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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS OF FENG-LIU TYPE Θ-CONTRACTIONS ON M-METRIC SPACES

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Abstract. In this paper, we present a new fixed point result for multivalued θ-contractions on M-complete M-metric spaces using Feng-Liu's technique. Our results extend and generalize some related fixed point theorems in the literature.

1. Introduction and preliminaries

Matthews [9] introduced the notion of the partial metric space, which is more general than the metric space, and presented a fundamental fixed point theorem on partial metric spaces. Then, Asadi, Karapınar and Salimi [5] extended the concept of partial metric spaces to M-metric spaces and presented some fixed point theorems for single valued mappings on M-metric spaces.

Definition 1.1 ([5])**.** *Let X be a nonempty set. A function $m : X \times X \to [0, \infty)$ is called an M-metric if the following conditions are satisfied: for all $x, y, z \in X$*

(m1) $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$,
(m2) $m_{xy} = \min\{m(x, x), m(y, y)\} \leq m(x, y)$,
(m3) $m(x, y) = m(y, x)$,
(m4) $m(x, x) - m_{xy} \leq m(x, z) - m_{xz} + m(z, y) - m_{zy}$.

Then, the pair (X, m) is called an M-metric space.

Next, Altun et al. [4] studied on the topological structures of M-metric space, and then presented some fixed point theorems for multivalued mappings of Feng-Liu type on M-metric space (see [4, 14, 15] and references therein). Let (X, m) be an M-metric space, $x \in X$ and $\varepsilon > 0$. The open ball with centered $x \in X$ and

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radius ε is defined by

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{xy} + \varepsilon\}.$$

Then, the family

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\}$$

is a base of a topology on X. This topology is defined by τ_m and the closure of a subset A of X with respect to τ_m by $\overline{A^m}$.

Example 1.1. *Let $X = \left\{\frac{1}{n^2} : n \in \{1,2,3,\cdots\}\right\} \cup \{0\}$ and $m : X \times X \to [0,\infty)$ be defined by $m(x,y) = \min\{x,y\}$. Then, (X,m) is a M-metric space. In this case, we have $\tau_m = \{\emptyset, X\}$.*

Definition 1.2. *Let (X,m) be an M-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then,*

(1) *$\{x_n\}$ is said to be M-converges to x if and only if*

$$\lim_{n\to\infty} [m(x_n,x) - m_{x_n x}] = 0.$$

(2) *$\{x_n\}$ is said to be M-Cauchy sequence if $\lim_{n,m\to\infty}[m(x_n,x_m) - m_{x_n x_m}]$ exists and is finite.*

(3) *(X,m) is said to be M-complete if every M-Cauchy sequence M-converges to a point $x \in X$.*

Note that the M-convergence of a sequence on an M-metric space coincides with the convergence with respect to τ_m.

Altun et al [4] proved the following fixed point theorem, which is M-metric version of Feng-Liu's fixed point theorem [12].

Theorem 1.1. *Let (X,m) be a M-complete M-metric space and $T : X \to C_m(X)$ (the family of all nonempty closed subsets of X) be a multivalued map. If there exist two constants $b,c \in (0,1)$ such that for all $x \in X$ with $m(x,Tx) > 0$ there is $y \in T_b^x(m)$ satisfying*

$$m(y,Ty) \le cm(x,y),$$

where

$$T_b^x(m) = \{y \in Tx : bm(x,y) \le m(x,Tx)\},$$

and

$$m(x,Tx) = \inf\{m(x,y) : y \in Tx\}.$$

Then, T has a fixed point in X provided that $c < b$ and the function $f(x) = m(x,Tx)$ is lower semicontinuous with respect to τ_m.

On the other hand, Jleli and Samet [12] introduced the concept of θ-contraction and then gave a fixed point theorem. So that, they generalize Banach contraction principle which is a quite different from many results in literature.

Let Θ be the family of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing;

(Θ2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim\limits_{n\to\infty} t_n = 0$ if and only if $\lim\limits_{n\to\infty} \theta(t_n) = 1$;

(Θ3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim\limits_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Example 1.2. *Let us consider the functions* $\theta_1(t) = e^{\sqrt{t}}$, $\theta_2(t) = e^{\sqrt{te^t}}$, $\theta_3(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{t^\alpha}\right)$ *for* $0 < \alpha < 1$ *and* $\theta_4(t) = e^{\sqrt{t^2+t}}$. *Then it can be seen that* $\theta_i \in \Theta$ *for* $i \in \{1, 2, 3, 4\}$.

Jleli and Samet [12] proved the following theorem.

Theorem 1.2. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping. Suppose that there exist* $\theta \in \Theta$ *and* $k \in (0, 1)$ *such that*

$$x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \theta\left(d(Tx, Ty)\right) \leq \left[\theta\left(d(x, y)\right)\right]^k.$$

Then, T *has a unique fixed point.*

Then, taking into account the family Θ, many authors have presented some fixed point results for both single valued and multivalued mappings on metric space. For example, in [2] the authors obtained a fixed point theorem for compact set valued mappings on metric space. Also, a similar result for closed set valued mappings on metric spaces have been provided by taking the following condition (Θ4) into consideration (see [1, 2, 3, 6, 7, 8, 10, 11, 13] and references therein):

(Θ4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by Ξ the set of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying (Θ1)-(Θ4).

In this paper, we present Feng-Liu type fixed point theorems for multivalued mappings considering the both families Θ and Ξ in M-metric spaces.

2. Main Result

Let (X, m) be an M-metric space. $P_m(X)$ and $C_m(X)$ denotes the family of all nonempty subsets and the family of all nonempty closed (w.r.t. τ_m) subsets of X, respectively. Also, we indicate the family of all subsets A of X satisfying the following property by $A_m(X)$: for all $x \in X$

$$\left.\begin{array}{l} m(x, A) = 0 \Rightarrow x \in A \\ \text{and} \\ m(x, A) > 0 \Rightarrow \exists a_x \in A,\ m(x, A) = m(x, a_x) \end{array}\right\}.$$

If (X, m) is a metric space, then it is clear that

$$A_m(X) = \{A \subseteq X : \forall x \in X, \exists a_x \in A, m(x, A) = m(x, a_x)\}$$

and also $A_m(X) \subseteq C_m(X)$. Let $T : X \to P_m(X)$ be a mapping, $\theta \in \Theta$ and $b \in (0, 1]$. For $x \in X$ with $m(x, Tx) > 0$, consider the set

$$\Theta_b^x(m) = \left\{y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \theta\left(m(x, Tx)\right)\right\}.$$

It is clear that if $b_1 \leq b_2$, then $\Theta_{b_1}^x(m) \subseteq \Theta_{b_2}^x(m)$ for fixed $x \in X$.

Theorem 2.1. *Let* (X, m) *be an* M*-complete* M*-metric space and* $T : X \to A_m(X)$ *be a multivalued map* $\theta \in \Theta$. *If there exists a constant* $k \in (0, 1)$ *such that for any* $x \in X$ *with* $m(x, Tx) > 0$, *there is* $y \in \Theta_b^x(m)$ *for* $b \in (0, 1]$ *satisfying*

$$\theta\left(m(y, Ty)\right) \leq \left[\theta\left(m(x, y)\right)\right]^k, \tag{2.1}$$

then T *has a fixed point in* X *provided that* $k < b$ *and the function* $f(x) = m(x, Tx)$ *is lower semi-continuous with respect to* τ_m.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $m(x, Tx) > 0$. Since $Tx \in A_m(X)$ for every $x \in X$, the set $\Theta_b^x(m)$ is nonempty for any $b \in (0,1]$. Let $x_0 \in X$ be any initial point, then there exists $x_1 \in \Theta_b^{x_0}(m)$ such that

$$\Theta\left(m\left(x_1, Tx_1\right)\right) \leq \left[\Theta\left(m(x_0, x_1)\right)\right]^k$$

and for $x_1 \in X$, there exists $x_2 \in \Theta_b^{x_1}(m)$ satisfying

$$\Theta\left(m\left(x_2, Tx_2\right)\right) \leq \left[\Theta\left(m(x_1, x_2)\right)\right]^k .$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in \Theta_b^{x_n}(m)$ and

$$\theta\left(m\left(x_{n+1}, Tx_{n+1}\right)\right)) \leq \left[\theta\left(m(x_n, x_{n+1})\right)\right]^k . \tag{2.2}$$

We will show that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in \Theta_b^{x_n}(m)$, we have

$$\left[\theta\left(m\left(x_n, x_{n+1}\right)\right)\right]^b \leq \theta\left(m(x_n, Tx_n)\right). \tag{2.3}$$

From (2.2) and (2.3), we have

$$\theta\left(m\left(x_{n+1}, Tx_{n+1}\right)\right) \leq \left[\theta\left(m\left(x_n, Tx_n\right)\right)\right]^{\frac{k}{b}}$$

and

$$\theta\left(m\left(x_{n+1}, x_{n+2}\right)\right) \leq \left[\theta\left(m\left(x_n, x_{n+1}\right)\right)\right]^{\frac{k}{b}} .$$

By the way, we can obtain

$$1 < \theta\left(m\left(x_n, x_{n+1}\right)\right) \leq \left[\theta\left(m\left(x_0, x_1\right)\right)\right]^{\left(\frac{k}{b}\right)^n} \tag{2.4}$$

and

$$1 < \theta\left(m\left(x_n, Tx_n\right)\right) \leq \left[\theta\left(m\left(x_0, Tx_0\right)\right)\right]^{\left(\frac{k}{b}\right)^n} . \tag{2.5}$$

Letting $n \to \infty$ in (2.4),

$$\lim_{n\to\infty} \theta\left(m\left(x_n, x_{n+1}\right)\right) = 1.$$

From (Θ_2),

$$\lim_{n\to\infty} m\left(x_n, x_{n+1}\right) = 0^+.$$

Similarly, we can obtain

$$\lim_{n\to\infty} m\left(x_n, Tx_n\right) = 0.$$

So from (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $\varepsilon = \ell/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\left| \frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} - \ell \right| \leq \varepsilon.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} \geq \ell - \varepsilon = \varepsilon.$$

Then, for all $n \geq n_0$,

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1\right],$$

where $A = \frac{1}{\varepsilon}$.

Suppose now that $\ell = \infty$. Let $\varepsilon > 0$ be arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$\frac{\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1}{\left(m\left(x_n, x_{n+1}\right)\right)^r} \geq \varepsilon.$$

This implies that, for all $n \geq n_0$,

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1\right],$$

where $A = \frac{1}{\varepsilon}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\theta\left(m\left(x_n, x_{n+1}\right)\right) - 1\right].$$

Using (2.4), we obtain for all $n \geq n_0$

$$n\left[m\left(x_n, x_{n+1}\right)\right]^r \leq An\left[\left[\theta\left(m\left(x_0, x_1\right)\right)\right]^{\left(\frac{k}{b}\right)^n} - 1\right].$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\left[m\left(x_n, x_{n+1}\right)\right]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$

$$m\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{1/r}}. \tag{2.6}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using (m4) and from (2.6), we have

$$\begin{aligned}
m\left(x_n, x_m\right) - m_{x_n x_m} &\leq \left[m\left(x_n, x_{n+1}\right) - m_{x_n x_{n+1}}\right] + \left[m\left(x_{n+1}, x_m\right) - m_{x_{n+1} x_m}\right] \\
&\leq \left[m\left(x_n, x_{n+1}\right) - m_{x_n x_{n+1}}\right] + \cdots + \left[m\left(x_{m-1}, x_m\right) - m_{x_{m-1} x_m}\right] \\
&\leq m\left(x_n, x_{n+1}\right) + m\left(x_{n+1}, x_{n+2}\right) + \cdots + m\left(x_{m-1}, x_m\right) \\
&\leq \sum_{i=n}^{m-1} m(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} m(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}$$

By the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$, letting to limit $n \to \infty$, we get

$$\lim_{n,m\to\infty} \left[m\left(x_n, x_m\right) - m_{x_n x_m}\right] = 0.$$

Hence, we find that $\{x_n\}$ is an M-Cauchy sequence. Because X is M-complete, one sees that there exists $z \in X$ such that

$$\lim_{n\to\infty} \left[m\left(x_n, z\right) - m_{x_n z}\right] = 0$$

that is, $\{x_n\}$ converges to z with respect to τ_m. Now, we show that z is fixed point of T. On the other hand, from (2.5) and (Θ_2), we have $\lim_{n\to\infty} m\left(x_n, Tx_n\right) = 0$. Since $f(x) = m(x, Tx)$ is lower semi-continuous with respect to τ_m, then

$$0 < m\left(z, Tz\right) = f(z) \leq \lim_{n\to\infty} \inf f\left(x_{n'}\right) = \lim_{n\to\infty} \inf m\left(x_n, Tx_n\right) = 0.$$

This is a contradiction. Hence, T has a fixed point. □

To give a fixed point result for $C_m(X)$ valued multivalued mappings, we will consider the family Ξ.

Theorem 2.2. *Let (X, m) be an M-complete M-metric space and $T : X \to C_m(X)$ be a multivalued map $\theta \in \Xi$. If there exists a constant $k \in (0, 1)$ such that for all any $x \in X$ with $m(x, Tx) > 0$, there is $y \in \Theta_b^x(m)$ for $b \in (0, 1)$ satisfying*

$$\theta\left(m(y, Ty)\right) \leq \left[\theta\left(m(x, y)\right)\right]^k.$$

Then, T has a fixed point in X provided that $k < b$ and the function $f(x) = m(x, Tx)$ is lower semi-continuous with respect to τ_m.

Proof. Suppose that T has no fixed point. Then, for all $x \in X$ we have $m(x, Tx) > 0$. Indeed, if $m(x, Tx) = 0$, then $x \in \overline{Tx}^m = Tx$. Since $\theta \in \Xi$, for any $x \in X$ with $m(x, Tx) > 0$, the set $\Theta_b^x(m)$ is nonempty for any $b \in (0, 1)$. Indeed, using the property (Θ_4), we obtain

$$\begin{aligned}
\Theta_b^x(m) &= \left\{y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \theta\left(m(x, Tx)\right)\right\} \\
&= \left\{y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \theta\left(\inf\left\{m(x, y) : y \in Tx\right\}\right)\right\} \\
&= \left\{y \in Tx : \left[\theta\left(m(x, y)\right)\right]^b \leq \inf\left\{\theta\left(m(x, y) : y \in Tx\right)\right\}\right\} \\
&\neq \emptyset.
\end{aligned}$$

The rest of the proof can be completed as in the proof of Theorem 2.1 by considering the $Tz \in C_m(X)$. □

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