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Discretization of Fractional Order Operator in Delta Domain

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Abstract

The fractional order operator is the backbone of the fractional order system (FOS). The fractional order operator (FOO) is generally represented as $s^{\pm\mu}(0 < \mu < 1)$. Discrete time FOS can be obtained through the discretization of the fractional order operator. The FOO is the general form of either fractional order differentiator (FOD) or integrator (FOI) depending upon the values of μ. Out of the two discretization methods, direct discretization outperforms the method of indirect discretization. The mapping between the continuous time and discrete time domain is done with the development of generating function. Continuous fraction expansion (CFE) is used expand the generating function for the rational approximation of the FOO. There is an inherent problem associated with the discretization of FOO in discrete z-domain particularly at very fast sampling rate. In the other hand, discretization using delta operator parameterization provides the continuous time and discrete time results in hand to hand, when the continuous time systems are sampled at very fast sampling rate and circumventing the problem with shift operator parameterization at fast sampling rate. In this work, a new generating function is proposed to discretize the FOO using the Gauss-Legendre 3-point quadrature rule and generating function is expanded using the CFE to form rational approximation of the FOO in delta domain. The benchmark fractional order systems are considered in this work for the simulation purpose and comparison of results are made to prove the efficacy of the proposed method using MATLAB.

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1. INTRODUCTION

Nowadays, scientists and researchers are paying much more attention to the theory and application of fractional calculus (Oldham & Spanier, 1974; Miller & Ross, 1993), though it was invented more than 300 years back. The researchers are really overwhelmed to rediscover the untouched and undiscovered part of engineering application as well as in the diversified classes of science using fractional calculus (Sun et al., 1984a, 1984b; Skaar et al., 1988; Caponetto et al., 2010). Fractional calculus plays the most important role in fractional order control study. Analytical design Fractional order PID controller for fractional order or integer order plant has been studied in (Yumuk et al., 2019). Robust fractional order controller was designed by (Yumuk et al., 2022) using the ideal Bode transfer functions using the fundamental property of fractional order calculus. The fractional order calculus extends its horizon even in the field of electromagnetics. In the study of plane wave diffraction by two strips, (Tabatadze et al., 2020) have applied fractional boundary conditions on strips having different fractional order. For the analysis of complex system, fractional order derivative and partial differential equations are widely used in the field of science and technology. Caputo version (Caputo, 1967) as well as Atangana-Baleanu derivative (Atangana & Baleanu, 2016) version of fractional order derivatives are used in the literature. Garden equation is a kind of nonlinear partial differential equation, commonly used to describe

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complex systems. To solve this garden equation, (Dokuyucu, 2020) have expanded the garden equation to Caputo derivative sense in order to make it fractional order.

The fundamental part of the fractional order calculus or fractional order differential equation is the fractional order operator (FOO), generally represented by $s^{\pm\mu}$. The operator can be termed as fractional order differentiator (FOD) or fractional order integrator (FOI) considering the positive and negative values of μ (Nakagawa & Sorimachi, 1992; Podlubny, 1999). The FOO needs to be converted into rational transfer function for the realization of the fractional order system (FOS). The rationalization may be done in continuous time domain or in discrete time domain. There are bench mark solutions for conversion of FOO into rational approximation in continuous time domain (Vinagre et al., 2000; Xue et al., 2006; Khanra et al., 2010).

Nowadays, there is a call for paradigm shift from the continuous time domain to discrete time domain realization of every system. This results in a serious attempt to get the FOS to be realized in discrete time domain. Therefore, the continuous time (Oustaloup, 1995) rational approximation of the FOO is to be converted into the discrete domain to realize FOS in the discrete time domain. This method is known as indirect discretization (Chen & Moore, 2002; Krishna, 2011; 2015; Maione, 2011; Keyser & Muresan, 2016) and it is having certain flaws. The problems associated with the indirect discretization is overcome with the direct discretization method.

In direct discretization method, generating functions are used to get the relationship between the s -domain and z -domain in shift operator parameterization. Euler, Tustin, Al-Alauoi are some of the renowned mathematicians to propose the new generating functions for the aforesaid application and finally the generating functions are expanded using continued fraction expansion (CFE) method (Chen et al., 2009).

As the demand is increasing day by day for the digital realization of the fractional order controller or fractional order system, the systems to be designed for maximum accuracy. To implement any FOS in discrete time domain, the continuous time systems need to be sampled with very high sampling rate. The increased sampling rate will make the results ill conditioned when implemented in discrete z -domain and also there will be issue with finite word length effect (Middleton & Goodwin, 1990a, 1990b; Chen & Moore, 2002). Therefore, the considerable features of the traditional continuous-time system cannot be produced by shift operator parameterization of discrete time systems at high sample rates. In other way around, delta operator parameterization (Middleton & Goodwin, 1990a, 1990b) provides meaningful outcome at very high sampling frequency and the results are found for both continuous time and discrete time in hand to hand rather than two special cases. The problems associated with traditional shift operator parameterization thus can be avoided using delta operator parameterization and is used in numerous application (Middleton & Goodwin, 1986; Cortés-Romero et al., 2013; Sarkar et al., 2016; Zhao & Zhang, 2017; Swarnakar et al., 2017; Gao et al., 2018; Lamrabet et al., 2020; Quezada-Téllez et al., 2020; Ganguli et al., 2021).

In this paper, the FOO is directly discretized using delta operator parameterization though direct discretization of the FOO using shift operator parameterization (Vinagre et al., 2003; Pan & Das, 2013) has been worked out in earlier work. The 3-point Gauss-Legendre quadrature rule (Khattri, 2009) is a powerful computational tool for numerical computation and this is capitalized in this paper to form the new generating function for direct discretization of FOO in delta domain. The developed generating function is expanded using the traditional CFE tool to get the rational approximation of the FOO in delta domain.

The following section discusses the significant contributions of this work. It can be observed from literature that the discretization of the fractional order system was made through the discretization of the fractional order operator in shift operator parameterization. This paper deals with the direct discretization of FOS using delta operator parameterization. At a very high sampling rate, the shift operator parameterized discrete time system fails to provide meaningful information whereas, delta operator parameterized discrete time system provides the same result as that of the continuous time system making the approach a unified one. A direct relationship between the continuous time variable and discrete delta domain variable is developed and that gives rise to a new generating function for rational approximation of the fractional order operator. The results obtained are compared with the results obtained using the other standard methods, to prove the efficacy of the proposed method.

The paper is organized as follows: Introduction to fractional order operator and direct discretization of FOO in delta domain are illustrated in section 2. In Section 3, three different examples are considered for simulation purpose to analyze the results. The section 4 is devoted for conclusion.

2. MATERIAL AND METHOD

2.1. Fractional order operator and fractional order system

A non-integer order system is literally said to be a fractional order system. Like the integer order system, any fractional order system can be described by its fractional order differential equation. A fractional order differential equation is described below (Pan & Das, 2013).

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + ... + a_0 D^{\alpha_0} y(t) = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + ... + b_0 D^{\beta_0} u(t) \quad (1)$$

where, a_i and b_j are the coefficients. D signifies the fractional derivative operator. The generalized operator used in (1) is known as integro-diffrerentiator operator and expressed as (Boubaker & Jafary, 2018):

$${}_g D_t^{\eta} = \begin{cases} \frac{d^{\eta}}{dt^{\eta}} (\eta > 0) \\ 1(\eta = 0) \\ \int_g^{\eta} (dt)^{\eta} (\eta < 0) \end{cases} \quad (2)$$

Taking the Laplace transform on both sides of (1) considering initial condition zero, the transfer function of the fractional order system is taking the form as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m D^{\beta_m} + b_{m-1} D^{\beta_{m-1}} + ... + b_0 D^{\beta_0}}{a_n D^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + ... + a_0 D^{\alpha_0}} \quad (3)$$

where, $L[y(t)] = Y(s), L[u(t)] = U(s)$. L stands for Laplace trasform.

Mathematical representations of integro-diffrerentiator operator are usually done with the help of Grünwald-Letnikov (GL), Riemann-Liouville (RL) and Caputo derivative based approach. R-L definition is utilized in this paper to represent any fractional order system.

The RL definition is mathematically represented as:

$${}_g D_t^{\eta} f(t) = \frac{1}{\Gamma(n-\eta)} \frac{d^n}{dt^n} \int_g^t (t-\tau)^{n-\eta-1} f(\tau) d\tau \quad (4)$$

where, g and t are the operating limits, n is an integer and $\Gamma(n-\alpha)$ is the gamma function of $(n-\alpha)$.

Taking the Laplace transform of (4) with initial condition zero, the transfer function of a FOS is obtained.

$$L\left\{ {}_g D_t^{\eta} f(t) \right\} = s^{\eta} F(s) \quad \text{for } 0 < \eta < 1 \quad (5)$$

From (5), it is observed that s^{η} plays the pivotal role for the realization of a FOS and this is known as fractional order operator (FOO), conceptualized using R-L definition.

2.2. Discretization of fractional order operator in delta domain

The delta operator is defined as:

$$\delta = \frac{q-1}{\Delta} \tag{6}$$

q is the forward shift operator and Δ is termed as sampling time. It is nothing but the scaled and shifted version of the forward shift operator. Applying the delta operator (δ) on any differentiable signal $\xi(t)$ gives the following relationship at high sampling limit ($\Delta \to 0$).

$$\underset{\Delta\to 0}{Lt}\, \delta\xi(t) = \underset{\Delta\to 0}{Lt}\, \frac{\xi(t+\Delta)-\xi(t)}{\Delta} \approx \frac{d\xi(t)}{dt} \tag{7}$$

In complex delta domain, γ is used to represent the frequency domain variable similar to z in shift operator parameterization. The relationship between these two variables are represented by (8) (Middleton & Goodwin, 1990a, 1990b)

$$\gamma = \frac{z-1}{\Delta} = \frac{e^{s\Delta}-1}{\Delta} \tag{8}$$

At high sampling time limits ($\Delta \to 0$), (8) is rewritten as

$$\underset{\Delta\to 0}{Lt}\,\gamma = \underset{\Delta\to 0}{Lt}\, \frac{e^{s\Delta}-1}{\Delta} \approx \underset{\Delta\to 0}{Lt}\, \frac{1+s\Delta+\frac{s^2\Delta^2}{2!}+....-1}{\Delta} = s \tag{9}$$

From (9), the relationship between the frequency domain variable γ in delta domain and frequency domain variable s in continuous time domain is established at fast sampling rate. It is observed that at fast sampling rate two domain results coincide making the approach a unified one and this philosophy is capitalized in this work. Equation (10) is established by rearranging (8).

$$e^{s\Delta} = \gamma\Delta + 1$$
$$\text{or, } s = \frac{1}{\Delta}\ln(1+\gamma\Delta) \tag{10}$$

For the discretization of fractional order operator in delta domain, (10) acts as the pivotal equation. CFE is used to get the rational transfer function using the generating function in delta domain. $\ln(1+\gamma\Delta)$ has to be approximated in its closed form such that CFE can be utilized for expansion. One of the best close form approximation of $\ln(1+x)$ is 3P -GILOG which is done using trapezoidal quadrature rule (Khattri, 2009) Applying 3P -GILOG rule, $\ln(1+x)$ is approximated with an error of 0.00002548744.

$$\ln(1+x) \approx \frac{60x+60x^2+11x^3}{60+90x+36x^2+3x^3} \tag{11}$$

Equation (10) is re-established in (12) by using (11)

$$s = \left\{\frac{1}{\Delta}\ln(1+\gamma\Delta)\right\} \approx \frac{1}{\Delta}\left\{\frac{60\Delta\gamma+60\Delta^2\gamma^2+11\Delta^3\gamma^3}{60+90\gamma\Delta+36\Delta^2\gamma^2+3\Delta^3\gamma^3}\right\}$$
$$\text{or, } s \approx \left\{\frac{60\gamma+60\Delta\gamma^2+11\Delta^2\gamma^3}{60+90\gamma\Delta+36\Delta^2\gamma^2+3\Delta^3\gamma^3}\right\} \tag{12}$$

The discrete-time frequency variable (γ) in delta domain coincides with the continuous- time frequency variable (s) at fast sampling limit ($\Delta \to 0$) which is again established from (12).Therefore, the direct relasionship between (γ) and (s) can be expressed by (12).Equation (12) is used as the generating function for rest of the paper.

Consider the general form of a fractional order differentiator (FOD):

$$G(s) = s^r \left(0 < r < 1\right) \tag{13}$$

Equation (13) can be used as the fractional order integrator (FOI) if r is replaced by $-r$ .In general $s^{\pm r}$ is the representation of a FOO. For the direct discretization of the FOO in delta domain, generating function developed in (12) is used. This will be called as CFE3P-GILOG method. The delta domain rational is expressed by (14).

$$s^r \approx G_\delta(\gamma) \approx CFE\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^r\right\} \tag{14}$$

The CFE approximation is mathematically formulated using (15).

$$(1+p)^q = 1 + \cfrac{qp}{1 + \cfrac{(1-q)p}{2 + \cfrac{(1+q)p}{3 + \cfrac{(2-q)p}{2 + \cfrac{(2+q)p}{5 + \cfrac{(3-q)p}{2 + ...}}}}}} \tag{15}$$

To obtain the standard form of CFE as given in (15), p is replaced by $\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right) - 1\right\}$ to get the rational approximation of fractional order trasfer function as described by (13).

Third and fifth order approximation of the r th order FOO in delta domain are obtained in this study and corresponding integer order transfer functions are given by (16) and (17) respectively.

$$s^r \approx G_{\delta 3}(\gamma) = CFE\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^r\right\} = \frac{a^0\gamma^{-3} + a^1\gamma^{-2} + a^2\gamma^{-1} + a^3}{b^0\gamma^{-3} + b^1\gamma^{-2} + b^2\gamma^{-1} + b^3} \tag{16}$$

$$s^r \approx G_{\delta 5}(\gamma) = CFE\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^r\right\} = \frac{a_0\gamma^{-5} + a_1\gamma^{-4} + a_2\gamma^{-3} + a_3\gamma^{-2} + a_4\gamma^{-1} + a_5}{b_0\gamma^{-5} + b_1\gamma^{-4} + b_2\gamma^{-3} + b_3\gamma^{-2} + b_4\gamma^{-1} + b_5} \tag{17}$$

The coefficients of fifth order approximation as given in (17) are tabulated in Table 1 and Table 2. Third order coefficients can also be generated using same method.

Table 1. *Numerator coefficients in Delta Domain (Fifth order approximation)*

$$D=\begin{pmatrix} 0.0000000000009*r^{20}+0.0000000002*r^{19}+0.0000000007*r^{18} \\ -0.0000000125*r^{17}-0.0000000938*r^{16}+0.0000002678*r^{15} \\ +0.0000041147*r^{14}+0.0000023404*r^{13}-0.0000832278*r^{12} \\ -0.0001815327*r^{11}+0.0008236971*r^{10}+0.0028742227*r^{9} \\ -0.0037642038*r^{8}-0.0219278504*r^{7}+0.0029948516*r^{6} \\ +0.0913906642*r^{5}+0.0456801094*r^{4}-0.1882914088*r^{3} \\ -0.1659256439*r^{2}+0.1311668715*r+0.1407588614 \end{pmatrix}$$

Coefficients	Numerator
a_0	(11/3)^r*(-0.0000000008*r^(19)+0.0000808873*r^(13) -0.0000000118*r^(18)+0.0189329987*r^(7)-0.0006421643*r^(11) +0.0000064551*r^(14)-0.1370707737*r^(5)+0.0000000002*r^(20) +0.3542170527*r^(3)-0.00000000000083*r^(21)-0.0347587723*r^(2) +0.0008899811*r^(9)-0.0002647606*r^(12)+0.0000001739*r^(16) +0.0943855158*r^(6)-0.2719257329*r-0.1426112993*r^(4)-0.0256920542*r^(8) -0.0000043825*r^(15)+0.0036979198*r^(10)+0.0000001064*r^(17) +0.1407588613)/(r+1)/(D)
a_1	(11/3)^r*(0.1644434976*Δ*r^(4)+0.0918686023*r^(7)*Δ -0.0003734650*Δ*r^(12)+0.0000000593*Δ*r^(17) -0.3878961892*r^(5)*Δ-0.0000000008*Δ*r^(19)+0.0007351380*Δ*r^(11) -0.0000066339*Δ*r^(13)-0.0118943514*Δ*r^(9)+0.0000000024*Δ*r^(18) +0.0000000000004*Δ*r^(20)-0.0000013710*Δ*r^(15)+0.0239456455*Δ*r^(6) +0.0037394061*Δ*r^(10)-0.5508586034*r*Δ+0.5687601324*Δ-0.0000004127*Δ*r^(16) +0.0000185012*Δ*r^(14)-0.6689646772*Δ*r^(2)+0.8039575376*Δ*r^(3) -0.0180971662*Δ*r^(8))/(r+1)/(D)
a_2	(11/3)^r*(-0.0000001311*r^(16)*Δ*2-0.0000276031*r^(13)*Δ*2 +0.3130657428*r^(4)*Δ*2+0.4267807131*r^(3)*Δ*2+0.0000044924*r^(14)*Δ*2 +0.0006386468*r^(11)*Δ*2-0.0000629870*r^(12)*Δ*2-0.0000000000009*ΔΔ2*r^(19) +0.0536655937*r^(7)*Δ*2+0.0000000015*r^(18)*Δ*2-0.0077595346*ΔΔ2*r^(9) -0.0000000005*r^(17)*Δ*2+0.0051773501*ΔΔ2*r^(8)+0.0000005116*r^(15)*Δ*2 -0.2848454910*r*Δr 2-0.7146110314*r^(2)*Δ*2-0.2137256757*r^(5)*Δ*2 +0.5083225658*ΔΔ2+0.0001514540*r^(10)*Δ*2-0.061671367*r^(6)*Δ*2)/(r+1)/(D)
a_3	(11/3)^r*(0.0001204050*r^(11)*Δ^3+0.0000001372*Δ^3*r^(15) -0.0014131743*r^(9)*Δ^3+0.0000093853*r^(12)*Δ^3 +0.0766858880*r^(3)*Δ^3-0.0000000204*Δ^3*r^(14)-0.0000057087*r^(13)*Δ^3 -0.0000000056*Δ^3*r^(16)+0.15779943*Δ^3+0.0000000000010*r^(18)*Δ^3 -0.0000000014*Δ^3*r^(17)-0.0509720473*r*Δ^3+0.1140054487*Δ^3*r^(4) +0.0096773395*r^(7)*Δ^3-0.2337211260*Δ^3*r^(2)-0.0002793147*r^(10)*Δ^3 -0.0384625577*r^(5)*Δ^3+0.0037561161*r^(8)*Δ^3 -0.0275056629*Δ^3*r^(6))/(r+1)/(D)
a_4	(11/3)^r*(-0.0029236890*r*Δ^4+0.0000064494*r^(11)*Δ^4 -0.0000465989*r^(10)*Δ^4+0.0043795008*r^(3)*Δ^4-0.0300922900*Δ^4*r^(2) -0.0000770354*r^(9)*Δ^4+0.0150541247*Δ^4*r^(4)+0.0000000005*Δ^4*r^(16) +0.0000000070*Δ^4*r^(15)-0.0037739418*Δ^4*r^(6)-0.0000003008*r^(13)*Δ^4 +0.0005489612*r^(8)*Δ^4+0.0005371903*r^(7)*Δ^4+0.0000022018*r^(12)*Δ^4 -0.0000000000007*Δ^4*r^(17)+0.0200212359*Δ^4-0.0000000526*Δ^4*r^(14) -0.0021709653*r^(5)*Δ^4)/(r+1)/(D)
a_5	(11/3)^r*(-0.0000000021*Δ^5*r^(14)+0.0006633505*Δ^5*r^(4) +0.0000000000002*Δ^5*r^(16)+0.0000000919*Δ^5*r^(12)-0.0001641415*Δ^5*r^(6) +0.0000235385*Δ^5*r^(8)-0.0000019706*Δ^5*r^(10)+0.0008933494*Δ^5 -0.0013381808*Δ^5*r^(2))/(r+1)/D)

Table 2. *Denominator coefficients in Delta Domain (Fifth order approximation)*

Coefficients	**Denominator**
b_0	(0.00000000008*r^(21)+0.000000002*r^(20)+0.0000000008*r^(19) -0.0000000118*r^(18)-0.0000010632*r^(17)+0.0000001739*r^(16)+0.0000043825*r^(15) +0.0000064551*r^(14)-0.0000808873*r^(13)-0.0002647606*r^(12)+0.0006421643*r^(11) +0.0036979198*r^(10)-0.0008899811*r^(9)-0.0256920542*r^(8)-0.0189329987*r^(7) +0.0943855158*r^(6)+0.1370707737*r^(5)-0.1426112993*r^(4)-0.3542170527*r^(3) -0.0347587723*r^(2)+0.2719257329*r+0.1407588613)/(r+1)/(D)
b_1	(0.1644434976*Δ*r^(4)-0.9186860239*r^(7)*Δ-0.0003734650*Δ*r^(12) -0.0000000593*Δ*r^(17)+0.3878961892*r^(5)*Δ+0.0000000009*Δ*r^(19) -0.0007351380*Δ*r^(11)+0.0000066339*Δ*r^(13)+0.0118943514*Δ*r^(9) +0.0000000023*Δ*r^(18)+0.00000000004*Δ*r^(20)+0.0000013710*Δ*r^(15) +0.0239456455*Δ*r^(6)+0.0037394061*Δ*r^(10)+0.5508586034*r*Δ +0.5687601324*Δ-0.0000004127*Δ*r^(16)+0.0000185012*Δ*r^(14) -0.6689646772*Δ*r^(2)-0.8039575376*Δ*r^(3) -0.0180971662*Δ*r^(8))/(r+1)/(D)
b_2	(-0.0000001311*r^(16)*Δ^2+0.0000276031*r^(13)*Δ^2 +0.3130657428*r^(4)*Δ^2-0.4267807131*r^(3)*Δ^2+0.0000044924*r^(14)*Δ^2 -0.0006386468*r^(11)*Δ^2-0.0000629870706*r^(12)*Δ^2+0.00000000009*Δ^2*r^(19) -0.0536655937*r^(7)*Δ^2+0.000000000014807*r^(18)*Δ^2+0.0077595346*Δ^2*r^(9) +0.0000000009*r^(17)*Δ^2+0.0051773501*Δ^2*r^(8)-0.0000005116*r^(15)*Δ^2 +0.2848454910*r*Δ^2-0.7146110314*r^(2)*Δ^2+0.2137256757*r^(5)*Δ^2 +0.5083225658*Δ^2+0.0001514540*r^(10)*Δ^2 -0.06167136*r^(6)*Δ^2)/(r+1)/(D)
b_3	(-0.0001204050*r^(11)*Δ^3-0.0000001372*Δ^3*r^(15)+0.0014131743*r^(9)*Δ^3 +0.0000093853*r^(12)*Δ^3-0.0766858880*r^(3)*Δ^3-0.0000000204*Δ^3*r^(14) +0.0000057087*r^(13)*Δ^3-0.0000000056*Δ^3*r^(16)+0.1577994381*Δ^3 +0.0000000010*r^(18)*Δ^3+0.0000000014*Δ^3*r^(17)+0.0509720473*r*Δ^3 +0.1140054487*Δ^3*r^(4)-0.0096773395*r^(7)*Δ^3-0.2337211260*Δ^3*r^(2) -0.0002793147*r^(10)*Δ^3+0.0384625577*r^(5)*Δ^3+0.0037561161*r^(8)*Δ^3 -0.0275056629*Δ^3*r^(6))/(r+1)/(D)
b_4	(0.0029236890*r*Δ^4-0.0000064494*r^(11)*Δ^4 -0.0000465989*r^(10)*Δ^4-0.0043795008*r^(3)*Δ^4-0.0300922900*Δ^4*r^(2) +0.0000770354*r^(9)*Δ^4+0.0150541247*Δ^4*r^(4)+0.0000000006*Δ^4*r^(16) -0.0000000054*Δ^4*r^(15)-0.0037739418*Δ^4*r^(6)+0.0000003008*r^(13)*Δ^4 +0.0005489612*r^(8)*Δ^4-0.0005371903*r^(7)*Δ^4+0.0000022011*r^(12)*Δ^4 +0.0000000007*Δ^4*r^(17)+0.0200212359*Δ^4-0.0000000526*Δ^4*r^(14) +0.0021709653*r^(5)*Δ^4)/(r+1)/(D)
b_5	(-0.0000000021*Δ^5*r^(14)+0.0006633505*Δ^5*r^(4)+0.0000000002*Δ^5*r^(16) +0.0000000919*Δ^5*r^(12)-0.0001641415*Δ^5*r^(6)+0.0000235385*Δ^5*r^(8) -0.00000019706*Δ^5*r^(10)+0.0008933494*Δ^5 -0.0013381808*Δ^5*r^(2))/(r+1)/(D)

3. SIMULATION AND RESULT ANALYSIS

Here, three different fractional order transfer functions are considered as examples for simulation study to find the efficacy of the proposed method.

Example 1:

A half order differentiator (1/2nd) is considered in this example (Swarnakar et al., 2017) with transfer function as shown below:

$$G(s) = s^{0.5} \tag{18}$$

In delta domain, the mathematical formulation of the 1/2nd order differentiator is expressed by (19) where, the sampling rate is considered as $\Delta = 0.01\sec$.

$$s^{0.5} \approx G_{CFE\text{-}3P\text{-}GILOGdel}(\gamma) \approx CFE\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^{0.5}\right\}_{\Delta=0.01} \tag{19}$$

For a sampling rate of $\Delta = 0.01$ sec, The third and fifth order approximation of $s^{0.5}$ in delta domain after continued fraction expansion of $\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^{0.5}$ results in Eq. (20) and Eq. (21) respectively.

$$s^{0.5} \approx G_{CFE\text{-}3P\text{-}GOLOGdel3}(\gamma) \approx$$

$$CFE\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^{0.5}\right\}_{\Delta=0.01} = \frac{0.3932\,\gamma^3 + 0.01202\,\gamma^2 + 3.91e\text{-}05\,\gamma^1 + 3.191e\text{-}08}{\gamma^3 + 0.01114\,\gamma^2 + 2.587e\text{-}05\,\gamma^3 + 1.667e\text{-}08} \tag{20}$$

$$s^{0.5} \approx G_{CFE\text{-}3P\text{-}GILOGdel5}(\gamma) \approx$$

$$CFE\left\{\left(\frac{60\gamma + 60\Delta\gamma^2 + 11\Delta^2\gamma^3}{60 + 90\gamma\Delta + 36\Delta^2\gamma^2 + 3\Delta^3\gamma^3}\right)^{0.5}\right\}_{\Delta=0.01} \tag{21}$$

$$= \frac{0.2495\,\gamma^5 + 0.001962\,\gamma^4 + 2.198e\text{-}06\,\gamma^3 + 7.738e\text{-}10\,\gamma^2 + 1.078e\text{-}13\,\gamma + 5.195e\text{-}18}{\gamma^5 + 0.002716\,\gamma^4 + 2.013e\text{-}06\,\gamma^3 + 5.587e\text{-}10\,\gamma^2 + 6.517e\text{-}14\,\gamma + 2.713e\text{-}18}$$

For $G_{CFE\text{-}3P\text{-}GILOGdel5}(\gamma)$, Table 1 and Table 2 are used to compute the coefficients for the numerator and denominator taking $r = 0.5$ and $\Delta = 0.01$ sec. The frequency responses of delta domain transfer functions for both 3rd order and 5th order approximations are demonstrated in Figure 1. The magnitude and phase error in frequency responses for $G_{CFE\text{-}3P\text{-}GILOGdel3}(\gamma)$ and $G_{CFE\text{-}3P\text{-}GILOGdel5}(\gamma)$ are shown in Figure 2. The error in magnitude and phase are calculated with respect to the frequency response characteristics of the continuous time half order differentiator. It can be observed that the higher order approximation gives more closer result to the original half order differentiator.

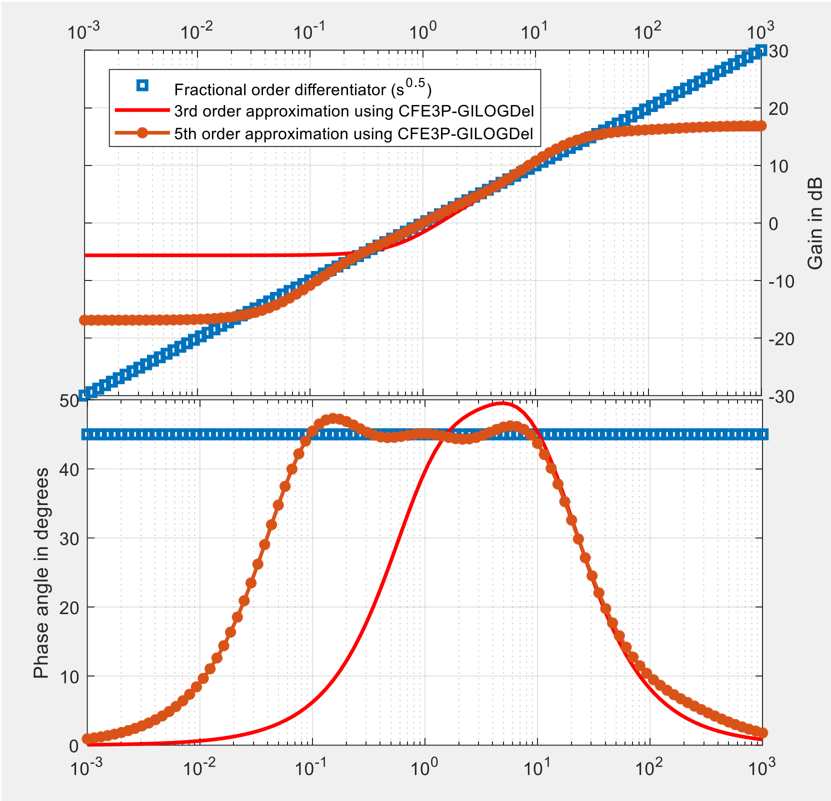

Figure 1. *Fifth order and third order approximation of $s^{0.5}$ in delta domain using proposed method (Frequency Response)*

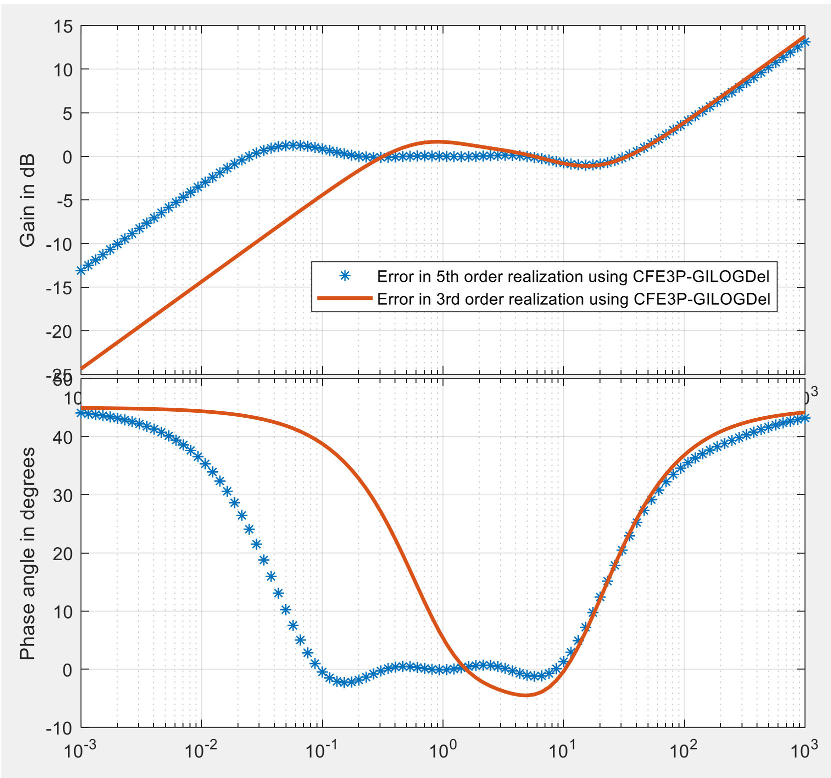

Figure 2. *Error comparison between fifth order and third order approximation of $s^{0.5}$ in delta domain using proposed method (magniude error and phase error)*

The magnitude response is more closer to the original half order diffrentiator than the phase response characteristics for a wide range of frequency. The error in frequency response characteristics between original system and rational approximation is compared on the basis of the maximum absolute values of magnitude and phase error and these values are tabulated in Table 3.

Table 3. *Absolute maximum phase error and magnitude error for discretization of 0.5th- order differentiator using CFE-3P-GILOGDel*

Approximation order	Maximum magnitude error (dB)	Maximum phase error (Degree)
Fifth	0.862	4.563
Third	1.02	27.5

The fifth order CFE approximation has been employed to develop the frequency responses for the various systems taken into consideration in this study since the approximation findings for the fifth order are more evident than those for the third order. At a sampling time of $\Delta = 0.01s$, the fifth order rational approximation of 1/2[nd] order differentiator is considered to be realized in discrete time domain based upon the four methods described in this paper namely CFE of Euler (CFEEuler), CFE of Tustin (CFETust), CFEDO and CFE of 3P-GILOG in Delta domain (CFE-3P-GILOGdel) and following rational transfer functions are obtained.

$$G_{CFEEuler5}(z)\Big|_{\Delta=0.01} = \frac{3940 - 6402\,z^{-1} + 1724\,z^{-2} + 1116\,z^{-3} - 324.2\,z_{-4} - 23.69\,z^{-5}}{124.6 - 109\,z^{-1} - 38.94\,z^{-2} + 30.9\,z^{-3} + 3.62\,z^{-4} - z^{-5}} \tag{22}$$

$$G_{CFETust5}(z)\Big|_{\Delta=0.01} = \frac{5572 - 9054\,z^{-1} + 2438\,z^{-2} + 1578\,z^{-3} - 458.4\,z^{-4} - 33.5\,z^{-5}}{124.6 - 109\,z^{-1} - 38.94\,z^{-2} + 30.9\,z^{-3} + 3.62\,z^{-4} - z^{-5}} \tag{23}$$

$$G_{CFEDO5}(\gamma)\Big|_{\Delta=0.01} = \frac{1.641e\text{-}06\,\gamma^{6} + 0.001001\gamma^{5} + 0.2051\gamma^{4} + 14.45\gamma^{3} + 66.61\gamma^{2} + 39.43\gamma + 1.875}{2.344e\text{-}07\,\gamma^{6} + 0.0001505\gamma^{5} + 0.03308\gamma^{4} + 2.676\gamma^{3} + 41.02\gamma^{2} + 66.02\gamma + 13.13} \tag{24}$$

$$G_{CFE\text{-}3P\text{-}GILOGdel5}(\gamma)\Big|_{\Delta=0.01} = \frac{\gamma^{5} + 0.02716\,\gamma^{4} + 0.0002013\gamma^{3} + 5.587e\text{-}07\,\gamma^{2} + 6.517e\text{-}10\,\gamma + 2.713e\text{-}13}{0.2495\,\gamma^{5} + 0.01962\,\gamma^{4} + 0.0002198\gamma^{3} + 7.738e\text{-}07\,\gamma^{2} + 1.078e\text{-}09\,\gamma + 5.195e\text{-}13} \tag{25}$$

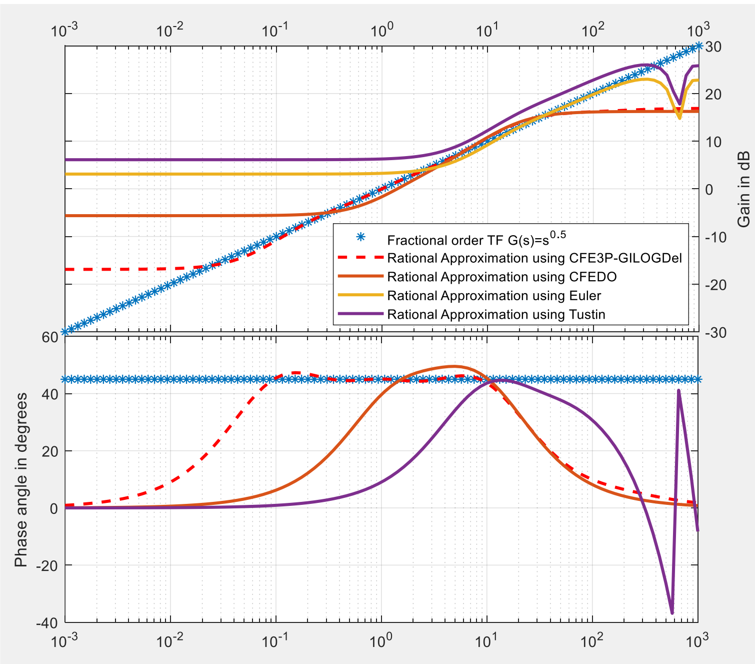

Figure 3. *Frequency response comparison after discretization of* $G(s)$ *using four methods for r=0.5 and* Δ*=0.01*

Example 2: A fractional order system is considered (Baranowski et al., 2015).

$$G_1(s) = \frac{0.5863}{0.2318 s^{0.958} + 1} \tag{26}$$

For the discretization of the above system, sampling time considered is $\Delta = 0.0001$ sec. The discretization of this fractional order transfer function results in four rational approximation T.F. as given by (27), (28), (29) and (30) by using four methods as mentioned in earlier section.

$$G_{1CFEEuler5}(z)\Big|_{\Delta=0.0001} = \frac{14.33 + 11.93\,z^{-1} - 10.72\,z^{-2} - 8.061\,z^{-3} + 1.169\,z^{-4} + 0.5863\,z^{-5}}{3.85e04 - 4.01e04\,z^{-1} - 2.275e04\,z^{-2} + 2.489e04\,z^{-3} + 1182\,z^{-4} - 1665\,z^{-5}} \tag{27}$$

$$G_{1CFETust5}(z)\Big|_{\Delta=0.0001} = \frac{14.33 + 11.93\,z^{-1} - 10.72\,z^{-2} - 8.061\,z^{-3} + 1.169\,z^{-4} + 0.5863\,z^{-5}}{7.476e04 - 7.792e04\,z^{-1} - 4.417e04\,z^{-2} + 4.837e04\,z^{-3} + 2294\,z^{-4} - 3236\,z^{-5}} \tag{28}$$

$$G_{1CFEDO5}(\gamma)\Big|_{\Delta=0.0001} = \frac{20.57\,\gamma^5 - 946.2\,\gamma^4 + 1.435e005\,\gamma^3 + 2.697e005\,\gamma^2 + 1.311e005\,\gamma + 1.27e004}{5061\,\gamma^5 + 5.024e004\,\gamma^4 + 3.515e005\,\gamma^3 + 5.168e005\,\gamma^2 + 2.233e005\,\gamma\gamma + 2.168e004} \tag{29}$$

$$G_{1\,CFE\text{-}3P\text{-}GILOGdel5}(\gamma)\Big|_{\Delta=0.0001} = \frac{0.5863\,\gamma^5 + 0.0001124\,\gamma^4 + 6.915e\text{-}09\,\gamma^3 + 1.719e\text{-}13\,\gamma^2 + 1.852e\text{-}18\,\gamma\gamma + 7.24e\text{-}24}{1.003\,\gamma^5 + 0.0002162\,\gamma^4 + 1.506e\text{-}08\,\gamma^3 + 4.202e\text{-}13\,\gamma^2 + 5.079e\text{-}18\,\gamma\gamma + 2.229e\text{-}23} \tag{30}$$

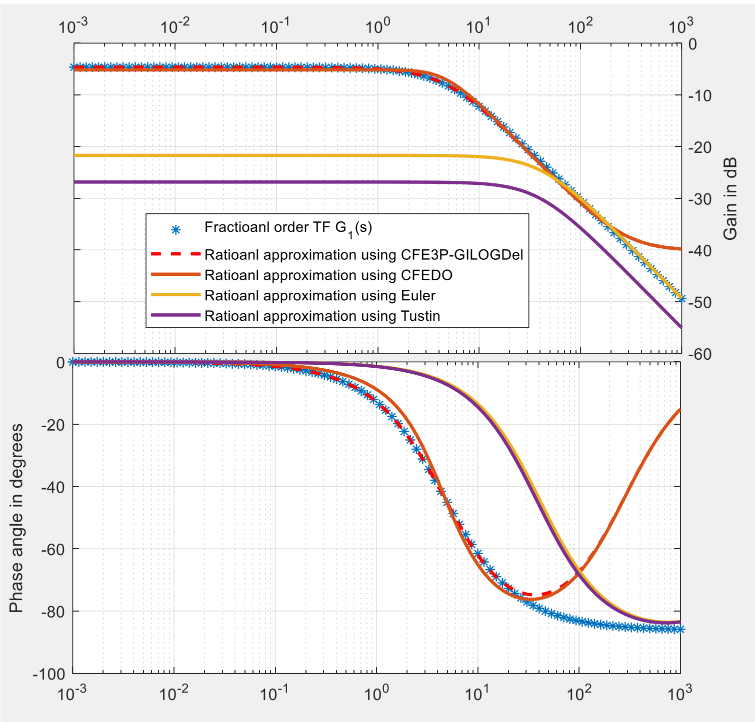

Figure 4. *Frequency response comparison after discretization of* $G_1(s)$ *using four methods for r=0.958 and* Δ*=0.0001*

Example 3:

The FO system (Kothari et al., 2019) is chosen and the transfer function is given as:

$$G_2(s) = \frac{0.979}{0.997 s^{0.716} + 0.9} \tag{31}$$

A higher sampling rate is considered in this case to discretize the system function represented by (31).The sampling rate is considered as $\Delta = 0.00001s$. CFEEuler, CFE Tust, CFEDO and CFE-3P-GILOGdel are the four methods used to discretize this continuous time transfer function results in four rational approximation T.F. as given by (32), (33), (34) and (35) respectively.

$$G_{2CFEEuler\,5}(z)\Big|_{\Delta=0.00001}=\frac{23.92+19.93\,z^{-1}-17.9\,z^{-2}-13.46\,z^{-3}+1.953\,z^{-4}+0.979\,z^{-5}}{1.655e05-1.726e05\,z^{-1}-9.778e04\,z^{-2}+1.071e05\,z^{-3}+5076\,z^{-4}-7165\,z^{-5}} \tag{32}$$

$$G_{2CFETustin5}(z)\Big|_{\Delta=0.00001}=\frac{23.92+19.93\,z^{-1}-17.9\,z^{-2}-13.46\,z^{-3}+1.953\,z^{-4}+0.979\,z^{-5}}{3.215e05-3.352e05\,z^{-1}-1.899e05\,z^{-2}+2.081e05\,z^{-3}+9858\,z^{-4}-1.392e04\,z^{-5}} \tag{33}$$

$$G_{2CFEDO5}(\gamma)\Big|_{\Delta=0.00001}=\frac{925.4\,\gamma^{5}+4.38e004\,\gamma^{4}+3.058e005\,\gamma^{3}+4.274e005\,\gamma^{2}+1.527e005\gamma+1.018e004}{1.13e004\,\gamma^{5}+1.995e005\,\gamma^{4}+7.43e005\,\gamma^{3}-2.905e030\,\gamma^{2}+1.982e005\,\gamma+1.118e004} \tag{34}$$

$$G_{2\,CFE\text{-}3P\text{-}GILOGdel5}(\gamma)\Big|=_{\Delta=0.00001}=\frac{0.979\,\gamma^{5}+2.301e\text{-}05\,\gamma^{4}+1.581e\text{-}10\,\gamma^{3}+4.188e\text{-}16\,\gamma^{2}+4.719e\text{-}22\,\gamma\gamma+1.909e\text{-}28}{1.093\,\gamma^{5}+3.777e\text{-}05\,\gamma^{4}+3.415e\text{-}10\,\gamma^{3}+1.101e\text{-}15\,\gamma^{2}+1.462e\text{-}21\,\gamma\gamma+6.851e\text{-}28} \tag{35}$$

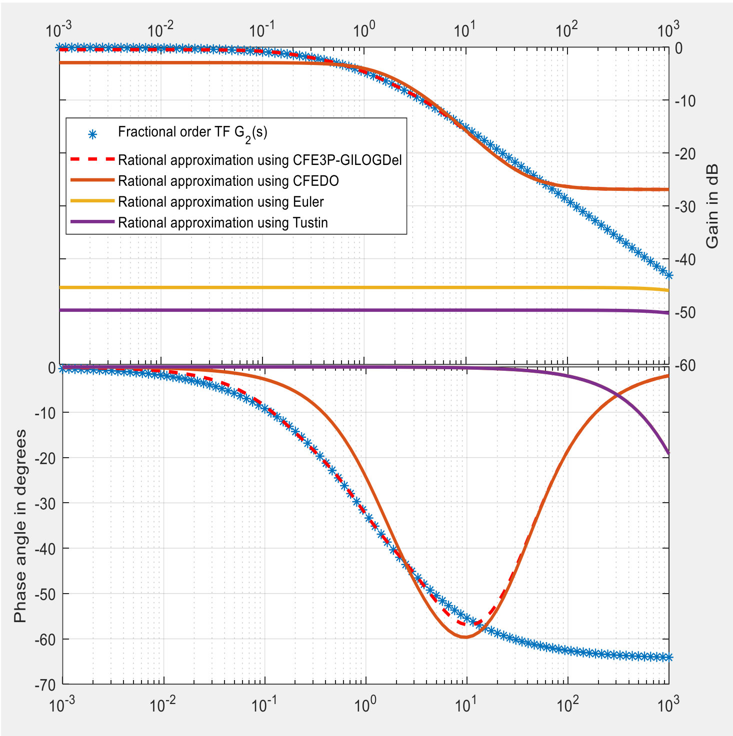

Figure 5. *Frequency response comparison after discretization of $G_2(s)$ using four methods for r=0.716 and Δ=0.00001*

Four different methods including the proposed method are used in this work to discretize the fractional order operator contained in three different continuous time fractional order systems. All systems' frequency responses (in fractional order), as well as the related discrete-time systems' frequency responses in shift as well as delta operator paramerterized domain are shown in Figure 3, Figure 4 and Figure 5 respectively. The magnitude approximation turns out to be superior over the phase approximation as can be shown from the above results.CFE-3P-GILOGDel produces the excellent frequency responses in the frequency range of (0.001 rad/s to 1000 rad/sec) for all the three cases mentioned above. The proposed method is thus more promising than the other three approaches for discretization of fractional order system or operator in discrete time domain. Additionally, the superiority of the suggested approach is demonstrated by a comparison of the results with another method created in the delta domain. From Figure 5, it can be observed that at high sampling time ($\Delta=0.00001$sec), the frequency responses using The CFE-3P-GILOGDel method is very much closer to the frequency responses of the corresponding continuous time system. Therefore, the continuous time findings and the discrete time results with high sampling rate in delta domain are obtained in hand to hand, leading to the development of a unified framework for direct discretization o FOO in complex delta domain.

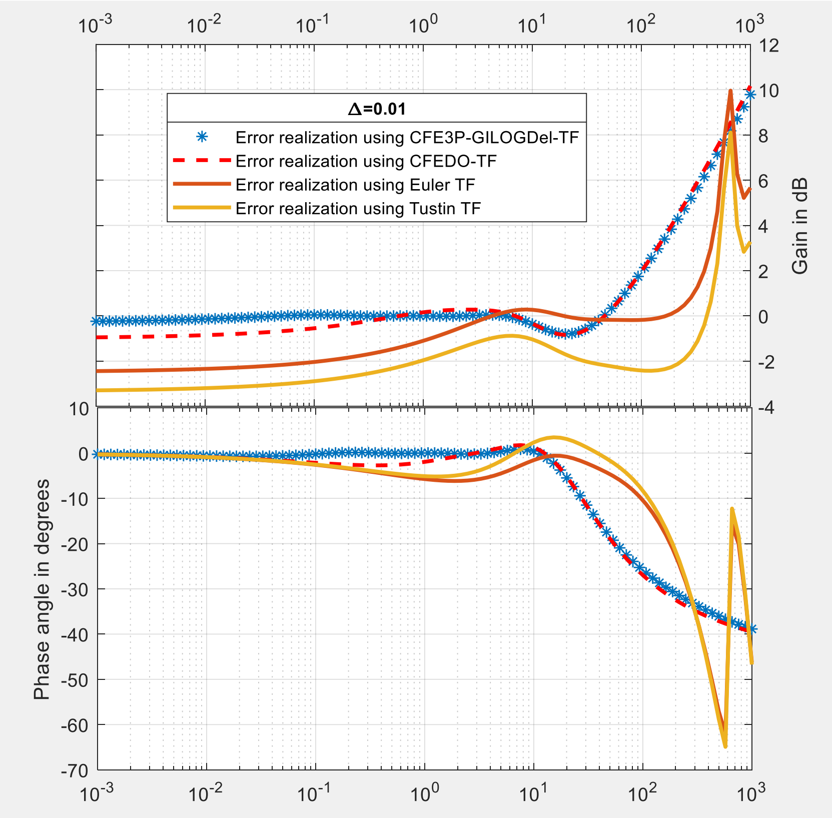

Figure 6. *Magnitude and phase error after discretization of* $G(s)$ *using four methods, at r=0.5 and* Δ*=0.01*

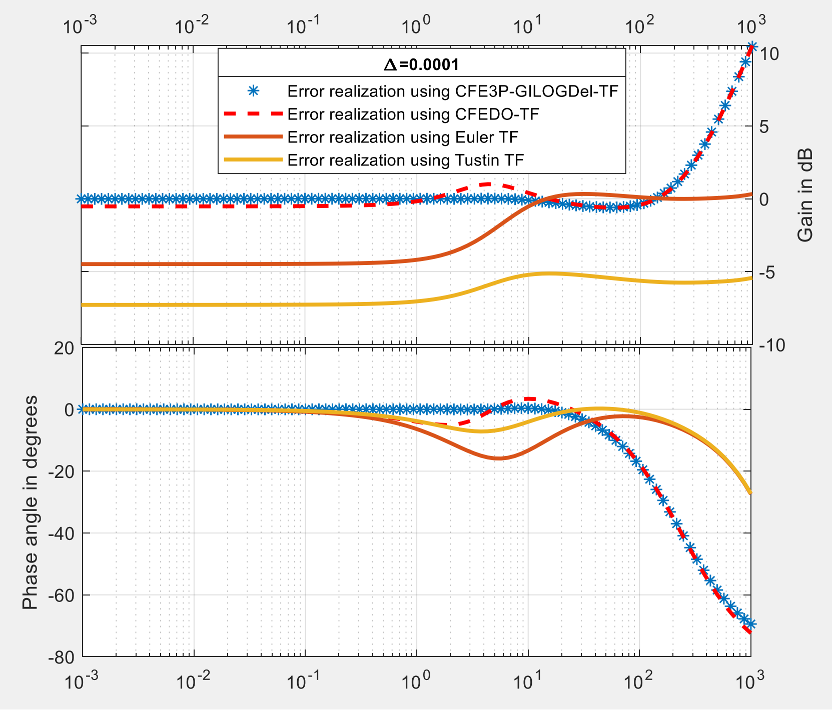

Figure 7. *Magnitude and phase error after discretization of* $G_1(s)$ *using four methods at r=0.958 and* Δ*=0.0001*

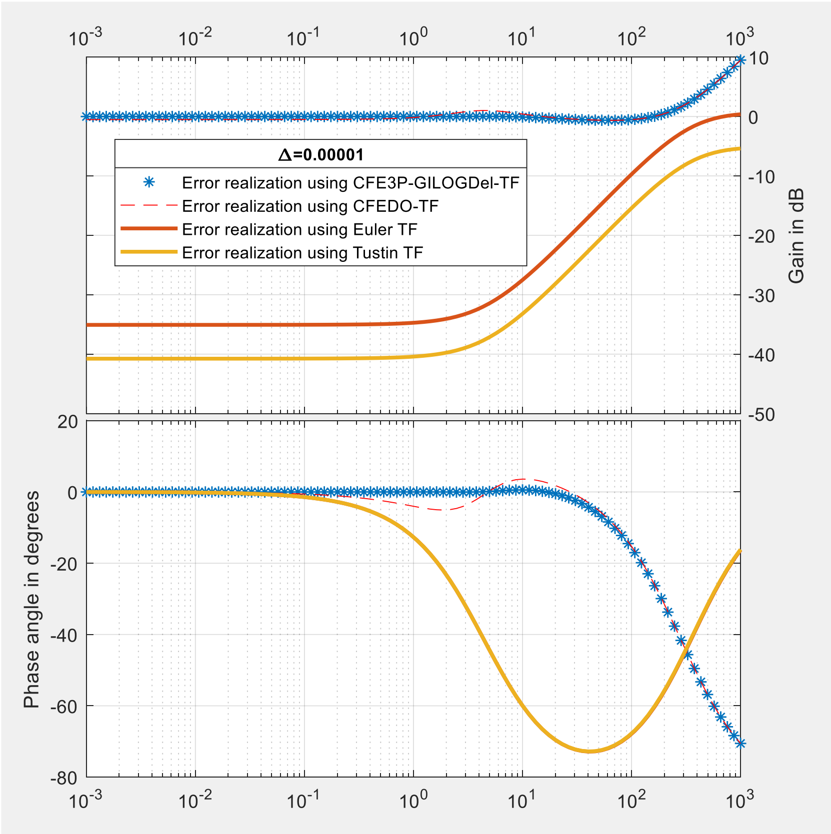

Figure 8. *Magnitude and phase error after discretization of* $G_2(s)$ *using four methods at r=0.716 and Δ=0.00001*

The error in approximation of the fractional order operators using four methods are shown in Figure 6, Figure 7 and Figure 8. In each system, magnitude and phase errors are calculated through the absolute value of maximum magnitude and phase error using each of the four methods and they are tabulated in Table 4.

Table 4. *Absolute maximum magnitude error and phase error for four discretization methods for different systems*

FOS	Maximum magnitude error (dB)				Maximum phase error (degree)			
	CFE-3P GILOG Del	CFEDO	Euler	Tustin	CFE-3P GLOGD el	CFEDO	Euler	Tustin
$G(s) = s^{0.5}$	13.1065	24.3833	53.0980	56.1083	44.0834	44.9375	45	45
$G_1(s) = \frac{0.5863}{0.2318s^{0.958}+1}$	9.5916	9.4648	17.0709	22.2271	47.0785	47.8811	10.7379	70.7052
$G_2(s) = \frac{0.979}{0.997s^{0.716}+0.985}$	16.2447	16.2449	31.2399	35.4573	54.7988	54.8412	62.1541	62.1493

For the systems described in example 1, example 2 and example 3, the errors in magnitude and phase are less when the system is discretized using CFE3PGILOGDel method of discretization in delta domain than the other methods. At a very high limiting value of sampling rate, $\Delta = 0.00001s$, the maximum absolute magnitude error and phase error is much higher in case of discretization using Tustin and Euler method in z -domain in comparison to the discretization using delta operator parameterization. The results is visualized be visualized from Figure 8. The proposed method is proved to be superior as compared to the other methods in the literature for all three examples as stated in this paper.

The pole and zero positions of the rational transfer functions obtained using four different methods are calculated and plotted in Figure 9, Figure 10, Figure 11 and Figure 12.

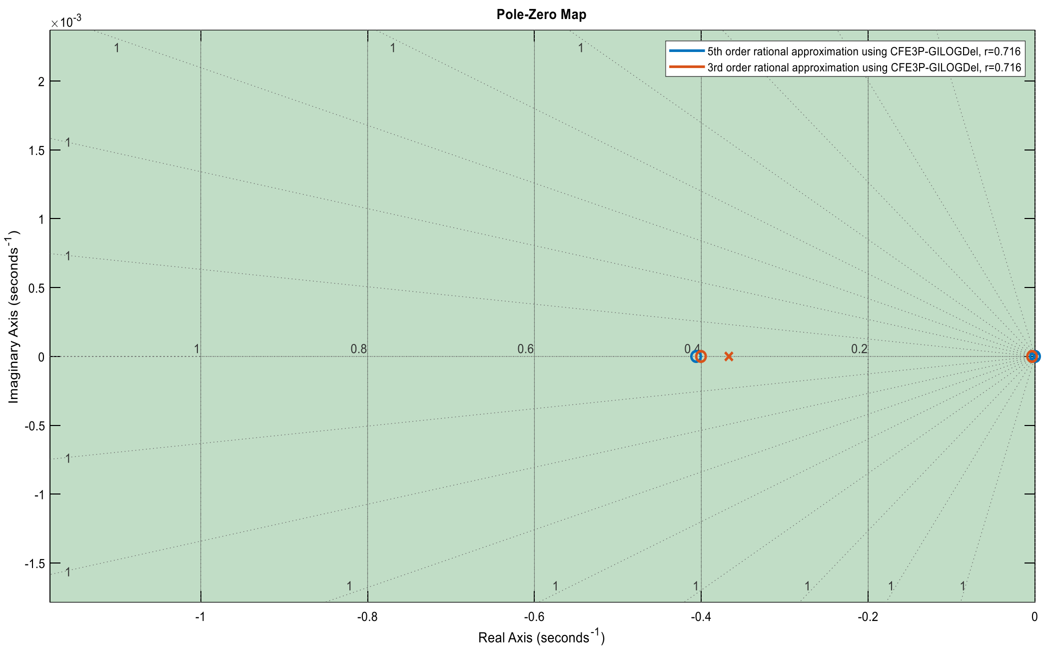

Figure 9. *The pole zero plot of third-order and fifth order approximation of* $s^{0.716}$*using CFE-3P-GILOGDel*

Another delta domain based discretization method (CFEDO) is adopted in this paper to discretize the fractional order operator. From Table 4, the superiority of the proposed method over CFEDO in terms of magnitude and phase error of approximation is explained. The pole–zero plot for the rational approximation of $s^{0.716}$ using CFE3GILOGDEl method and CFEDO method are shown in Figure 9 and Figure 10 respectively. It can be observed from Figure 10 that the system's rational transfer function is unstable because the system's poles are located in an unstable area, as opposed to how the poles were produced using third-order and fifth-order approximations using CFE3GILOGDEl are lying in the stable region as can be observed in Figure 9. Therefore, it is evident that the proposed method delivers preferable approximation amidst all the four discretization methods and is a viable alternative in the literature of direct discretization of fractional order operator in delta domain.

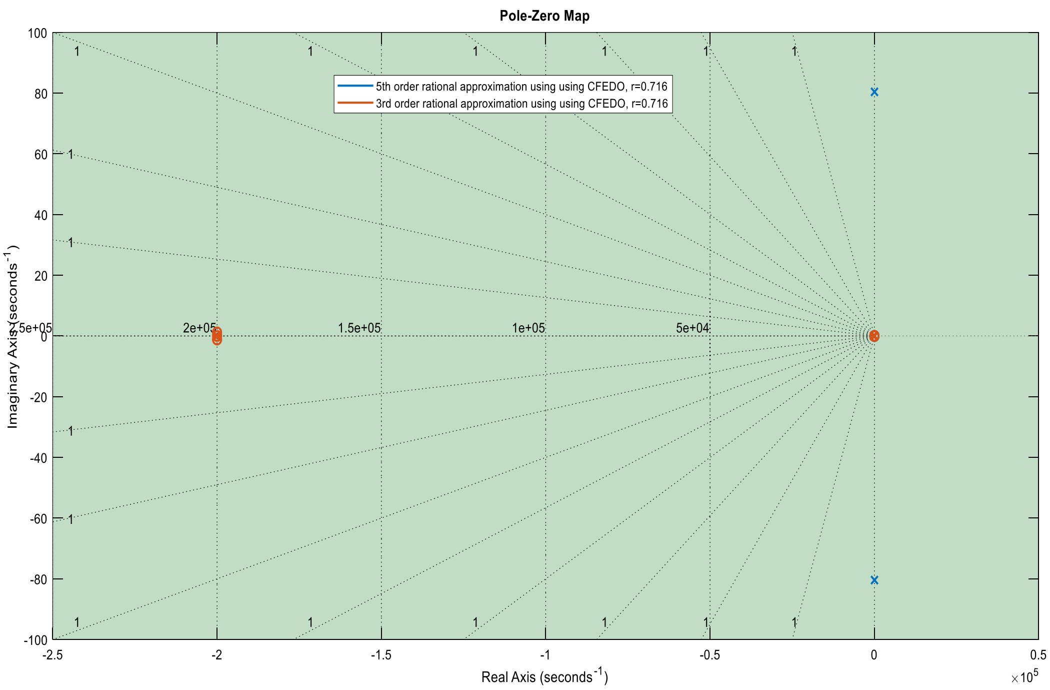

Figure 10. *The pole zero plot of third-order and fifth order approximation of* $s^{0.716}$*using CFE-DO method*

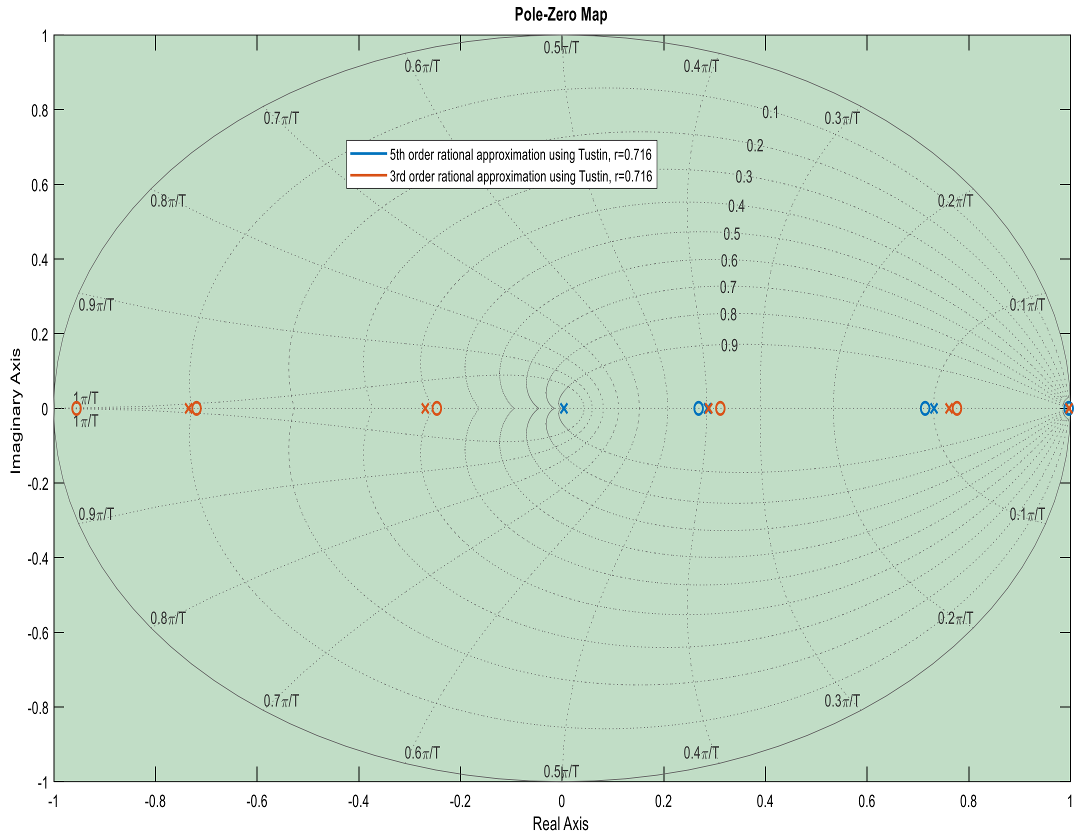

Figure 11. *The pole zero plot of third-order and fifth order approximation of $s^{0.716}$ using Tustin method*

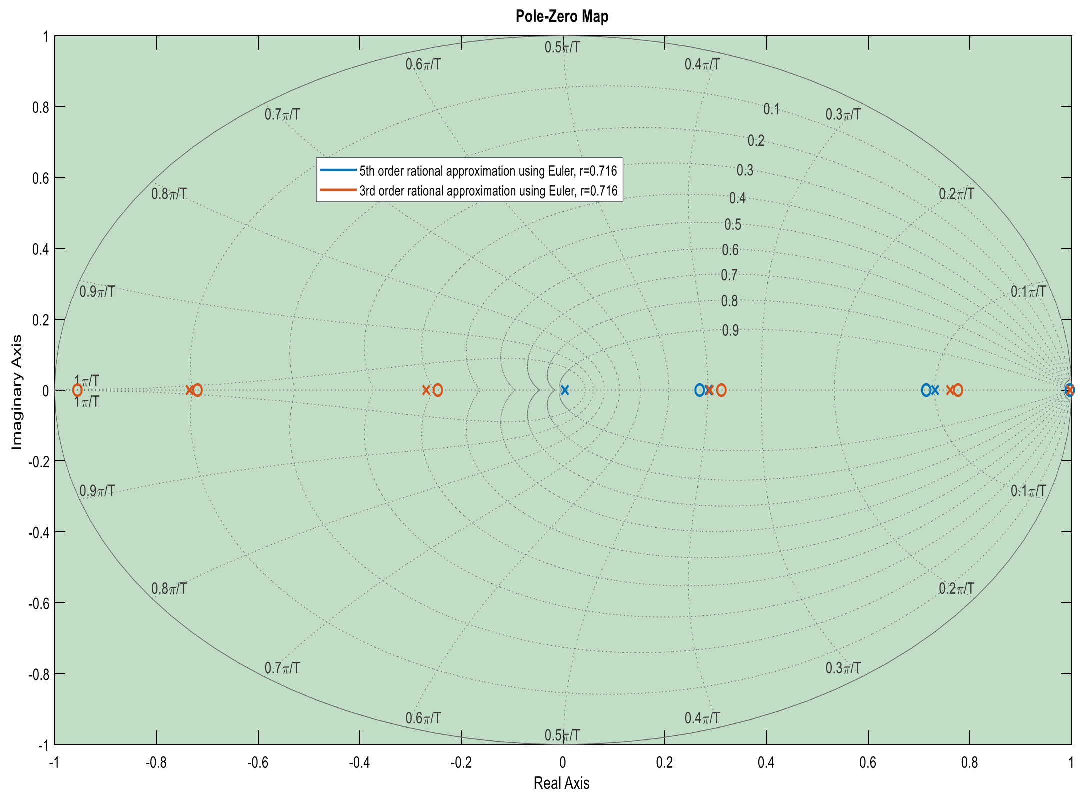

Figure 12. *The pole zero plot of third-order and fifth order approximation of $s^{0.716}$ using Euler method*

An anlaysis is made in this section to show that direct discretization of the fractional order operator is preferable ($s^{\pm\mu}, 0 < \mu < 1$) over the indirect discretization in complex delta domain. For the illustration purpose a 1/2nd order differentiator is considered for the discretization purpose. This operator is discretized using indirect discretization approach proposed by Oustaloup approximation (Baranowski et al., 2015) method as an intermediate step.

Rational approximation of $s^{0.5}$ is obtained using (Azarmi et al., 2016) as given in (36).

$$s^{0.5}\Big|_{Ous} = \frac{\begin{array}{c}10\,s^{11} + 940.9\,s^{10} + 2.674e04\,s^{9} + 2.909e05\,s^{8} \\ + 1.303e06\,s^{7} + 2.473e06\,s^{6} + 2.006e06\,s^{5} + \\ 6.956e05\,s^{4} + 1.021e05\,s^{3} + 6178\,s^{2} + 143\,s + 1\end{array}}{\begin{array}{c}s^{11} + 143\,s^{10} + 6178\,s^{9} + 1.021e05\,s^{8} \\ + 6.956e05\,s^{7} + 2.006e06\,s^{6} + 2.473e06\,s^{5} + \\ 1.303e06\,s^{4} + 2.909e05\,s^{3} + 2.674e04\,s^{2} + 940.9\,s + 10\end{array}} \tag{36}$$

Equation. (36) is discretized in delta domain to get the rational approximation of $s^{0.5}$ with $\Delta = 0.0001$ sec.

$$s^{0.5}\Big|_{ind} = \frac{1.04e004\,\gamma^{5} + 1.56e005\,\gamma^{4} + 4.366e005\,\gamma^{3} - 2.914e032\,\gamma^{2} + 4.474e004\,\gamma + 945}{947.2\,\gamma^{5} + 4.477e004\,\gamma^{4} + 3.125e005\,\gamma^{3} + 4.366e005\,\gamma^{2} + 1.559e005\,\gamma + 10395} \tag{37}$$

The rational approximation of $s^{0.5}$ in delta domain using proposed direct discretization method (CFE3GILOGDEl) is illustrated in (38) with $\Delta = 0.0001$ sec.

$$s^{0.5}\Big|_{dir} = \frac{0.2495\,\gamma^{5} + 0.0001962\gamma^{4} + 2.198e\text{-}08\,\gamma^{3} + 7.738e\text{-}13\,\gamma^{2} + 1.078e\text{-}17\,\gamma + 5.195e\text{-}23}{\gamma^{5} + 0.0002716\gamma^{4} + 2.013e\text{-}08\,\gamma^{3} + 5.587e\text{-}13\,\gamma^{2} + 6.517e\text{-}18\,\gamma\gamma + 2.713e\text{-}23} \tag{38}$$

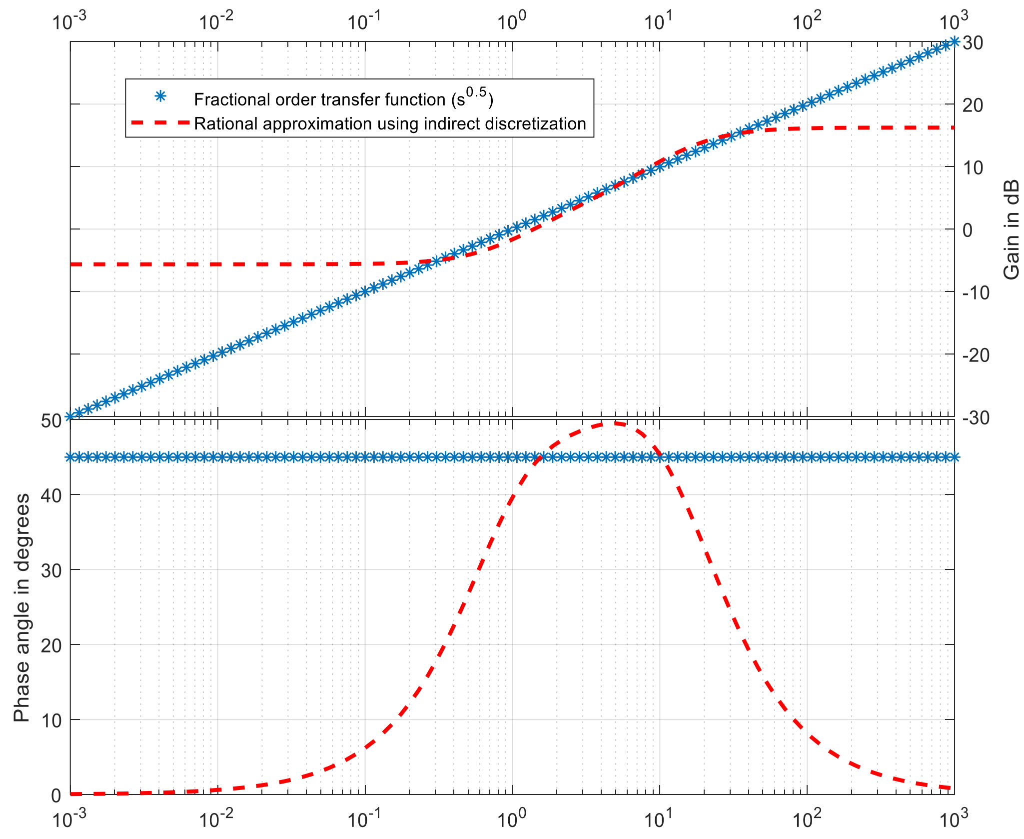

Figure 13. *Frequency response of transfer function obtained using indirect discretization of $s^{0.5}$ at Δ=0.0001 sec*

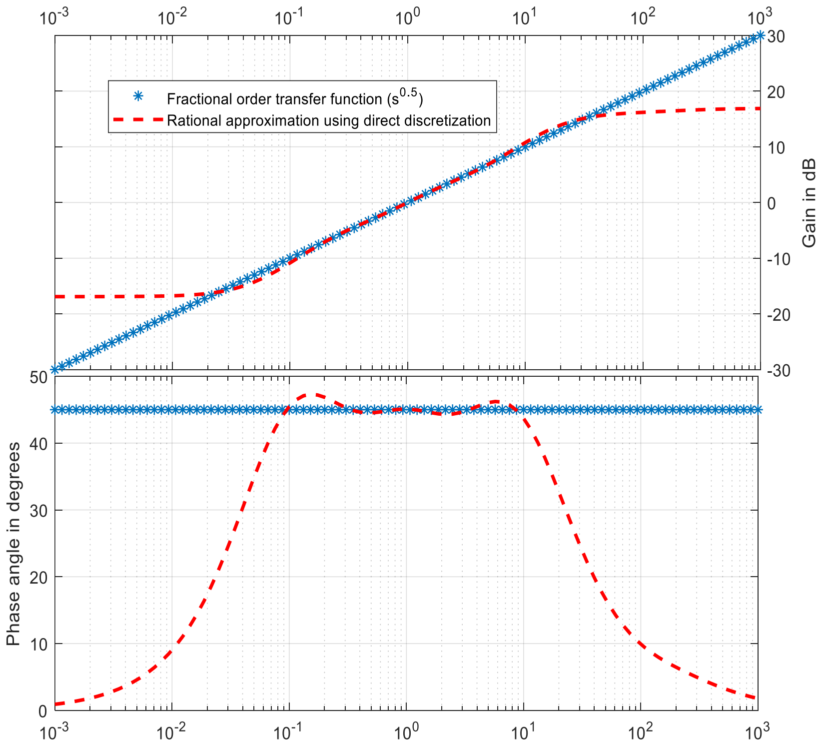

Figure 14. *Frequency response transfer function obtained using direct discretization of $s^{0.5}$ at Δ=0.0001 sec*

From the Figure 13 and Figure 14, it is clearly seen that the magnitude and phase plot obtained using the direct discretization in delta domain resembles to that of the 1/2nd order differentiator continuous time domain. In otherway, there is a notable deviation in the magnitude and phase plots of the rational transfer function obtained using indirect discretization via Oustaloup approximation technique with respect to the frequency reponse characterstics of 1/2nd order differentiator continuous time domain. Thus, direct discretization of the fractional operator in delta domain proves to be superior over indirect discretization of fractional order operator.

4. CONCLUSION

In this paper, a new direct discretization method for fractional order operator is proposed. The discretization of the fractional order operator or system using tradional discrete z-domain can not provide meaningful information when sampled at a very fast sampling frequency. The corrsponding delta operator parameterized systems provides the continuous time results at high sampling frequency. For obtaining rational transfer function, mapping between the continuous time and delta domain variables are required. In this work, an approximation mapping between the s-domain and δ-domain is established through trapezoidal quadrature rule and traditional CFE method is used to obtain the rational transfer function corresponding to the fractional order operator in discrete delta domain.

From the simulation results, it is observed that the the proposed discretization method using delta operator is producing gratifying frequency response for the approximated transfer functions in delta domain. At fast sampling rate (Δ=0.00001 sec), the delta operator parameterized system produces almost same result as that of the response obtained with original fractional order systems (Figure 5). The Table 4 illustrates the minimum errors in magnitude and phase for delta operator parameterized approximation of FOO. The superiority of the proposed method over the indirect discretization method is also verified as can be observed from Figure 13 and Figure 14. Therefore, the method proposed is said to be a viable alternate method of the direct discretization in discrete delta domain for discretizing the fractional order operator or systems available in the literature.

CONFLICT OF INTEREST

The authors declare no conflict of interest.

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