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Research Article

A fast converging sampling operator

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ABSTRACT. We construct a sampling operator with the property that the smoother a function is, the faster its approximation is. We establish a direct estimate and a weak converse estimate of its rate of approximation in the uniform norm by means of a modulus of smoothness and a K-functional. The case of weighted approximation is also considered. The weights are positive and power-type with non-positive exponents at infinity. This sampling operator preserves every algebraic polynomial.

Keywords: Sampling operator, sampling series, weighted approximation, direct estimate, weak converse estimate, modulus of smoothness, K-functional.

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1. INTRODUCTION

The general form of the sampling series or operator of the function $f : \mathbb{R} \to \mathbb{R}$ is given for $w > 0$ by

$$(G_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \quad x \in \mathbb{R}. \tag{1.1}$$

Here $\chi : \mathbb{R} \to \mathbb{R}$ is referred to as the kernel of the operator. Under certain assumptions on χ, we have that $G_w^\chi f$ is well-defined for any f in a given function class and $G_w^\chi f$ converges to f either point-wise, or in norm, as w tends to infinity. For example, if χ is continuous on $\mathbb{R}$, has compact support and

$$\sum_{k\in\mathbb{Z}} \chi(u-k) = 1, \quad u \in \mathbb{R}, \tag{1.2}$$

then (see [10, Theorem 1])

$$\lim_{w\to\infty} G_w^\chi f(x) = f(x)$$

at any point $x \in \mathbb{R}$ at which f is continuous, as, moreover, the convergence is uniform on $\mathbb{R}$ provided that f is bounded and uniformly continuous on $\mathbb{R}$. More general conditions on the kernel, which provide such approximation, can be found e.g. in [5, 6, 9, 14, 15, 17].

Clearly, (1.2) implies that G_w^χ reproduces the constant functions. Given any positive integer r, Butzer and Stens [10, pp. 165–168] constructed a kernel of compact support such that the corresponding sampling operator reproduces the algebraic polynomials up to degree $r-1$. Another approach to achieve the same goal is given in [6, Section 3.2]. The purpose of the present paper is to introduce a sampling operator which reproduces *all* algebraic polynomials.

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As a consequence, this sampling operator has the property that the smoother the function is, the faster its approximation is. We estimate the rate of approximation in unweighted and weighted uniform norm on the real line. The weights are of power-type with non-positive exponents at infinity.

The contents of the paper are organized as follows. In the next section, we construct the kernel of the sampling operator. We will consider and show that this sampling operator is well-defined for a certain broad class of continuous functions. Then, in Section 3, we state our main results about estimating its rate of approximation by a modulus of smoothness. Section 4 contains basic properties of the sampling operator, from which the main results are derived. In the last section, we provide proofs of the main results.

2. THE DEFINITION OF THE SAMPLING OPERATOR

Let $C(\mathbb{R})$ denote the space of the continuous (not necessarily bounded) functions on $\mathbb{R}$ and $CB(\mathbb{R})$ the space of the continuous bounded functions on $\mathbb{R}$. Further, let $\|\circ\|$ stand for the uniform norm in $CB(\mathbb{R})$. Let $C^r(\mathbb{R})$ and $C^\infty(\mathbb{R})$ be the spaces of the functions that are r-times and infinitely many times, respectively, continuously differentiable on $\mathbb{R}$. Also, as usual, let $L(\mathbb{R})$ denote the space of the Lebesgue summable functions on $\mathbb{R}$.

Let the function $\eta : \mathbb{R} \to \mathbb{R}$ be defined by

$$\eta(v) := \begin{cases} 1, & v = 0, \\ e^{-\dfrac{1}{e^{1/v^2} - e}}, & |v| < 1,\ v \neq 0, \\ 0, & |v| \geq 1. \end{cases}$$

Lemma 2.1. *We have that $\eta \in C^\infty(\mathbb{R})$.*

Proof. The assertion of the lemma is established by elementary calculus. For the sake of completeness, we include it.

Clearly, $\eta(v)$ is continuous on $\mathbb{R}$ and, for any $j \in \mathbb{N}_+$, $\eta^{(j)}(v)$ exists and is continuous on $\mathbb{R}\backslash\{0, \pm 1\}$. It remains to demonstrate that $\eta^{(j)}(v)$ exists and is continuous at $v = 0, \pm 1$. We set

$$\xi(v) := \frac{1}{e^{1/v^2} - e}, \quad v \in (-1, 1)\backslash\{0\}.$$

First, by means of Faà di Bruno's formula we get

$$\eta^{(j)}(v) = \eta(v) \sum_{m_1, m_2, \ldots, m_j} \frac{j!\,(-1)^{m_1+m_2\cdots+m_j}}{(m_1!\,1!^{m_1})(m_2!\,2!^{m_2})\cdots(m_j!\,j!^{m_j})} \times \prod_{n=1}^{j} \left(\xi^{(n)}(v)\right)^{m_n}, \quad v \in (-1, 1)\backslash\{0\}, \tag{2.1}$$

where the sum is over all non-negative integers $m_1, m_2, \ldots, m_j$ such that

$$1m_1 + 2m_2 + \cdots + jm_j = j.$$

Next, we verify by induction that

$$\xi^{(n)}(v) = \frac{\xi^{n+1}(v)}{v^{3n}} \sum_{\ell=1}^{n} e^{\ell/v^2} p_{n,\ell}(v^2), \quad v \in (-1, 1)\backslash\{0\}, \tag{2.2}$$

where $p_{n,\ell}(x)$ are algebraic polynomials of degree $n - 1$.

Now, using (2.1)-(2.2), we see by induction on $j \in \mathbb{N}_+$ that $\eta^{(j)}(v)$ exists at $v = 0, \pm 1$ and is equal to 0. Here, we use that $\eta(v)$ is continuous at these points and $\lim_{v\to 0,\pm 1\mp 0} \eta^{(j)}(v) = 0$ for all $j \in \mathbb{N}_+$. ☐

We need the Fourier transform of functions in $L(\mathbb{R})$. We use it in the form

$$\hat{f}(v) := \int_{\mathbb{R}} f(u) e^{-ivu} du, \quad u \in \mathbb{R}.$$

Lemma 2.2. *There exists $\theta \in L(\mathbb{R})$ such that $\hat{\theta}(v) = \eta(v)$, $v \in \mathbb{R}$. Moreover,*

$$\theta(u) = \frac{1}{\pi} \int_0^1 \eta(v) \cos uv \, dv, \quad u \in \mathbb{R}, \tag{2.3}$$

$\theta \in C^\infty(\mathbb{R})$ and $\theta^{(j)}(u) = O(|u|^{-n})$ as $u \to \pm\infty$ for all $j, n \in \mathbb{N}_0$.

Proof. The existence of θ as well as its representation (2.3) can be established by means of e.g. [8, Proposition 6.3.10] (see also its proof; let us note that the Fourier transform is normalized differently in [8]). The last two assertions of the lemma are established directly from the theorem for differentiation under the integral sign and by integration by parts, as it is more convenient to write $\theta(u)$ in the form

$$\theta(u) = \frac{1}{2\pi} \int_{-1}^1 \eta(v) \cos uv \, dv$$

and use that $\eta^{(n)}(\pm 1) = 0$ for all $n \in \mathbb{N}_0$. ☐

We will consider the sampling operator

$$G_w := G_w^\theta,$$

where G_w^θ is defined in (1.1) with $\chi := \theta$ given in (2.3). As we will establish now, $G_w f(x)$, $x \in \mathbb{R}$, is well defined for any $f \in C(\mathbb{R})$ of at most polynomial growth at infinity. Actually, more is valid.

Proposition 2.1. *If $f \in C(\mathbb{R})$ is such that $f(x) = O(|x|^\nu)$ as $x \to \pm\infty$ with some $\nu \in \mathbb{N}_0$, then $G_w f \in C^\infty(\mathbb{R})$, $w > 0$, as, moreover,*

$$(G_w f)^{(j)}(x) = w^j \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \theta^{(j)}(wx - k), \quad x \in \mathbb{R}, \ j \in \mathbb{N}_0, \tag{2.4}$$

the series being uniformly convergent on the compact intervals of $\mathbb{R}$.

Proof. Below, we will denote by c positive constants, not necessarily the same at each occurrence, which are independent of $x \in \mathbb{R}$ and k. We have that

$$|f(x)| \le c(1 + |x|)^\nu, \quad x \in \mathbb{R}. \tag{2.5}$$

Let $j \in \mathbb{N}_0$. By Lemma 2.2, we have

$$|\theta^{(j)}(x)| \le c(1 + |x|)^{-\nu-2}, \quad x \in \mathbb{R}. \tag{2.6}$$

Let $[a, b]$ be an arbitrary compact subinterval of $\mathbb{R}$. Let $\gamma := w \max\{|a|, |b|\}$. Then for all $x \in [a, b]$ and $k \in \mathbb{Z}$ such that $|k| \ge \gamma$, we have

$$1 + |wx - k| \ge 1 + |k| - \gamma \ge \frac{|k| + 1}{\gamma + 1};$$

hence, using (2.5) and (2.6), we arrive at the estimate

$$\left| f\left(\frac{k}{w}\right) \theta^{(j)}(wx-k) \right| \le c \left(1+\frac{|k|}{w}\right)^{\nu} (1+|wx-k|)^{-\nu-2}$$
$$\le \frac{c}{(|k|+1)^2}, \quad x \in [a,b],\ |k| \ge \gamma.$$

Now, the Weierstrass M-test implies that the series

$$\sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \theta^{(j)}(wx-k)$$

is uniformly convergent on $[a,b]$ for each $j \in \mathbb{N}_0$. Consequently, $G_w f \in C^\infty(\mathbb{R})$ for every $w > 0$ and (2.4) holds. ☐

Remark 2.1. *As it follows from Propositions 4.2 and 4.4 below, $(G_w f)^{(j)}(x)$, $j \in \mathbb{N}_0$, is at most of the same polynomial growth at infinity as $f(x)$.*

3. Estimates of the rate of approximation of G_w

We will consider approximation by G_w in the weighted uniform norm with the weight

$$\rho_{\alpha,\beta}(x) := \begin{cases} |x|^{-\alpha}, & x < -1, \\ 1, & -1 \le x \le 1, \\ x^{-\beta}, & x > 1, \end{cases}$$

where $\alpha, \beta \ge 0$. Let us explicitly note that the results obtained include the unweighted case $\rho_{0,0}(x) \equiv 1$. In the case $\alpha = \beta$, we can instead write $\rho_{\alpha,\alpha}$, equivalently, in the concise form

$$\rho_{\alpha,\alpha}(x) := \frac{1}{1+|x|^\alpha}, \quad x \in \mathbb{R}.$$

As we observed earlier (Proposition 2.1), $G_w f$ is a well-defined infinitely continuously differentiable function on $\mathbb{R}$ for any $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$ with some $\alpha, \beta \ge 0$.

Let $f \in C(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$. We will use the modulus of smoothness of order $r \in \mathbb{N}_+$ of f, defined for $t > 0$ by

$$\omega_r(f,t)_{\alpha,\beta} := \sup_{0<h\le t} \|\rho_{\alpha,\beta} \Delta_h^r f\|,$$

where $\Delta_h f(x) := f(x+h/2) - f(x-h/2)$, $x \in \mathbb{R}$, $h > 0$, and $\Delta_h^r := \Delta_h(\Delta_h^{r-1})$. Clearly, $\rho_{\alpha,\beta}\Delta_h^r f \in CB(\mathbb{R})$ for every $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and every $h > 0$.

We will establish the following direct estimate of the rate of approximation of G_w.

Theorem 3.1. *Let $\alpha, \beta \ge 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w \ge 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| \le c\, \omega_r(f, 1/w)_{\alpha,\beta}.$$

Above c is a positive constant whose value is independent of f and w.

This theorem and basic properties of the modulus of smoothness, or more directly Proposition 4.3 below imply that if $f \in C^\infty(\mathbb{R})$ and $\rho_{\alpha,\beta} f^{(r)} \in CB(\mathbb{R})$ for all $r \in \mathbb{N}_0$, then

$$\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-r}) \text{ as } w \to \infty \quad \forall r \in \mathbb{N}_+;$$

in particular, if $f \in C^\infty(\mathbb{R})$ and $f^{(r)} \in CB(\mathbb{R})$ for all $r \in \mathbb{N}_0$, then

$$\|G_w f - f\| = O(w^{-r}) \text{ as } w \to \infty \quad \forall r \in \mathbb{N}_+.$$

Also, let us note that Theorem 3.1 yields that G_w preserves any algebraic polynomial (see also Corollary 4.1 and Remark 4.2 below).

Estimates of the rate of approximation of general sampling operators (1.1) in spaces of continuous functions associated with the weight $\rho_{2,2}$ have been recently obtained in [1]. Similar results for integral modifications of the general sampling operator were established in [2, 3]. Also, such results were proved for an integral form of general exponential sampling operators in function spaces equipped with a logarithmic weight in [4, Section 5].

The assertion of Theorem 3.1 in the unweighted case follows from the one-dimensional form of the general assertion in [15, Theorem 6, (8)]. A direct estimate of a different type than the one in Theorem 3.1 was established for a very general class of sampling operators, in particular, in the essential supremum norm with the weight $\rho_{\alpha,\alpha}$ under certain additional assumptions on f in [13, Theorem 31 and Remark 34]. There the rate of approximation of a general class of multivariate quasi-projection operators in weighted L_p-spaces was considered.

The direct estimate in Theorem 3.1 is essentially best possible–the following equivalence result holds.

Theorem 3.2. *Let $\alpha, \beta \geq 0$, $r \in \mathbb{N}_+$, $0 < \lambda < r$ and $f \in C(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$. Then*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-\lambda}) \qquad \Longleftrightarrow \qquad \omega_r(f,t)_{\alpha,\beta} = O(t^{\lambda}).$$

We will prove these theorems in the last section.

4. Basic relations and estimates

We will often apply the following auxiliary result (cf. [1, Proposition 1]).

Lemma 4.3. *Let $\alpha, \beta \geq 0$ and $j, \ell \in \mathbb{N}_0$. Then for all $x \in \mathbb{R}$ and $w \geq 1$, there holds*

$$\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell} |\theta^{(j)}(wx-k)| \leq c\rho_{\alpha,\beta}(x)^{-1}. \tag{4.1}$$

Above c is a positive constant whose value is independent of $x \in \mathbb{R}$ and $w \geq 1$.

Proof. First, let us note that it is sufficient to establish (4.1) for $x \geq 0$. This readily follows from the relations

$$\rho_{\alpha,\beta}(x) = \rho_{\beta,\alpha}(-x), \quad x \in \mathbb{R},$$

and

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell} |\theta^{(j)}(wx-k)| \\ &= \sum_{k\in\mathbb{Z}} \rho_{\beta,\alpha}\left(\frac{k}{w}\right)^{-1} |w(-x)-k|^{\ell} |\theta^{(j)}(w(-x)-k)|, \quad x \in \mathbb{R}.\end{aligned}$$

In the latter formula, we have taken into account that $\theta(u)$ is even; hence $\theta^{(2\ell)}(u)$ are even too, and $\theta^{(2\ell+1)}(u)$ are odd. Thus, let $x \geq 0$. We will estimate the sum on the negative k. We have

$$\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} \leq 1 + \left|\frac{k}{w}\right|^{\alpha} \leq 2|k|^{\alpha} \leq 2(wx-k)^{\alpha}, \quad k \leq -1,\ w \geq 1. \tag{4.2}$$

Let $n \in \mathbb{N}_+$ be such that $\alpha + \ell - n < -1$. By virtue of Lemma 2.2, for any fixed $j \in \mathbb{N}_0$ there exists a positive constant c such that for all $u \geq 1$ there holds

$$|\theta^{(j)}(u)| \leq cu^{-n}. \tag{4.3}$$

Since $wx - k \geq 1$ for $k \leq -1$, using (4.2) and (4.3), we get

$$\begin{aligned}\sum_{k\leq -1} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell}|\theta^{(j)}(wx-k)| &\leq c \sum_{k\leq -1}(wx-k)^{\alpha+\ell-n}\\ &\leq c\sum_{k\geq 1} k^{\alpha+\ell-n} \leq c.\end{aligned} \tag{4.4}$$

To estimate the sum on the non-negative k, we take into account that

$$\begin{aligned}\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} &\leq 1 + \left(\frac{k}{w}\right)^{\beta} \leq 1 + c\left(\left|\frac{k}{w} - x\right|^{\beta} + x^{\beta}\right)\\ &\leq c(1+|wx-k|^{\beta}+x^{\beta}), \quad k\geq 0,\ w\geq 1.\end{aligned}$$

Let $n \in \mathbb{N}_+$ be such that $\beta+\ell-n < -1$. By virtue of Lemma 2.2, there exists a positive constant c such that

$$|\theta^{(j)}(u)| \leq c(1+|u|)^{-n}, \quad u \in \mathbb{R}.$$

Consequently, similarly as in the previous case, we arrive at

$$\begin{aligned}&\sum_{k\geq 0} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell}|\theta^{(j)}(wx-k)|\\ &\leq c\sum_{k\geq 0}(1+|wx-k|^{\beta}+x^{\beta})|wx-k|^{\ell}(1+|wx-k|)^{-n}\\ &\leq c\sum_{k\geq 0}(1+|wx-k|)^{\beta+\ell-n} + cx^{\beta}\sum_{k\geq 0}(1+|wx-k|)^{\ell-n}\\ &\leq c(1+x^{\beta})\sum_{k\geq 0}(1+|wx-k|)^{\beta+\ell-n}\\ &\leq c(1+x^{\beta}),\end{aligned}$$

as at the last estimate we have taken into consideration that the series $\sum_{k\geq 0}(1+|u-k|)^{\beta+\ell-n}$ is convergent for every $u \geq 0$ and its sum is bounded on $[0,\infty)$. The latter can be easily verified if we consider instead the series $\sum_{k\in\mathbb{Z}}(1+|u-k|)^{\beta+\ell-n}$ for $u \in \mathbb{R}$. Clearly, it is uniformly convergent on each compact interval; hence its sum is a continuous function on $\mathbb{R}$. In addition, the sum is 1-periodic; consequently, it is bounded on $\mathbb{R}$. Combining the estimates we established above on the sums on $k < 0$ and $k \geq 0$, we arrive at

$$\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |wx-k|^{\ell}|\theta^{(j)}(wx-k)| \leq c(1+x^{\beta}), \quad x\geq 0;$$

hence (4.1) follows for $x \geq 0$. In view of the observation, we made in the beginning about the symmetry of the cases $x \leq 0$ and $x \geq 0$, the proof of the lemma is complete. □

We proceed to the basic properties of the operator G_w, which we will later use to establish estimates of its rate of approximation. We begin with showing that the family of operators $\{G_w\}_{w\geq 1}$ is uniformly bounded in the weighted spaces of continuous functions associated with the uniform norm with the weight $\rho_{\alpha,\beta}$.

Henceforward, c denotes positive constants, not necessarily the same at each occurrence, which are independent of the function and the operator order w.

Proposition 4.2. *Let $\alpha, \beta \geq 0$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta} G_w f\| \leq c\, \|\rho_{\alpha,\beta} f\|.$$

Proof. We have

$$|\rho_{\alpha,\beta}(x) G_w f(x)| \leq \rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} |\theta(wx-k)|\, \|\rho_{\alpha,\beta} f\|, \quad x \in \mathbb{R}.$$

Then the assertion follows immediately from Lemma 4.3 with $j = \ell = 0$. □

The discrete moment of θ of order $j \in \mathbb{N}_0$ is defined by

$$m_j(u) := \sum_{k\in\mathbb{Z}} (k-u)^j\, \theta(u-k), \quad u \in \mathbb{R}.$$

The following assertions for the discrete moments of θ holds true.

Lemma 4.4. *We have $m_0(u) = 1$ and $m_j(u) = 0$, $j \in \mathbb{N}_+$, for all $u \in \mathbb{R}$.*

Proof. The proof is standard–based on the Poisson summation formula (see e.g. [16, Theorem 4.2.8] or [8, Propositions 4.1.5, 5.1.28 and 5.1.29, and (3.1.22)]) and connected with certain Strang-Fix type conditions on θ (see e.g. [10, Lemma 3]). For the readers' convenience we include it. We apply the Poisson summation formula to the function $\theta_j(u) := u^j \theta(u)$, $j \in \mathbb{N}_0$. We have $\theta_j \in L(\mathbb{R})$ by virtue of Lemma 2.2. The Fourier transform of θ_j is

$$\widehat{\theta_j}(v) = i^j \hat{\theta}^{(j)}(v) = i^j \eta^{(j)}(v), \quad v \in \mathbb{R}, \tag{4.5}$$

where we have taken into account Lemma 2.2. Trivially, the series $\sum_{k\in\mathbb{Z}} |\widehat{\theta_j}(2\pi k)|$ is convergent.

Now, the Poisson summation formula, (4.5), $\eta^{(j)}(0) = 0$ for $j \geq 1$ (see Lemma 2.1) and $\eta^{(j)}(v) = 0$ for $|v| > 1$ and $j \geq 0$ yield

$$\begin{aligned} m_j(u) &= (-1)^j \sum_{k\in\mathbb{Z}} \theta_j(u-k) = (-1)^j \sum_{k\in\mathbb{Z}} \widehat{\theta_j}(2\pi k) e^{i2\pi k u} \\ &= (-i)^j \sum_{k\in\mathbb{Z}} \eta^{(j)}(2\pi k) e^{i2\pi k u} \\ &= \begin{cases} 1, & u \in \mathbb{R},\ j = 0, \\ 0, & u \in \mathbb{R},\ j \in \mathbb{N}_+. \end{cases} \end{aligned}$$

□

The following Jackson-type inequality holds true for G_w.

Proposition 4.3. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $g \in C^r(\mathbb{R})$ such that $\rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$, and all $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w g - g)\| \leq \frac{c}{w^r}\, \|\rho_{\alpha,\beta} g^{(r)}\|.$$

Proof. We expand $g(k/w)$ by Taylor's formula at the point $x \in \mathbb{R}$ to get

$$g\left(\frac{k}{w}\right) = \sum_{j=0}^{r-1} \frac{g^{(j)}(x)}{j!} \left(\frac{k}{w} - x\right)^j + \frac{1}{(r-1)!} \int_x^{k/w} \left(\frac{k}{w} - u\right)^{r-1} g^{(r)}(u)\, du. \tag{4.6}$$

Then, taking into account Lemma 4.4, we get the relation

$$(4.7)\qquad (G_w g)(x) - g(x) = \frac{1}{(r-1)!}\sum_{k\in\mathbb{Z}}\int_x^{k/w}\left(\frac{k}{w}-u\right)^{r-1} g^{(r)}(u)\,du\,\theta(wx-k), \quad x\in\mathbb{R}.$$

Consequently,

$$(4.8)\quad |(G_w g)(x) - g(x)| \le \frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{(r-1)!}\sum_{k\in\mathbb{Z}}\left|\int_x^{k/w}\left|\frac{k}{w}-u\right|^{r-1}\rho_{\alpha,\beta}(u)^{-1}du\right||\theta(wx-k)|, \quad x\in\mathbb{R}.$$

The function $\rho_{\alpha,\beta}(u)^{-1}$ is positive, decreasing on $(-\infty, 0]$, and increasing on $[0, +\infty)$; hence

$$\rho_{\alpha,\beta}(u)^{-1} \le \rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1} \quad \text{for } u \text{ between } x \text{ and } \frac{k}{w}.$$

Therefore, we deduce from (4.8) that

$$\begin{aligned}
&|(G_w g)(x) - g(x)|\\
\le&\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{(r-1)!}\sum_{k\in\mathbb{Z}}\left|\int_x^{k/w}\left(\frac{k}{w}-u\right)^{r-1}du\right|\left(\rho_{\alpha,\beta}(x)^{-1}+\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1}\right)|\theta(wx-k)|\\
=&\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{r!\,w^r}\sum_{k\in\mathbb{Z}}|wx-k|^r\left(\rho_{\alpha,\beta}(x)^{-1}+\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1}\right)|\theta(wx-k)|\\
=&\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{r!\,w^r}\rho_{\alpha,\beta}(x)^{-1}\sum_{k\in\mathbb{Z}}|wx-k|^r|\theta(wx-k)|\\
+&\frac{\|\rho_{\alpha,\beta} g^{(r)}\|}{r!\,w^r}\sum_{k\in\mathbb{Z}}\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1}|wx-k|^r|\theta(wx-k)|, \quad x\in\mathbb{R}.
\end{aligned}$$

Now, the assertion of the proposition follows from Lemma 4.3 with $j = 0$ and $\ell = r$, as to estimate the sum $\sum_{k\in\mathbb{Z}}|wx-k|^r|\theta(wx-k)|$, we apply it with $\alpha = \beta = 0$. □

If p is an algebraic polynomial of degree at most n, then $\rho_{n,n}p \in CB(\mathbb{R})$ and Proposition 4.3 with $r = n+1$ implies that G_w preserves p for all $w \ge 1$.

Corollary 4.1. *We have $G_w p = p$ for any algebraic polynomial p and all $w \ge 1$.*

Remark 4.2. *Actually, as it is quite easy to see, the assertion of the corollary holds for all $w > 0$.*

We will need a Bernstein-type inequality for G_w.

Proposition 4.4. *Let $\alpha, \beta \ge 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta}f \in CB(\mathbb{R})$, and all $w \ge 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w f)^{(r)}\| \le cw^r\,\|\rho_{\alpha,\beta}f\|.$$

Proof. Let us first recall that $G_w f \in C^\infty(\mathbb{R})$ (see Proposition 2.1). Then, by virtue of (2.4), we have

$$|\rho_{\alpha,\beta}(x)(G_w f)^{(r)}f(x)| \le w^r\rho_{\alpha,\beta}(x)\sum_{k\in\mathbb{Z}}\rho_{\alpha,\beta}\left(\frac{k}{w}\right)^{-1}|\theta^{(r)}(wx-k)|\,\|\rho_{\alpha,\beta}f\|, \quad x\in\mathbb{R}.$$

Now, the estimate in the proposition follows from Lemma 4.3 with $j = r$ and $\ell = 0$. □

The last auxiliary result for G_w, we will need, is an estimate of the weighted uniform norm of the derivatives of $G_w g$ for smooth g. In order to establish it, we will make use of a property of the discrete moments of the derivatives of θ, which is similar to Lemma 4.4. We set

$$m_{r,j}(u) := \sum_{k\in\mathbb{Z}} (k-u)^j\, \theta^{(r)}(u-k), \quad u \in \mathbb{R}.$$

The following assertions for the discrete moments of $\theta^{(r)}$ holds true.

Lemma 4.5. *Let $r \in \mathbb{N}_+$. We have $m_{r,j}(u) = 0$ for all $u \in \mathbb{R}$, where $j = 0, \dots, r-1$.*

Proof. Just similarly as in the proof of Lemma 4.4, we apply the Poisson summation formula but to the function $\theta_{r,j}(u) := u^j \theta^{(r)}(u)$, $j \in \mathbb{N}_0$. Since $\widehat{\theta^{(r)}}(v) = (iv)^r \hat{\theta}(v) = (iv)^r \eta(v)$, $v \in \mathbb{R}$ (recall Lemma 2.2), we get

$$\widehat{\theta_{r,j}}(v) = i^j \widehat{\theta^{(r)}}^{(j)}(v) = i^{r+j} \big(v^r \eta(v)\big)^{(j)}, \quad v \in \mathbb{R}.$$

We have for $j = 0, \dots, r-1$

$$\big(v^r \eta(v)\big)^{(j)} = \sum_{\ell=0}^{j} \binom{j}{\ell} r(r-1)\cdots(r-\ell+1) v^{r-\ell} \eta^{(j-\ell)}(v);$$

hence, we get $\widehat{\theta_{r,j}}(2\pi k) = 0$ for all $k \in \mathbb{Z}$.

Now, the Poisson summation formula yields

$$m_{r,j}(u) = (-1)^j \sum_{k\in\mathbb{Z}} \theta_{r,j}(u-k) = (-1)^j \sum_{k\in\mathbb{Z}} \widehat{\theta_{r,j}}(2\pi k) e^{i2\pi k u} \equiv 0.$$

$\square$

Proposition 4.5. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $g \in C^r(\mathbb{R})$ such that $\rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$, and all $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta} (G_w g)^{(r)}\| \leq c\, \|\rho_{\alpha,\beta} g^{(r)}\|.$$

Proof. Since $\rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$, then $g(x) = O(|x|^\nu)$ as $x \to \pm\infty$ with some $\nu \in \mathbb{N}_+$. Then, by (2.4), we have

$$(G_w g)^{(r)}(x) = w^r \sum_{k\in\mathbb{Z}} g\left(\frac{k}{w}\right) \theta^{(r)}(wx-k), \quad x \in \mathbb{R}. \tag{4.9}$$

We substitute $g(k/w)$ with its Taylor's expansion (4.6) and apply Lemma 4.5 to arrive at

$$(G_w g)^{(r)}(x) = \frac{w^r}{(r-1)!} \sum_{k\in\mathbb{Z}} \int_x^{k/w} \left(\frac{k}{w} - u\right)^{r-1} g^{(r)}(u)\, du\, \theta^{(r)}(wx-k), \quad x \in \mathbb{R}.$$

We complete the proof with the same argument, used to establish Proposition 4.3, but with $\theta^{(r)}$ in place of θ and we apply Lemma 4.3 with $j = \ell = r$. $\square$

5. Estimates of the rate of approximation of G_w by a K-functional

In this section, we will establish a direct inequality and a matching weak converse inequality for the rate of approximation of G_w in the uniform norm on $\mathbb{R}$ with the weight $\rho_{\alpha,\beta}$ by means of a K-functional. These estimates follow from the basic properties of the operator given in the preceding section by means of standard techniques (see e.g. [11, Chapter 7, §§ 3 and 5] or [12, Chapters 9 and 10]).

The K-functional we will use is defined for $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and $t > 0$ by

$$K_r(f,t)_{\alpha,\beta} := \inf\{\|\rho_{\alpha,\beta}(f-g)\| + t\,\|\rho_{\alpha,\beta} g^{(r)}\| : g \in C^r(\mathbb{R}),\ \rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})\}.$$

We proceed to the direct estimate.

Theorem 5.3. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| \leq c\, K_r(f, w^{-r})_{\alpha,\beta}.$$

Proof. Let $g \in C^r(\mathbb{R})$ be such that $\rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$. Then, by virtue of Propositions 4.2 and 4.3, we get

$$\begin{aligned}\|\rho_{\alpha,\beta}(G_w f - f)\| &\leq \|\rho_{\alpha,\beta} G_w (f-g)\| + \|\rho_{\alpha,\beta}(G_w g - g)\| + \|\rho_{\alpha,\beta}(g-f)\| \\ &\leq c\left(\|\rho_{\alpha,\beta}(f-g)\| + \frac{1}{w^r}\,\|\rho_{\alpha,\beta} g^{(r)}\|\right).\end{aligned}$$

Now, we take the infimum on g to arrive at the assertion of the theorem. □

The following weak converse inequality holds.

Theorem 5.4. *Let $\alpha, \beta \geq 0$ and $r \in \mathbb{N}_+$. Then for all $f \in C(\mathbb{R})$ such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $w, v \geq 1$ there holds*

$$K_r(f, w^{-r})_{\alpha,\beta} \leq \|\rho_{\alpha,\beta}(G_v f - f)\| + c\left(\frac{v}{w}\right)^r K_r(f, v^{-r})_{\alpha,\beta}.$$

Proof. By virtue of Propositions 2.1, 4.2 and 4.4, we have $G_v f \in C^r(\mathbb{R})$ and $\rho_{\alpha,\beta} G_v f, \rho_{\alpha,\beta}(G_v f)^{(r)} \in CB(\mathbb{R})$. Then

$$K_r(f, w^{-r})_{\alpha,\beta} \leq \|\rho_{\alpha,\beta}(f - G_v f)\| + \frac{1}{w^r}\,\|\rho_{\alpha,\beta}(G_v f)^{(r)}\|. \tag{5.1}$$

Let $g \in C^r(\mathbb{R})$ be such that $\rho_{\alpha,\beta} g, \rho_{\alpha,\beta} g^{(r)} \in CB(\mathbb{R})$. Then, we use Propositions 4.4 and 4.5 to estimate the second term on the right above as follows:

$$\begin{aligned}\|\rho_{\alpha,\beta}(G_v f)^{(r)}\| &\leq \|\rho_{\alpha,\beta}(G_v(f-g))^{(r)}\| + \|\rho_{\alpha,\beta}(G_v g)^{(r)}\| \\ &\leq c v^r\left(\|\rho_{\alpha,\beta}(f-g)\| + \frac{1}{v^r}\|\rho_{\alpha,\beta} g^{(r)}\|\right).\end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we arrive at

$$K_r(f, w^{-r})_{\alpha,\beta} \leq \|\rho_{\alpha,\beta}(G_v f - f)\| + c\left(\frac{v}{w}\right)^r \left(\|\rho_{\alpha,\beta}(f-g)\| + \frac{1}{v^r}\|\rho_{\alpha,\beta} g^{(r)}\|\right).$$

Finally, we take the infimum on g to derive the assertion of the theorem. □

Theorems 5.3 and 5.4 yield the following characterization of the rate of the approximation of G_w.

Corollary 5.2. *Let $\alpha, \beta \geq 0$, $r \in \mathbb{N}_+$, $0 < \lambda < r$ and $f \in C(\mathbb{R})$ be such that $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$. Then*

$$\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-\lambda}) \quad \Longleftrightarrow \quad K_r(f,t)_{\alpha,\beta} = O(t^{\lambda/r}).$$

Proof. If $K_r(f,t)_{\alpha,\beta} = O(t^{r/\lambda})$, then Theorem 5.3 implies $\|\rho_{\alpha,\beta}(G_w f - f)\| = O(w^{-\lambda})$.

To establish the inverse implication, we will use the Berens-Lorentz Lemma [7]. We will apply it in the form given in [11, Chapter 10, Lemma 5.2]. We set $\phi(x) := K_r(f,x^2)_{\alpha,\beta}$, $0 < x \leq 1$, and $\mu := 2\lambda/r \in (0,2)$. Then Theorem 5.4 implies

$$\phi(x) \leq c_f \left(y^\mu + \frac{x^2}{y^2}\,\phi(y) \right), \quad 0 < x \leq y \leq 1,$$

where c_f is a positive constant whose value may depend on f, but not on x and y. Now, the Berens-Lorentz Lemma yields

$$\phi(x) \leq c' c_f x^\mu, \quad 0 < x \leq 1,$$

with some positive constant c'; hence $K_r(f,t)_{\alpha,\beta} = O(t^{\lambda/r})$. $\square$

The K-functional above and the modulus of smoothness given in Section 3 are equivalent, that is, there exist constants $c, t_0 > 0$ such that for all $f \in C(\mathbb{R})$ with $\rho_{\alpha,\beta} f \in CB(\mathbb{R})$, and all $t \in (0, t_0]$ there hold (see [12, Theorem 6.1.1 with $\varphi \equiv 1$ and $p = \infty$])

$$c^{-1}\omega_r(f,t)_{\alpha,\beta} \leq K_r(f,t^r)_{\alpha,\beta} \leq c\,\omega_r(f,t)_{\alpha,\beta}. \tag{5.3}$$

Actually, it can be shown by means of the standard method to prove the above equivalence in the unweighted case (see e.g. [11, p. 177]) that it holds for any fixed positive t_0 (with c depending on t_0).

Combining Theorem 5.3 and Corollary 5.2 with relations (5.3) with $t_0 = 1$, we immediately get Theorems 3.1 and 3.2.

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