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A new characterization of the Aminov surface with regards to its Gauss map in $\mathbb{E}^4$

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Abstract
In this study, we focus on the Aminov surface with regard to its Gauss map in $\mathbb{E}^4$. Firstly, we write the covariant derivatives according to linear combinations of orthonormal vectors and separate the equalities using the Gauss and Weingarten formulas. Then, we get the Laplacian of the Gauss map. After giving some conditions, we yield the following results: Aminov surfaces can not have a harmonic Gauss map and can not have a pointwise one-type Gauss map of the first kind in $\mathbb{E}^4$. Further, we give an example of a helical cylinder which is also congruent to an Aminov surface. Lastly, we obtain the conditions of having a pointwise one-type Gauss map of the second kind.

1. Introduction

Surfaces given with Monge patch which are also called digital graph surfaces have many advantages by means of visualization. These types of surfaces can be covered by just a few atlas that are produced with Monge patches. The presentation of $3-$dimensional form is $\varphi(u,v)=(u,v,g(u,v))$ where g is a differentiable function [7].

Digital graph surfaces (Monge surfaces) in $4-$dimensional spaces have also attracted attention as $3-$dimensional spaces. These surfaces are given by $z=g(u,v)$, $w=h(u,v)$ where u,v,z,w are the cartezian coordinates [1,3]. Some of them are translation surfaces, factorable (homothetical) surfaces, $TF-$type surfaces etc.[13, 14]. In particular, translation surface has many applications in architecture. They have a quadrilateral form and thanks to this property, they are used for free form glass structures [8].

The idea of finite (limited) type submanifolds was announced by Chen in the 1970s and has grown into a widely used concept in studies of Euclidean and semi-Euclidean spaces. This concept has been extended to differentiable transformations, especially to the Gauss map of submanifolds.The condition for a surface (or a submanifold) to have a pointwise one-type Gauss map is

$$\Delta G=\lambda\left(G+\vec{C}\right) \tag{1}$$

where λ is a differentiable function and $\vec{C}$ is a constant vector in the $n-$dimensional Euclidean (or semi-Euclidean) space. If $\vec{C}=0$, the surface is said to have a pointwise one-type Gauss map of the first kind, otherwise the second kind [6].
One of the popular surfaces, Aminov surface, can be represented by a Monge patch

$$z(u,v)=r(u)\cos v,\quad w(u,v)=r(u)\sin v, \tag{2}$$

where $r(u)$ is a differentiable function [1, 3]. In [3, 4], the authors handled Aminov surfaces according to their curvatures in $4-$dimensional Euclidean and Minkowski spaces. The other studies about some surfacesin $\mathbb{E}^4$ can be found in [9, 10, 11, 12].

In this study, we evaluate Aminov surfaces with regards to their Gauss maps in $\mathbb{E}^4$. In section 3, we obtain the covariant derivatives of

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orthonormal vectors on the surface and arrange them in accordance with the Gauss and Weingarten formulas. We write the shape operator matrices of the surface. In Section 4, we compute the Laplacian of the Gauss map of these surfaces. Then, we prove that Aminov surfaces can not have a harmonic Gauss map and can not have a pointwise one-type Gauss map of the first kind in $\mathbb{E}^4$. In an example, we get the Laplace transform of the Gauss map of a helical cylinder. Further, we present the conditions for Aminov surfaces to have a pointwise one-type Gauss map of the second kind.

2. Basic Concepts

Let $M:\varphi(u,v)$ denote a surface patch in Euclidean $4-$space $\mathbb{E}^4$. $\{\varphi_u,\varphi_v\}$ spans the tangent space of M. The first fundamental form coefficients are calculated by $F=\langle\varphi_u,\varphi_v\rangle, E=\langle\varphi_u,\varphi_u\rangle, G=\langle\varphi_v,\varphi_v\rangle$. Hence, M is known as regular in case of $W^2=EG-F^2\neq 0.$

Euclidean $4-$space can be considered as a decomposition of tangential and normal component of M for each point p:

$$\mathbb{E}^4=T_pM\oplus T_p^{\perp}M.$$

Let the orthonormal tangent vectors and normal vector field of the surface be represented by φ_1,φ_2 and η, respectively. $\tilde{\nabla}$ and ∇ denotes the Levi-Civita connections, then the Weingarten and Gauss formulas are known as

$$\tilde{\nabla}_{\varphi_1}\eta=-A_\eta\varphi_1+D_{\varphi_1}\eta, \tag{3}$$

$$\tilde{\nabla}_{\varphi_1}\varphi_2=\nabla_{\varphi_1}\varphi_2+h(\varphi_1,\varphi_2),$$

where A_η is the shape operator, D is the normal connection and h is the second fundamental tensor [2, 5].

Assuming that $\varphi_u=\frac{\partial}{\partial u}\varphi(u,v)$ and $\varphi_v=\frac{\partial}{\partial v}\varphi(u,v)$ are orthogonal, the orthonormal tangent vectors are

$$\varphi_1=\frac{\varphi_u}{E},\quad \varphi_2=\frac{\varphi_v}{G} \tag{4}$$

The normal frame field $\{\eta_1,\eta_2\}$ is chosen as $\langle\eta_1,\eta_1\rangle=1, \langle\eta_2,\eta_2\rangle=1, \langle\eta_1,\eta_2\rangle=0,$ and the quadruple $\{\varphi_1,\varphi_2,\eta_1,\eta_2\}$ is positively oriented in $\mathbb{E}^4$. Thus, according to orthogonal tangent vectors φ_1 and φ_2, the second fundamental form is written as follows

$$\begin{aligned} h(\varphi_1,\varphi_1)&=h_{11}^1\eta_1+h_{11}^2\eta_2,\\ h(\varphi_1,\varphi_2)&=h_{12}^1\eta_1+h_{12}^2\eta_2,\\ h(\varphi_2,\varphi_2)&=h_{22}^1\eta_1+h_{22}^2\eta_2,\end{aligned} \tag{5}$$

where $h_{ij}^k\ (i,j,k=1,2)$ are the coefficients of the second fundamental form.

With the help of Gauss and Weingarten formulas, $K=det(A_{\eta_1})+det(A_{\eta_2})$ gives the Gauss curvature of M and $H=\frac{trh}{2}$ gives the mean curvature. Therefore, the surface M is known as minimal (flat), if mean curvature (Gauss curvature) vanishes[5].

In $n-$dimensional Euclidean space, let $\{e_1,e_2\}$ be tanget vector fields of a surface, and the normal vectors denoted by $\{e_3,\ldots,e_n\}$, for the orthonormal frame $\{e_1,e_2,...,e_n\}$. Then, Gauss map of the surface is given by

$$G(p)=(e_1\wedge e_2)(p), \tag{6}$$

and the Laplace of any differentiable function ψ on M is known as

$$\Delta\psi=-\left(\tilde{\nabla}_{\varphi_i}\tilde{\nabla}_{\varphi_i}\psi-\tilde{\nabla}_{\nabla_{\varphi_i\varphi i}}\psi\right). \tag{7}$$

(see, [5]).

3. Aminov Surfaces in $4-$dimensional Euclidean Space

Definition 1: Let $M:\varphi(u,v)$ be a regular surface in $\mathbb{E}^4$. If M is parametrized by the Monge patch

$$\varphi(u,v)=(u,v,r(u)\cos v, r(u)\sin v), \tag{8}$$

where $r(u)$ is a differentiable function, then this surface is called as Aminov surface in $\mathbb{E}^4$ [1, 3].

Assume that M is an Aminov surface in four-dimensional Euclidean space. Then, the vector fields

$$\varphi_u=(1,0,r'(u)\cos v, r'(u)\sin v),$$

$$\varphi_v=(0,1,-r(u)\sin v, r(u)\cos v),$$

are tangent to M. Thus, the coefficients of first fundamental form are

$$E=(r')^2+1, \tag{9}$$
$$F=0,$$
$$G=r^2+1.$$

We set $W^2=EG-F^2=EG=(r^2+1)((r')^2+1)\neq 0$, i.e., it is regular.
Since these vectors are orthogonal, the orthonormal tangent vectors are written as

$$\varphi_1=\frac{1}{\sqrt{E}}\varphi_u=\frac{1}{\sqrt{(r')^2+1}}(1,\ 0,\ r'(u)\cos v, r'(u)\sin v),$$
$$\tag{10}$$
$$\varphi_2=\frac{1}{\sqrt{G}}\varphi_v=\frac{1}{\sqrt{r^2+1}}(0,\ 1,\ -r(u)\sin v,\ r(u)\cos v).$$

and the vectors

$$\eta_1=\frac{1}{\sqrt{\tilde{E}}}(-r'\cos v, r\sin v, 1, 0), \tag{11}$$
$$\eta_2=\frac{1}{\sqrt{\tilde{E}EG}}(-r'G\sin v, -rE\cos v, -\tilde{F}, \tilde{E})$$

are obtained as unit normal vector fields, where

$$\tilde{E}=1+r'^2\cos^2 v+r^2\sin^2 v,$$
$$\tilde{F}=\cos v\sin v(r'^2-r^2).$$

Furthermore, with the help of Weingarten and Gauss formulas, we get

$$\tilde{\nabla}_{\varphi_1}\varphi_1=k_1\eta_1+k_2\eta_2,$$
$$\tilde{\nabla}_{\varphi_1}\varphi_2=-k_6\eta_1+k_7\eta_2,$$
$$\tilde{\nabla}_{\varphi_2}\varphi_1=k_3\varphi_2-k_6\eta_1+k_7\eta_2,$$
$$\tilde{\nabla}_{\varphi_2}\varphi_2=-k_3\varphi_1-k_4\eta_1-k_5\eta_2, \tag{12}$$
$$\tilde{\nabla}_{\varphi_1}\eta_1=-k_1\varphi_1+k_6\eta_1+k_8\eta_2,$$
$$\tilde{\nabla}_{\varphi_2}\eta_1=k_6\varphi_1+k_4\varphi_2-k_9\eta_2,$$
$$\tilde{\nabla}_{\varphi_1}\eta_2=-k_2\varphi_1-k_7\varphi_2-k_8\eta_1,$$
$$\tilde{\nabla}_{\varphi_2}\eta_2=-k_7\varphi_1+k_5\varphi_2+k_9\eta_1,$$

where the differentiable functions $k_i\ (i=1,...,9)$ satisfy

$$k_1=\frac{r''\cos v}{E\sqrt{\tilde{E}}},\quad k_2=\frac{r''G\sin v}{E\sqrt{\tilde{E}EG}},\quad k_3=\frac{r'r}{G\sqrt{E}},$$

$$k_4=\frac{r\cos v}{G\sqrt{\tilde{E}}},\quad k_5=\frac{r\sin v}{\sqrt{\tilde{E}EG}},\quad k_6=\frac{r'\sin v}{\sqrt{\tilde{E}EG}}, \tag{13}$$

$$k_7=\frac{r'\cos v}{G\sqrt{\tilde{E}}},\quad k_8=\frac{r'\cos v\sin v(r''G-rE)}{\tilde{E}E\sqrt{G}},$$
$$k_9=\frac{EG-\tilde{E}}{\tilde{E}G\sqrt{E}}.$$

Thus, these relations can be decompose into tangent and normal components as

$$\begin{aligned} &\nabla_{\varphi_1}\varphi_1=0, && A_{\eta_1}\varphi_1=k_1\varphi_1-k_6\varphi_2,\\ &\nabla_{\varphi_2}\varphi_2=-k_3\varphi_1, && A_{\eta_1}\varphi_2=-k_6\varphi_1-k_4\varphi_2,\\ &\nabla_{\varphi_1}\varphi_2=0, && A_{\eta_2}\varphi_1=k_2\varphi_1+k_7\varphi_2,\\ &\nabla_{\varphi_2}\varphi_1=k_3\varphi_2, && A_{\eta_2}\varphi_2=k_7\varphi_1-k_5\varphi_2, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &h(\varphi_1,\varphi_1)=k_1\eta_1+k_2\eta_2, && D_{\varphi_1}\eta_1=k_8\eta_2,\\ &h(\varphi_1,\varphi_2)=-k_6\eta_1+k_7\eta_2, && D_{\varphi_1}\eta_2=-k_8\eta_1,\\ &h(\varphi_2,\varphi_2)=-k_4\eta_1-k_5\eta_2, && D_{\varphi_2}\eta_1=-k_9\eta_2,\\ & && D_{\varphi_2}\eta_2=k_9\eta_1. \end{aligned} \tag{15}$$

Moreover, with the help of (5) and (15), we obtain the coefficients h_{ij}^k:

$$h_{11}^1=\frac{r''\cos v}{E\sqrt{\tilde{E}}},\quad h_{11}^2=\frac{r''G\sin v}{E\sqrt{\tilde{E}EG}},$$
$$h_{12}^1=\frac{-r'\sin v}{\sqrt{\tilde{E}EG}},\quad h_{12}^2=\frac{r'\cos v}{G\sqrt{\tilde{E}}}, \tag{16}$$
$$h_{22}^1=\frac{-r\cos v}{G\sqrt{\tilde{E}}},\quad h_{22}^2=\frac{-r\sin v}{\sqrt{\tilde{E}EG}}.$$

Lemma 2: Let the equality (8) respresent an Aminov surface in $\mathbb{E}^4$. Then, the shape operator matrices are given by

$$A_{\eta_1} = \begin{pmatrix} \frac{r'' \cos v}{E\sqrt{\tilde{E}}} & \frac{-r' \sin v}{\sqrt{\tilde{E}EG}} \\ \frac{-r' \sin v}{\sqrt{\tilde{E}EG}} & \frac{-r \cos v}{G\sqrt{\tilde{E}}} \end{pmatrix}, \quad (17)$$

$$A_{\eta_2} = \begin{pmatrix} \frac{r'' G \sin v}{E\sqrt{\tilde{E}EG}} & \frac{r' \cos v}{G\sqrt{\tilde{E}}} \\ \frac{r' \cos v}{G\sqrt{\tilde{E}}} & \frac{-r \sin v}{\sqrt{\tilde{E}EG}} \end{pmatrix}.$$

Theorem 3:[3]Let M be an Aminov surface in $\mathbb{E}^4$ given by the Monge patch (8). Then, the Gaussian curvature and the mean curvature vectors are

$$K = \frac{-r'' rG - (r')^2 E}{W^4} \quad (18)$$

and

$$H = \frac{\cos v (r'' G - rE)}{2\sqrt{\tilde{E}}W^2}\eta_1 + \frac{\sin v (r'' G - rE)}{2E\sqrt{\tilde{E}}W}\eta_2 \quad (19)$$

respectively.

Proof. With the help of Lemma 2, and the Gaussian curvature of a surface in $\mathbb{E}^4$ $\left(K = \det(A_{\eta_1}) + \det(A_{\eta_2})\right)$; we yield

$$K = \frac{1}{EG}\left[\left(\frac{-r'' r \cos^2 v - r'^2 \sin^2 v}{\tilde{E}}\right) + \left(\frac{-r'' rG^2 \sin^2 v - r'^2 E^2 \cos^2 v}{\tilde{E}EG}\right)\right]$$

$$= \frac{\left(-r'' rG - r'^2 E\right)\left(1 + r'^2 \cos^2 v + r^2 \sin^2 v\right)}{\tilde{E}E^2G^2}$$

$$= \frac{-r'' rG - (r')^2 E}{W^4}.$$

In addition, with the help of the mean curvature vector of a surface $\left(H = \frac{trh}{2}\right)$, we compute

$$H = \frac{r'' G \cos v - rE \cos v}{2EG\sqrt{\tilde{E}}}\eta_1 + \frac{r'' G \sin v - rE \sin v}{2E\sqrt{\tilde{E}EG}}\eta_2$$

$$= \frac{\cos v\left(r'' G - rE\right)}{2\sqrt{\tilde{E}}W^2}\eta_1 + \frac{\sin v\left(r'' G - rE\right)}{2E\sqrt{\tilde{E}}W}\eta_2.$$

It completes the proof.

Corollary 4: Let the equality (8) represent an Aminov surface in $\mathbb{E}^4$. Then, it is minimal if and only if

$$r(u) = \frac{1}{2c_1}\left((c_1)^2 e^{\pm\frac{2(u+c_2)}{c_1}} + (c_1)^2 - 1\right)e^{\pm\frac{(u+c_2)}{c_1}}, \quad (20)$$

where c_i, $(i = 1,2)$ are real constants.

Proof. The surface is minimal, i.e., $H = 0$ in (19) if and only if

$$-rE + r'' G = 0.$$

Substituting the first fundamental form coefficients into (9), we get the differential equation

$$-r\left(1 + (r')^2\right) + r''\left(1 + r^2\right) = 0$$

which has the solution (20). It completes the proof.

Example 5: The surface

$$M : \varphi(u,v) = \left(u, v, e^u \cos v, e^u \sin v\right)$$

is congruent to minimal Aminov surface with $r(u) = e^u$ with the constants $c_i = 1$ in (20). One can plot by Maple:

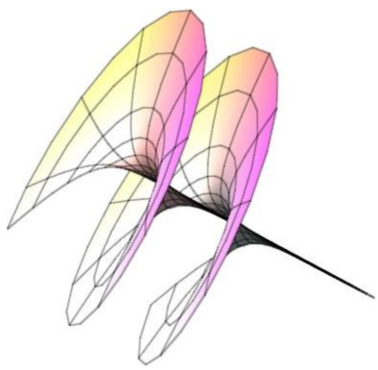

Figure 1: Minimal Aminov surface

4. A characterization of Aminov Surface with regards to its Gauss map in $\mathbb{E}^4$

Let $\{\varphi_1, \varphi_2, \eta_1, \eta_2\}$ denote the orthonormal frame of a regular surface $M : \varphi(u,v)$ in $\mathbb{E}^4$. Then, the Gauss map of the surface is

$$G = \varphi_1 \wedge \varphi_2.$$

By the use of the relation (7), the Laplacian operator of this Gauss map in $\mathbb{E}^4$ can be written as

$$-\Delta G = \tilde{\nabla}_{\varphi_1}\tilde{\nabla}_{\varphi_1}G + \tilde{\nabla}_{\varphi_2}\tilde{\nabla}_{\varphi_2}G - \tilde{\nabla}_{\nabla_{\varphi_1}\varphi_1}G - \tilde{\nabla}_{\nabla_{\varphi_2}\varphi_2}G. \quad (21)$$

Theorem 6: Let M be an Aminov surface in $\mathbb{E}^4$ given by the Monge patch (8). Then, the Laplace of G of M is given by

$$\begin{aligned}
-\Delta G &= \left(-(k_1)^2-(k_2)^2-(k_4)^2-(k_5)^2-2(k_6)^2-2(k_7)^2\right)\varphi_1\wedge\varphi_2\\
&+\left(-\varphi_1[k_6]-\varphi_2[k_4]-k_7k_8-k_5k_9-2k_3k_6\right)\varphi_1\wedge\eta_1\\
&+\left(\varphi_1[k_7]-\varphi_2[k_5]-k_6k_8+k_4k_9+2k_3k_7\right)\varphi_1\wedge\eta_2 \quad (22)\\
&+\left(-\varphi_1[k_1]+\varphi_2[k_6]+k_2k_8-k_1k_3-k_3k_4-k_7k_9\right)\varphi_2\wedge\eta_1\\
&+\left(-\varphi_1[k_2]-\varphi_2[k_7]-k_1k_8-k_2k_3-k_3k_5-k_6k_9\right)\varphi_2\wedge\eta_2\\
&+2\left(k_7(k_1+k_4)+k_6(k_2+k_5)\right)\eta_1\wedge\eta_2
\end{aligned}$$

where $\varphi_i[k_j]$ are correspond to directional derivatives with respect to φ_i and the functions k_j $(j = 1,...,9)$ are given by (13).

Proof. With the help of Gauss and Weingarten formulas and their components, the derivatives $\tilde{\nabla}_{\varphi_i}\tilde{\nabla}_{\varphi_i}G$ and $\tilde{\nabla}_{\nabla_{\varphi_i}\varphi_i}G$ $(i = 1,2)$ are yielded as

$$\begin{aligned}
\tilde{\nabla}_{\varphi_1}\tilde{\nabla}_{\varphi_1}G &= \left(-(k_1)^2-(k_2)^2-(k_6)^2-(k_7)^2\right)\varphi_1\wedge\varphi_2\\
&+\left(-\varphi_1[k_6]-k_7k_8\right)\varphi_1\wedge\eta_1+\left(\varphi_1[k_7]-k_6k_8\right)\varphi_1\wedge\eta_2\\
&+\left(-\varphi_1[k_1]+k_2k_8\right)\varphi_2\wedge\eta_1+\left(-\varphi_1[k_2]-k_1k_8\right)\varphi_2\wedge\eta_2\\
&+2\left(k_1k_7+k_2k_6\right)\eta_1\wedge\eta_2,
\end{aligned}$$

$$\begin{aligned}
\tilde{\nabla}_{\varphi_2}\tilde{\nabla}_{\varphi_2}G &= \left(-(k_4)^2-(k_5)^2-(k_6)^2-(k_7)^2\right)\varphi_1\wedge\varphi_2\\
&+\left(-\varphi_2[k_4]-k_3k_6-k_5k_9\right)\varphi_1\wedge\eta_1\\
&+\left(-\varphi_2[k_5]+k_3k_7+k_4k_9\right)\varphi_1\wedge\eta_2\\
&+\left(\varphi_2[k_6]-k_3k_4-k_7k_9\right)\varphi_2\wedge\eta_1\\
&+\left(-\varphi_2[k_7]-k_3k_5-k_6k_9\right)\varphi_2\wedge\eta_2\\
&+2\left(k_4k_7+k_5k_6\right)\eta_1\wedge\eta_2,
\end{aligned}$$

$$\tilde{\nabla}_{\nabla_{F_1}F_1}G = 0,$$

$$\tilde{\nabla}_{\nabla_{F_2}F_2}G = (k_3k_6)\varphi_1\wedge\eta_1-(k_3k_7)\varphi_1\wedge\eta_2+(k_1k_3)\varphi_2\wedge\eta_1+(k_2k_3)\varphi_2\wedge\eta_2.$$

Then, substitute these derivatives into (21), we obtain the desired result.

Example 7: The helical cylinder

$$M : \varphi(u,v) = (0, v, \cos v, \sin v) + (u, 0, 0, 0) \quad (23)$$

also corresponds to Aminov surface with $r(u) = 1$ and the Laplace of G of M is calculated as

$$\Delta G = \left(\frac{1}{4}\right)\varphi_1\wedge\varphi_2 - \frac{\sin v}{2\sqrt{2}}\left(\frac{1-2\sqrt{2}\cos^2 v}{(1+\sin^2 v)^{\frac{3}{2}}}\right)\varphi_1\wedge\eta_1 + \frac{\cos v}{4}\left(\frac{1-2\sqrt{2}\cos^2 v}{(1+\sin^2 v)^{\frac{3}{2}}}\right)\varphi_1\wedge\eta_2.$$

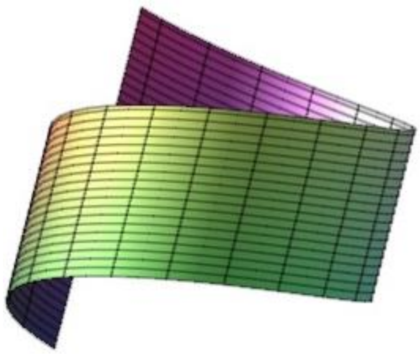

Figure 2: Aminov surface with r(u)=1

Theorem 8: Aminov surfaces can not have harmonic Gauss map in $\mathbb{E}^4$.

Proof. Let M be an Aminov surface in $\mathbb{E}^4$ given by the Monge patch (8). The Laplacian of Gauss map can be written as

$$\Delta G = \alpha_1(\varphi_1 \wedge \varphi_2) + \alpha_2(\varphi_1 \wedge \eta_1) + \alpha_3(\varphi_1 \wedge \eta_2) + \alpha_4(\varphi_2 \wedge \eta_1) + \alpha_5(\varphi_2 \wedge \eta_2) + \alpha_6(\eta_1 \wedge \eta_2),$$

where α_i (i=1,…,6) are indicated in (22). If M has a harmonic Gauss map, then $\Delta G = 0$, i.e., $\alpha_i = 0$. Then, we get

$$\alpha_1 = (k_1)^2 + (k_2)^2 + (k_4)^2 + (k_5)^2 + 2(k_6)^2 + 2(k_7)^2 = 0,$$

It means that $r(u) = 0$ in the surface parametrization (8). Hence, this contradicts with the regularity of the surface which completes the proof.

Theorem 9: Let M be an Aminov surface in $\mathbb{E}^4$ given by the representation (8). Then, M has pointwise one-type Gauss map of first kind if and only if

$$\frac{(k_1)^2 + (k_2)^2 + (k_4)^2 + (k_5)^2 + 2(k_6)^2 + 2(k_7)^2}{\lambda} - 1 = 0,$$

$$\frac{\varphi_1[k_6] + \varphi_2[k_4] + k_7k_8 + k_5k_9 + 2k_3k_6}{\lambda} = 0,$$

$$\frac{-\varphi_1[k_7] + \varphi_2[k_5] + k_6k_8 - k_4k_9 - 2k_3k_7}{\lambda} = 0, \qquad (24)$$

$$\frac{\varphi_1[k_1] - \varphi_2[k_6] - k_2k_8 + k_1k_3 + k_3k_4 + k_7k_9}{\lambda} = 0,$$

$$\frac{\varphi_1[k_2] + \varphi_2[k_7] + k_1k_8 + k_2k_3 + k_3k_5 + k_6k_9}{\lambda} = 0,$$

$$\frac{-2k_7}{\lambda}(k_1 + k_4) + \frac{-2k_6}{\lambda}(k_2 + k_5) = 0,$$

where λ is a non-zero smooth function.

Proof. Let M be an Aminov surface in $\mathbb{E}^4$ given by Monge patch (8). With the help of (1) and (22), we can write

$$\lambda + \lambda\langle \vec{C}, \varphi_1 \wedge \varphi_2 \rangle = (k_1)^2 + (k_2)^2 + (k_4)^2 + (k_5)^2 + 2(k_6)^2 + 2(k_7)^2,$$

$$\lambda\langle \vec{C}, \varphi_1 \wedge \eta_1 \rangle = \varphi_1[k_6] + \varphi_2[k_4] + k_7k_8 + k_5k_9 + 2k_3k_6,$$

$$\lambda\langle \vec{C}, \varphi_1 \wedge \eta_2 \rangle = -\varphi_1[k_7] + \varphi_2[k_5] + k_6k_8 - k_4k_9 - 2k_3k_7,$$

$$\lambda\langle \vec{C}, \varphi_2 \wedge \eta_1 \rangle = \varphi_1[k_1] - \varphi_2[k_6] - k_2k_8 + k_1k_3 + k_3k_4 + k_7k_9,$$

$$\lambda\langle \vec{C}, \varphi_2 \wedge \eta_2 \rangle = \varphi_1[k_2] + \varphi_2[k_7] + k_1k_8 + k_2k_3 + k_3k_5 + k_6k_9,$$

$$\lambda\langle \vec{C}, \eta_1 \wedge \eta_2 \rangle = -2k_7(k_1 + k_4) - 2k_6(k_2 + k_5),$$

where λ is non-zero differentiable function. By using the equality (1), the constant vector $\overleftarrow{C}$ can be considered as

$$\vec{C} = C_1\varphi_1 \wedge \varphi_2 + C_2\varphi_1 \wedge \eta_1 + C_3\varphi_1 \wedge \eta_2 + C_4\varphi_2 \wedge \eta_1 + C_5\varphi_2 \wedge \eta_2 + C_6\eta_1 \wedge \eta_2 \qquad (25)$$

where

$$C_1(u,v) = \frac{(k_1)^2 + (k_2)^2 + (k_4)^2 + (k_5)^2 + 2(k_6)^2 + 2(k_7)^2}{\lambda} - 1,$$

$$C_2(u,v) = \frac{\varphi_1[k_6] + \varphi_2[k_4] + k_7k_8 + k_5k_9 + 2k_3k_6}{\lambda},$$

$$C_3(u,v) = \frac{-\varphi_1[k_7] + \varphi_2[k_5] + k_6k_8 - k_4k_9 - 2k_3k_7}{\lambda}, \qquad (26)$$

$$C_4(u,v) = \frac{\varphi_1[k_1] - \varphi_2[k_6] - k_2k_8 + k_1k_3 + k_3k_4 + k_7k_9}{\lambda},$$

$$C_5(u,v) = \frac{\varphi_1[k_2] + \varphi_2[k_7] + k_1k_8 + k_2k_3 + k_3k_5 + k_6k_9}{\lambda},$$

$$C_6(u,v) = \frac{-2k_7}{\lambda}(k_1 + k_4) + \frac{-2k_6}{\lambda}(k_2 + k_5)$$

are differentiable functions. If M has a pointwise one-type Gauss map of the first kind, then $C = 0$ in the equation (1). By the use of (25) and (26), we get the desired result.

Corollary 10: Aminov surfaces can not have a pointwise one-type Gauss map of the first kind in $\mathbb{E}^4$.

Proof. Assume that an Aminov surface M has a pointwise one-type Gauss map of the first kind in $\mathbb{E}^4$. Then, the equation system (24) is hold. By using the last equation in (24), we get

$$\frac{2r'\left(r''G+rE\right)}{\lambda E^2G^2}=0.$$

Hence, there are two cases: $r'(u)=0$ or $r''(u)G+r(u)E=0.$ Using the first one ($r=const.$) in the second equation of (24), we yield

$$\frac{-r\sin v}{\left(1+r^2\sin^2 v\right)^{\frac{3}{2}}}\left(1-\frac{r^2\cos^2 v}{\left(1+r^2\right)^{\frac{3}{2}}}\right)=0.$$

Using the second one, i.e., substituting $r''=-\frac{rE}{G}$ into the second equation in (24), we obtain

$$-\frac{(r')^2G^2+\left(2(r')^2+r^2\right)E^2}{E^2G^2}=0.$$

These relations are satisfied if and only if $r(u)=0,$ but it contradicts with the regularity of the surface. Thus, we get the desired result.

Theorem 11 Let M be Aminov surface in $\mathbb{E}^4$ given by the Monge patch (8). Then, M has a pointwise one-type Gauss map of the second kind if and only if

$$\begin{aligned}
&\varphi_1[C_1]+C_2k_6-C_3k_7+C_4k_1+C_5k_2=0,\\
&\varphi_1[C_2]-C_1k_6-C_3k_8+C_6k_2=0,\\
&\varphi_1[C_3]+C_1k_7+C_2k_8-C_6k_1=0,\\
&\varphi_1[C_4]-C_1k_1-C_5k_8+C_6k_7=0,\\
&\varphi_1[C_5]-C_1k_2+C_4k_8+C_6k_6=0,\\
&\varphi_1[C_6]-C_2k_2+C_3k_1-C_4k_7-C_5k_6=0
\end{aligned}\tag{27}$$

and

$$\begin{aligned}
&\varphi_2[C_1]+C_2k_4+C_3k_5-C_4k_6+C_5k_7=0,\\
&\varphi_2[C_2]-C_1k_4+C_3k_9-C_4k_3+C_6k_7=0,\\
&\varphi_2[C_3]-C_1k_5-C_2k_9-C_5k_3+C_6k_6=0,\\
&\varphi_2[C_4]+C_1k_6+C_2k_3+C_5k_9-C_6k_5=0,\\
&\varphi_2[C_5]-C_1k_7+C_3k_3-C_4k_9+C_6k_4=0,\\
&\varphi_2[C_6]-C_2k_7-C_3k_6+C_4k_5-C_5k_4=0
\end{aligned}\tag{28}$$

are satisfied, where $\varphi_i[C_j]$ are correspond to directional derivatives with respect to φ_i.

Proof. Let M be an Aminov surface in $\mathbb{E}^4$ which has a pointwise one-type Gauss map of second kind. It means that $\vec{C}$ is a constant in (1). With the help of the equation (25) and (12), the directional derivatives of the vector $\vec{C}$ are obtained as

$$\begin{aligned}
\tilde{\nabla}_{\varphi_1}C=&\left(\varphi_1[C_1]+C_2k_6-C_3k_7+C_4k_1+C_5k_2\right)\varphi_1\wedge\varphi_2\\
&+\left(\varphi_1[C_2]-C_1k_6-C_3k_8+C_6k_2\right)\varphi_1\wedge\eta_1\\
&+\left(\varphi_1[C_3]+C_1k_7+C_2k_8-C_6k_1\right)\varphi_1\wedge\eta_2\\
&+\left(\varphi_1[C_4]-C_1k_1-C_5k_8+C_6k_7\right)\varphi_2\wedge\eta_1\\
&+\left(\varphi_1[C_5]-C_1k_2+C_4k_8+C_6k_6\right)\varphi_2\wedge\eta_2\\
&+\left(\varphi_1[C_6]-C_2k_2+C_3k_1-C_4k_7-C_5k_6\right)\eta_1\wedge\eta_2
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\nabla}_{\varphi_2}C=&\left(\varphi_2[C_1]+C_2k_4+C_3k_5-C_4k_6+C_5k_7\right)\varphi_1\wedge\varphi_2\\
&+\left(\varphi_2[C_2]-C_1k_4+C_3k_9-C_4k_3+C_6k_7\right)\varphi_1\wedge\eta_1\\
&+\left(\varphi_2[C_3]-C_1k_5-C_2k_9-C_5k_3+C_6k_6\right)\varphi_1\wedge\eta_2\\
&+\left(\varphi_2[C_4]+C_1k_6+C_2k_3+C_5k_9-C_6k_5\right)\varphi_2\wedge\eta_1\\
&+\left(\varphi_2[C_5]-C_1k_7+C_3k_3-C_4k_9+C_6k_4\right)\varphi_2\wedge\eta_2\\
&+\left(\varphi_2[C_6]-C_2k_7-C_3k_6+C_4k_5-C_5k_4\right)\eta_1\wedge\eta_2
\end{aligned}$$

Due to fact that the vector $\vec{C}$ is constant, these derivatives vanish and the equalities (27) and (28) are obtained. This completes the proof.

Now, we will also visualize the Aminov surface with the following example:

Example 12: Suppose that an Aminov surface is given by the parameterization (8). By taking $r(u)=\ln u$, one can plot the projection of this surface with the help of Maple command:

$$plot3d([u+v, \ln u \cos v, \ln u \sin v], u=-2*pi..2*pi, v=-2*pi..2*pi)$$

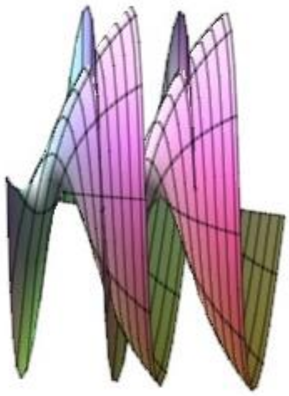

Figure 3: Aminov surface with r(u)=lnu

Conclusion

Aminov surfaces are the first described by Yu. A. Aminov in four-dimensional Euclidean space. In this paper, we calculate the Laplace of the Gauss map of these types of surfaces and examine them with respect to having pointwise one-type Gauss map.In future studies, this examination can be done for all surfaces given with Monge patch

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References

[1] Y.A. Aminov, "Surfaces in E^4 with a Gaussian curvature coinciding with a Gaussian torsion up to sign," *Mathematical Notes*, vol. 56, pp. 1211-1215, 1994.

[2] Y.A. Aminov,*The Geometry of Submanifolds*. Singapore: Gordon and Breach Science Publishers, 2001.

[3] B. Bulca, K. Arslan, "Surfaces Given with the Monge Patch in E^4," *Journal of Mathematical Physics, Analysis, Geometry*, vol. 9, pp. 435-447, 2013.

[4] S. Büyükkütük, G. Öztürk, "A new type timelike surface given with Monge patch in E_1^4," *TWMS J. App. and Eng. Math.,* vol.11, 176–183, 2021.

[5] B.Y. Chen,*Geometry of Submanifolds*. New York: Dekker, 1973.

[6] B.Y. Chen, M. Choi, Y.H. Kim, "Surfaces of revolution with pointwise 1-type Gauss map.,"*J. Korean Math. Soc.*vol. 42, 447–455, 2005.

[7] A. Gray,"A Monge Patch.," *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed. Boca Raton. FL: CRC Press. pp. 398–401, 1997.

[8] J. Glymph, D. Schelden, C. Ceccato,J.Mussel, H. Schober, "A parametric strategy for free-form glass structures using quadrilateral planar facets,"*Automation in Construction*,vol.13, 187–202, 2004.

[9] E. Güler, Ö. Kişi,"The Second Laplace-Beltrami Operator on Rotational Hypersurfaces in the Euclidean 4-Space,"*Mathematica Aeterna*. Vol.8, 1–12, 2018.

[10] E. Güler, Ö. Kişi,"Weingarten Map of the Henneberg-Type Minimal Surfaces in 4-Space,*"Konuralp J Math*. Vol.9, 132–136, 2021.

[11] E. Güler, Ö. Kişi,C. Konaxis, "Implicit Equations of the Henneberg-Type Minimal Surface in the Four-Dimensional Euclidean Space,*"Mathematics*. Vol.9, 279, 2018.

[12] İ. Kişi, G. Öztürk, "A new type tubular surface having pointwise 1-type Gauss map in Euclidean 4-space E4,*"Journal of Korean Mathematical Society*. Vol.55, 923-938, 2018.

[13] E.A. Pamuk, B. Bulca,"Translation-Factorable Surfaces in 4-dimensional Euclidean Space,"*Konuralp Journal of Mathematics*. Vol.9, 193–200, 2021.

[14] L.Verstraelen,J.Walrave,S.Yaprak, "The Minimal Translation Surface in Euclidean Space," Soochow J. Math., vol.20, 77–82, 1994.