a(\beta_{b-a}p_{d-a,c-a} - p_{c-a,b+d-2a} - p_{d-a,b+c-2a})$$

$$g_{43} = \sum_{a<b<c<d} \varepsilon_{abdc}\alpha_4^a(\beta_{d-a}p_{c-a,b-a} - p_{c-a,b+d-2a} - p_{b-a,d+c-2a})$$

and that $g_4 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \le a, b]$. Consequently,

$$g = g_1 + g_2 + g_3 + g_4 \in K[\alpha_1, \alpha_2, \alpha_3, \alpha_4, p_{ab} \mid 0 \le a, b].$$

□

4. Conclusion

In this study, generators for the algebra of D_4-invariants were provided. This might be an initial approach for determining generators for algebras of G-invariants, where G is a subgroup of S_n, $n \ge 4$.

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