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Research Article

A First Countable T_1 Topology as related to Statistical Metric Spaces

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Abstract: In this paper, we study the conditions under which one can obtain a first countable and T_1 topology without the left-continuity and symmetry, which have an important role in the statistical metric space theory.

İstatistiksel Metrik Uzaylarla ilgili Birinci Sayılabilir T_1 Topolojisi

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Anahtar Kelimeler
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Öz: Bu çalışmada, istatiksel metrik uzay teorisinde önemli bir rolü olan soldan süreklilik ve simetri koşulları olmadan, hangi koşullar altında, birinci sayılabilir ve T_1 olan bir topoloji elde edilebildiği incelenmiştir.

1. Introduction

The theory of statistical metric space initially was started in 1942 by Menger who introduced statistical metric space while studying about some physical measurements (Menger, 1942). Schweizer and Sklar introduced Menger statistical metric space by using the condition of triangle inequality defined by Menger via t-norm (Schweizer & Sklar, 1960), (Schweizer et al., 1960). In these studies, they defined a different statistical metric space by using triangle functions, instead of t-norm, to obtain the triangle inequality. Moreover, they proved that it is possible to determine a uniformity in this space under certain conditions. Thus, a topology was defined on a statistical metric space induced by the triangle functions.

Let X be a nonempty set. First of all, we state some basic notions of statistical metric spaces. Recall that a function F, defined on extended real numbers, is called a distribution function, if it is monotone increasing and $F(-\infty) = 0$, $F(+\infty) = 1$.

A distribution function is called a distance function if $F(0) = 0$. For example, the unit step function μ_θ, defined by,

$$\mu_\theta(t) = \begin{cases} 0, & if\ t \leq \theta \\ 1, & if\ t > \theta \end{cases} \tag{1}$$

where θ is any real number, is a left-continuous distance function. The collection of all distance and all left-continuous distance functions is denoted by Δ^+ and Δ_L^+ respectively.

Definition 1.1 A function $\varepsilon : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if the following are satisfied

- **(i)** $\varepsilon(\varepsilon(c,d), e) = \varepsilon(c, \varepsilon(d,e))$
- **(ii)** $\varepsilon(c,d) = \varepsilon(d,c)$
- **(iii)** $\varepsilon(c,d) \leq \varepsilon(e,f)\ whenever\ c \leq e\ and\ d \leq f$
- **(iv)** $\varepsilon(c,1) = c$

for all $c, d, e, f \in [0,1]$.

Example 1.1 The following are t-norms: $\varepsilon_{\min}(c,d) = \min\{c,d\}$, $\varepsilon_{\max}(c,d) = \max\{c + d - 1, 0\}\ and\ \varepsilon^*(c,d) = c.d$, for $c, d \in [0,1]$.

Definition 1.2 Let $F : X \times X \to \Delta_L^+$ be a function and t a t-norm. By F_{cd}, we will denote the value F at the point (c,d). Consider the following conditions, for $c, d, e \in X$ and $r, s > 0$,

- **(S1)** $F_{cd} = \mu_0\ if\ and\ only\ if\ \ c = d$
- **(S2)** $F_{cd} = F_{dc}$
- **(S3)** $F_{ce}(r+s) = 1\ \ whenever\ \ F_{cd}(r) = 1\ and\ F_{de}(s) = 1$
- **(S4)** $F_{cd}(r+s) \geq t(\,F_{ce}(r), F_{ed}(s))$.

The couple (X, F) provided that the conditions (S1), (S2) and (S3) is called Statistical metric space. Satisfying (S1), (S2) and (S4), the triple (X, F, t) is called Statistical Menger space.

It is proved by Schweizer et al. (1960) that a statistical metric space is metrizable if

$$sup_{a<1} t(a,a) = 1. \tag{2}$$

Definition 1.3 A positive real valued function ξ, defined on $X \times X$ is called quasi-metric if $\xi(c,c) = 0$ and $\xi(c,e) \leq \xi(c,d) + \xi(d,e)$, for $c, d, e \in X$. We say that X is quasi-metrizable if the collection $\{B_\varepsilon(c) : \varepsilon > 0\}$ is a local basis at each point $c \in X$, here $B_\varepsilon(c) = \{d \in X : \xi(c,d) < \varepsilon\}$ where ξ is a quasi-metric.

Let us show that the collection of all subsets of X by 2^X and the diagonal set by Δ. Let $A, B \in 2^{X \times X}$. By $A^{-1}\ and\ A \circ B$, it is shown that the set of all points (c,d) provided that $(d,c) \in A$ and that the set of all points (c,e) such that there exists $d \in X$ satisfying $(c,d) \in B$ and $(d,e) \in A$, respectively. Recall that a sub-family $\mathfrak{U}$ of $2^X \backslash \{\emptyset\}$ is called a filter on X if the intersection of two elements of it belongs to $\mathfrak{U}$ and if $A \subseteq X, U \epsilon\, \mathfrak{U}$ with $U \subseteq A$, then $A \in \mathfrak{U}$.

Definition 1.4 A filter S on $X \times X$ is said to be a quasi-uniformity if the following are satisfied

- **(i)** $\Delta \subseteq C\ \ for\ all\ \ C \in S$

(ii) $For\ each\ C \in S,\ there\ exists\ D \in S\ such\ that\ D \circ D \subseteq C.$

If there exists a quasi-uniformity on X, then X is called a quasi-uniform space.

Each quasi-uniformity generates a topology and this topology is more interesting than the topology generated by a uniformity which is always regular and T_1. But the topology generated by a quasi-uniformity is T_1 if and only if the intersection of all members is the diagonal set and it is not regular. The existence of a quasi-uniformity for a given topology was first proved in (Krishnan, 1955) then in (Császár, 1960). The direct topological proof was given in (Pervin, 1962).

Recall that a sub-collection $\mathfrak{B}$ of a given quasi-uniformity S is said to be a basis for S, if each element of S contains at least one element of $\mathfrak{B}$.

Theorem 1.1 There exists a quasi-uniformity having a sub-collection $\mathfrak{B}$ of $2^{X\times X}$ as a basis if and only if the following are satisfied

- **(i)** Each element of $\mathfrak{B}$ contains the diagonal
- **(ii)** For each $C \in \mathfrak{B}$ there exists $D \in \mathfrak{B}$ such that $D \circ D \subseteq C$
- **(iii)** For any $D_1, D_2 \in \mathfrak{B}$, there exists $D_3 \in \mathfrak{B}$ such that $D_3 \subseteq D_1 \cap D_2$.

Our main goal is to obtain a first countable T_1 topology generated by a quasi-uniformity induced by a triple (X, F, t) under weaker conditions on F, which is not left continuous, than the conditions (S_i) for $i = 1,2,3,4$ given in Definition 1.2.

For the terminology of quasi-uniform spaces, not explained or proved in this paper, we refer to (Kelley, 1975; Fletcher & William, 1982) and for probabilistic metric spaces theory we refer to (Schweizer & Sklar, 1983).

2. Material and Methods

The notions we used in the following results defined in (Schweizer et al., 1960) and (Shi-sheng, 1988). Moreover, a detailed review of quasi-uniformity and statistical metric space can be found in (Bilgin, 2021).

3. Results

For a nonempty set X, consider the function $F : X \times X \to \Delta^+$ (not necessarily left-continuous) and a t-norm t with $t \geq \varepsilon_{\max}$. Define the set for $\lambda > 0$, $U_\lambda = \{(x, y) \in X \times X : F_{xy}(\lambda) > 1 - \lambda\}$ and the function $\omega : X \times X \to [0, \infty[$, for $d, e \in X$, by

$$\omega(d, e) = \sup\{\alpha \in \mathbb{R} : (d, e) \notin U_\alpha\}. \tag{3}$$

We first remark the following trivial proposition.

Proposition 3.1 Let F, U_λ and ω be as above, where $\lambda > 0$ any real number. Then, for any all $d, e \in X$ and α, β positive real numbers, the following are satisfied

- **(i)** $(d, e) \in U_\lambda \Longrightarrow \omega(d, e) \leq \lambda$
- **(ii)** $\omega(d, e) < \lambda \Longrightarrow (d, e) \in U_\lambda$
- **(iii)** $U_\alpha \subseteq U_\beta\ for\ \alpha \leq \beta.$

We will refer to the following conditions: for any $c, d, e \in X$,

(Q1) If $c = d$ then $F_{cd} = \mu_0$
(Q2) $F_{ce}(r + s) \geq t\big(F_{cd}(r), F_{de}(s)\big)$ for $r, s > 0$
(Q3) If $F_{cd} = \mu_0$. then $c = d$.
Proposition 3.2 Let α, β be positive real numbers. If (Q2) is satisfied, then $U_\alpha \circ U_\beta \subseteq U_{\alpha+\beta}$

Proof Let $(c,d) \in U_\alpha \circ U_\beta$. Then there exists $e \in X$ satisfying $(c,e) \in U_\beta$ and $(e,d) \in U_\alpha$. As $t \geq \varepsilon_{\max}$, the condition (Q2) implies that $F_{cd}(\beta+\alpha) \geq F_{ce}(\beta) + F_{ed}(\alpha) - 1 > 1-(\alpha+\beta)$. By definition, $(c,d) \in U_{\alpha+\beta}$.

Proposition 3.3 The function ω, defined as above, satisfies the triangle inequality if (Q2) holds.

Proof Let $c,d,e \in X$ and $\lambda > 0$. By Proposition 3.1(ii), we get

$$(c,e) \in U_{\omega(c,e)+\frac{\lambda}{4}} \text{ and } (e,d) \in U_{\omega(e,d)+\frac{\lambda}{4}}. \tag{4}$$

Hence $(c,d) \in U_{\omega(e,d)+\frac{\lambda}{4}} \circ U_{\omega(c,e)+\frac{\lambda}{4}}$.

By Proposition 3.2, we get $(c,d) \in U_{\omega(c,e)+\omega(c,e)+\frac{\lambda}{2}}$ and by Proposition 3.1(i) we conclude that

$$\omega(c,d) \leq \omega(c,e) + \omega(e,d) + \frac{\lambda}{2} \quad for\ all\ \lambda > 0. \tag{5}$$

Thus $\omega(c,d) \leq \omega(c,e) + \omega(e,d)$.

Proposition 3.4 Suppose that the condition (Q1) holds. Then $\omega(c,c) = 0$, where ω defined as above.

Proof Let $\lambda > 0$ and $c \in X$. As $F_{cc}\left(\frac{\lambda}{2}\right) = \mu_0\left(\frac{\lambda}{2}\right) = 1 > 1 - \frac{\lambda}{2}$ under the hypothesis, we have $(c,c) \in U_{\lambda/2}$. It follows from Proposition 3.1(i) that $\omega(c,c) < \lambda$. Thus $\omega(c,c) = 0$.

Now it follows from Proposition 3.3 and 3.4 that the following corollary.

Corollary 3.1 The function ω, defined as above, is a quasi-metric on X if the conditions (Q1) and (Q2) are satisfied.

Theorem 3.1 X is quasi-uniformizable under the conditions (Q1) and (Q2).

Proof We will prove that there exists a quasi-uniformity on X having the collection $\mathfrak{A} = \{U_\gamma : \gamma > 0\}$ as a basis. Let us prove that $\mathfrak{A}$ satisfies the conditions of Theorem 1.1. Indeed, let $\gamma > 0$. By Proposition 3.4, $\Delta \subseteq U_\gamma$, by Proposition 3.2 $U_{\gamma/2} \circ U_{\gamma/2} \subseteq U_\gamma$, and by Proposition 3.1(iii) $U_{\min\{\alpha,\beta\}} \subseteq U_\alpha \cap U_\beta$ for each $\alpha, \beta > 0$. Thus for each $c \in X$, the collection $\mathfrak{A}(c) = \{U_\gamma(c) : \gamma > 0\}$ is a local basis at c, where $U_\gamma(c) = \{d \in X : (c,d) \in U\}$. We also remark that the collection $\mathfrak{B}(c) = \{B_\varepsilon(c) : \varepsilon > 0\}$ is a local basis at c, where $B_\varepsilon(c)$ is the set defined in Definition 1.3 by using the quasi-metric $\omega(c,d) = \sup\{\alpha \in \mathbb{R} : (c,d) \notin U_\alpha\}$. Indeed, Proposition 3.1(i) and (ii) imply that $U_{\varepsilon/2}(c) \subseteq B_\varepsilon(c)$ and $B_\varepsilon(c) \subseteq U_\varepsilon(c)$, respectively.

Corollary 3.2 Let (X,F,t) be the triple satisfying the conditions (Q1), (Q2) and (Q3). Then there exists a first countable T_1 topology on X.

Proof It follows from Theorem 3.1 that there exists a quasi-uniformity on X. Thus there exists a topology τ on X induced by this quasi-uniformity. This topology is first countable since the family $\left(B_{1/n}(c)\right)_{n\in\mathbb{N}}$ is a countable local basis at each $c \in X$ as we proved in Theorem 3.1. To prove τ is T_1, it is enough to show that the inclusion $\bigcap\{U_\lambda : \lambda > 0\} \subseteq \tau$ holds as the converse inclusion holds by Proposition 3.4. Let $(c,d) \in U_\lambda$ for all $\lambda > 0$. Then

$$F_{cd}(\lambda) \geq F_{cd}\left(\frac{1}{m}\right) \geq F_{cd}\left(\frac{1}{n}\right) \quad \text{for all } n \geq m \tag{6}$$

where $\frac{1}{m} < \lambda$. Hence $F_{cd} = \mu_0$. Taking into account the condition (Q3), we conclude that $(c, d) \in \Delta$.

4. Discussion and Conclusion

Our discussion answers the question as to under what conditions related to statistical metric spaces a set can be topologized as a first countable and T_1. In this regard, we determined three conditions and showed that the left-continuity and symmetry, in the sense of statistical metric space, are not necessary conditions.

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