Universal Journal of Mathematics and Applications, 5 (3) (2022) 82-88
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.1173595

Study on the Applications of Semi-Analytical Method for the Construction of Numerical Solutions of the Burgers' Equation

Mine Babaoğlu[1]

[1]*Department of Mathematics and Science Education, Faculty of Education, Kahramanmaraş Sütçü Imam University, Kahramanmaraş, Turkey*

Article Info

Keywords: *Auxiliary parameter, Approximate solution, Burgers' equation, Homotopy analysis method.*
2010 AMS: *35G20, 35G31, 65N99.*
Received: *10 September 2022*
Accepted: *24 September 2022*
Available online: *30 September 2022*

Abstract

In the present paper explores, the Burgers' Equation which is the considerable partial differential equation that emerges in nonlinear science. Also, Homotopy Analysis Method (HAM) has been implemented to Burgers' equation with given initial conditions. The efficieny of the proposed method is analyzed by using some illustrative problems. We are compared approximate solutions acquired via HAM with the exact solutions. As a result of comparisons, it is demonstrated that the gained solutions are in excellent agreement. Additionally, 2D-3D graphs and tables of attained results are drawn by means of a ready-made package program. The recent obtained results denote that HAM is extremely efficient technique. Nonlinear partial differential equations can be solved with the help of presented method.

1. Introduction

Nonlinear phenomena have important effects on different fields of applied mathematics and science. These phenomena frequently occur naturally in various fields of study such as in physics, applied mathematics and engineering. Many mathematicians and scientists have drawn too much attention to the extensive effectiveness of these equations. Because of this, most of methods have been constructed and applied to solve these problems [1–12]. Therefore, developing methods for finding the analytical and numerical solutions to these types of equations is very important. HAM is an essential semi-analytical method and has a considerable role among them [13, 14].
The Burgers' equation is considered one of the fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of gas dynamics, chemical reaction, heat conduction, elasticity and traffic flow.

Burgers' equation has the following general structure,

$$u_t + u u_x = v u_{xx}\,,\ x \in \mathbb{R},\ \ t > 0, \tag{1.1}$$

where v is a constant. It describes the kinematic viscosity. Mentioned equation is named inviscid Burgers' equation when $v = 0$. This equation has a significant role in gas dynamics. The equation (1.1) indicates the connection between diffusion and convection procedures.

Numerical solutions of the nonlinear partial equations have an important role in mathematics, physics, applied sciences, etc. Several analytical and semi-analytical methods have been developed to solve these problems. For the importance of these solutions, there are many studies are available in literature. In manuscript [15], authors analyzed the behaviors of forced KdV equation describing the free surface critical flow over a hole by finding the solution with the help of q-homotopy analysis transform technique (q-HATT). Also, the partial differential equations were transformed into nonlinear ordinary differential equations by introducing relevant similarity variables and approximate analytical solution was determined operating the homotopy analysis method [16]. Influence of different relevant parameters such as Deborah number, stratification, chemical reaction and variable thermophysical parameters on temperature, velocity and concentration distributions was shown to highlight the specifics of heat and mass transfer flow characteristics by authors [16]. The conformable fractional Adomian decomposition method (CFADM) and conformable fractional modified homotopy perturbation method (CFMHPM) redefined to gain the

Email address and ORCID number: mnbabaoglu@gmail.com, 0000-0003-0819-1166 (M. Babaoğlu)

approximate-analytical solution of fractional partial differential equations (PDEs) by using conformable fractional derivative [17]. In the framework [18], the coupled mathematical model of the atmosphere-ocean system called El Nino-Southern Oscillation (ENSO) was analyzed with the aid of Adams-Bashforth numerical scheme. The non-linear regularized long wave (RLW) equation was solved by semi-inverse method [19]. Authors investigated novel solutions of fractional-order option pricing models and their fundamental mathematical analyses in [20]. The modified Laplace decomposition method (MLDM) defined in the sense of Caputo, Atangana-Baleanu and Caputo-Fabrizio (in the Riemann sense) operators was used in securing the approximate-analytical solutions of the nonlinear model in reference [21].

Some analytical, numerical and approximate analytical methods were investigated by considering time-fractional nonlinear Burger–Fisher equation (FBFE) [22]. $(1/G')$-expansion method, finite difference method (FDM) and Laplace perturbation method (LPM) were considered to solve the FBFE. Authors attained the analytical solution of the mentioned problem via $(1/G')$-expansion method [22]. In the mentioned paper, authors indicated that the finite difference method was a lower error level than the Laplace perturbation method. In our work, similarly, we demonstrated the effectiveness, validity and strength of our proposed method namely HAM to solve the Burgers' equation.

The goal of this study, we examine the equation in detail and HAM will be used to acquire new analytic solutions of the Burgers' equation. The method is powerful and effective and avoids the complexity involved in other purely numerical methods.

2. General Structure of Homotopy Analysis Method (HAM)

For the purpose of explaining the methodology of HAM, focus on the subsequent differential equation in a general form;

$$\mathrm{N}\left[\varphi\left(x,p\right)\right]=0,$$

where N is a nonlinear operator, φ is an unknown function. Then, x and p are independent variables. For convenience, all initial or boundary conditions are ignored. By using the present method, firstly, the one-parameter family of equations is constructed has the following form [1]

$$\left(1-q\right)L\left[\Phi\left(x,p;q\right)-\varphi_0\left(x,p\right)\right]=q\hbar H\left(x,p\right)\mathrm{N}\left[x,p;q\right], \tag{2.1}$$

where $q\in[0,1]$ is the embedding parameter, $\Phi\left(x,p;q\right)$ is an unknown function, $\varphi_0\left(x,p\right)$ is an initial guess of $\varphi\left(x,p\right)$, $\hbar\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator and $H\left(x,p\right)$ defines a non-zero auxiliary function.

Expressly when $q=0$ and $q=1$, equation (2.1) holds that

$$\Phi\left(x,p;0\right)=\varphi_0\left(x,p\right),\quad \Phi\left(x,p;1\right)=\varphi\left(x,p\right).$$

respectively [1]. Hence, defining

$$\varphi_n\left(x,p\right)=\left.\frac{1}{n!}\frac{\partial^n\Phi\left(x,p;q\right)}{\partial q^n}\right|_{q=0}, \tag{2.2}$$

and expanding $\Phi\left(x,p;q\right)$ in Taylor series with respect to the embedding parameter q, we have

$$\Phi\left(x,p;q\right)=\varphi_0\left(x,p\right)+\sum_{n=1}^{+\infty}\varphi_n\left(x,p\right)q^n, \tag{2.3}$$

If the series (2.3) converge when the value $q=1$, one has

$$\varphi\left(x,p\right)=\varphi_0\left(x,p\right)+\sum_{n=1}^{+\infty}\varphi_n\left(x,p\right), \tag{2.4}$$

which is proven by Liao [1]. Describe the vectors

$$\vec{\varphi}_m=\left\{\varphi_0\left(x,p\right),\varphi_1\left(x,p\right),\cdots,\varphi_m\left(x,p\right)\right\}.$$

After essential mathematical operations and regulations have been performed, the nth-order deformation equation is obtained which has the following form:

$$L\left[\varphi_n\left(x,p\right)-\chi_n\varphi_{n-1}\left(x,p\right)\right]=\hbar\Re\left(\vec{\varphi}_{n-1}\right), \tag{2.5}$$

where

$$\Re\left(\vec{\varphi}_{n-1}\right)=\left.\frac{1}{(n-1)!}\frac{\partial^{n-1}\mathrm{N}\left[\Phi\left(x,p;q\right)\right]}{\partial q^{n-1}}\right|_{q=0}, \tag{2.6}$$

and

$$\chi_n=\begin{cases}0, & n\leq 1,\\ 1, & n>1.\end{cases} \tag{2.7}$$

It should be indicated that the nth-order deformation equations form a set of linear differential equations which can be solved via help of a ready-made package program such as Maple, Mathematica, Matlab and others.

3. Implementation of the Method and Numerical Results

In this section of the paper, the Homotopy Analysis Method (HAM) is developed to acquire the approximate solutions of the Burgers' equation is used and we discuss the obtained results. Primarily, we describe the algorithm of the HAM as it applies to the Burgers' equation. By using nth-order deformation equation, we consider the following operator form for applying HAM to Burgers' equation (1.1),

$$L[u_n(x,t)-\chi_n u_{n-1}(x,t)]=\hbar H(x,t)R_n[\vec{u}_{n-1}(x,t)], \quad n\geq 1, \tag{3.1}$$

where $\hbar$ is a non-zero auxiliary parameter and $H(x,t)$ is a non-zero auxiliary function, respectively.

$$R_n[u_{n-1}(x,t)]=\frac{\partial u_{n-1}(x,t)}{\partial t}+u_{n-1}(x,t)\frac{\partial u_{n-1}(x,t)}{\partial x}-\nu\frac{\partial^2 u_{n-1}(x,t)}{\partial x^2}, \tag{3.2}$$

$$\chi_n=\begin{cases}0, & n\leq 1\\ 1, & n>1\end{cases}, \quad H(x,t)=1. \tag{3.3}$$

Assuming the inverse of the operator exists which is denoted L^{-1} and putting the initial condition $u_n(x,0)$, we write (3.1) for $n=1,2,3,\ldots$ and yields the components $u_1,u_2,u_3,u_4,u_5,\ldots$ etc. of Burgers' equation's solutions by using (3.2)-(3.3). In view of the obtained components, we attain the approximate solution of initial value problem in series form by $u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$. On the other hand, accuracy level could be improved via elegantly computing further components.

Example 3.1. Consider the subsequent Burgers' equation

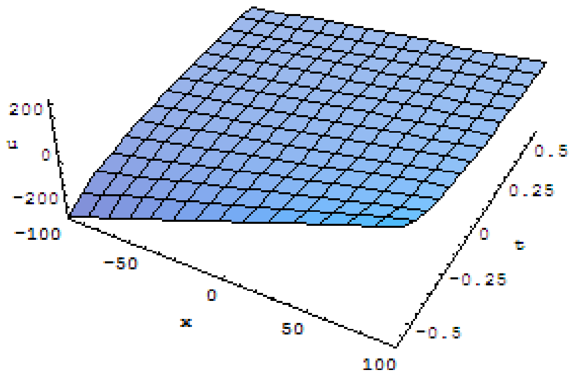
$$u_t+uu_x=u_{xx}, \tag{3.4}$$
$$u(x,0)=x.$$

By using the manner explained above, six components of the series were acquired of which $u(x,t)$ was evaluated to have the following expansion

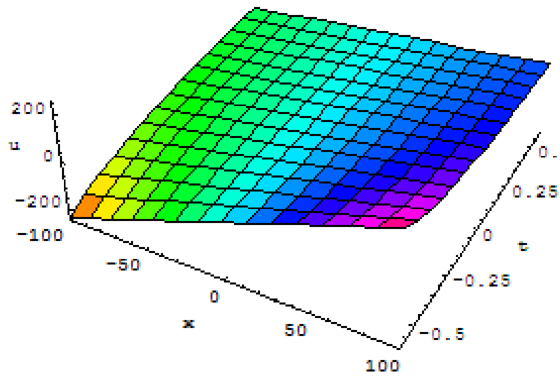
$$u(x,t)=x+xt\hbar+xt\hbar(t\hbar+\hbar+1)+xt\hbar(t\hbar+\hbar+1)^2+xt\hbar(t\hbar+\hbar+1)^3+xt\hbar(t\hbar+\hbar+1)^4+\ldots \tag{3.5}$$

Also, the exact solution of the equation (3.4) is given by

$$u(x,t)=\frac{x}{1+t}.$$



(a) Exact solution function's graph of the equation (3.4)



(b) (3.5) approximate solution for $u(x,t)$ as a graphical sketch ($\hbar=-1$)

Figure 3.1: Exact and approximate solution function of Burgers' equation by means of HAM

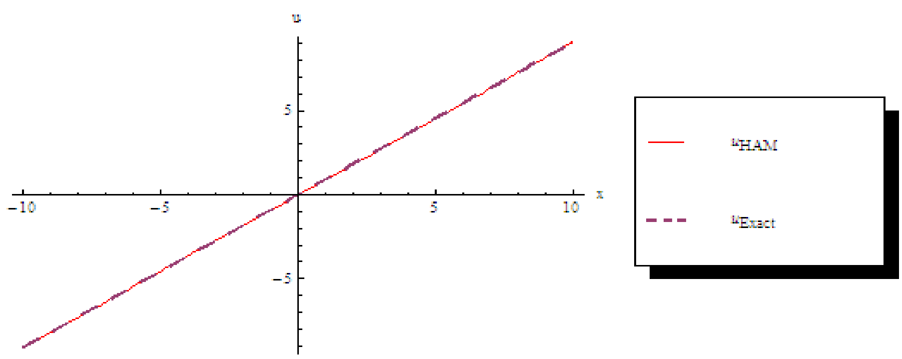


Figure 3.2: Comparison between HAM and exact solution when $\hbar=-1$

In this part of the work, we will compare the approximate solutions of Burgers' equation obtained by HAM with the exact solution. The results presented in Figure 3.1 and Figure 3.2, respectively, clearly demonstrate the good accuracy of the HAM with exact solution and the good agreement between HAM and exact solution of the equation. Additionally, we have the following results:

t/x	0.1	0.2	0.3	0.4	0.5
0.1	9.09091×10^{-8}	5.33333×10^{-6}	0.0000560769	0.000292571	0.00104167
0.2	1.81818×10^{-7}	0.0000106667	0.000112154	0.000585143	0.00208333
0.3	2.72727×10^{-7}	0.000016	0.000168231	0.000877714	0.003125
0.4	3.63636×10^{-7}	0.0000213333	0.000224308	0.00117029	0.00416667
0.5	4.54545×10^{-7}	0.0000266667	0.000280385	0.00146286	0.00520833

Table 1: Numerical solution's absolute error according to exact solution by using HAM when $\hbar=-1$

t	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$	$\hbar=-0.8$
0.1	0.0109227	0.00386764	0.00104167	0.000175073	0.0000106667
0.2	0.0218453	0.00773527	0.00208333	0.000350146	0.0000213333
0.3	0.032768	0.0116029	0.003125	0.000525219	0.000032
0.4	0.0436907	0.0154705	0.00416667	0.000700292	0.0000426667
0.5	0.0546133	0.0193382	0.00520833	0.000875365	0.0000533333

Table 2: Absolute error of Burgers' equation's numerical solution for different values of auxiliary parameter $\hbar$

The detailed results are demonstrated in Table 1 and Table 2. In Table 1, fix $\hbar=-1$, we compared the six component approximation of HAM with the exact solution. From Table 1, we see that numerical approximation by means of HAM show good agreement with the exact solution. In Table 1, it is seen that numerical solutions also obtained for six components by means of HAM are very convergent to exact solution. Furthermore, interval of the convergence can be found by drawing auxiliary parameter's graph on HAM solutions. In Figure 3.3, curve of the auxiliary parameter $\hbar$ is attained for series solution of the equation. As seen in this figure, interval of the convergence is approximately $-1.1\le\hbar\le-0.8$ for the series solution. The best value of the approximation on this interval can be found by giving values of $\hbar$.

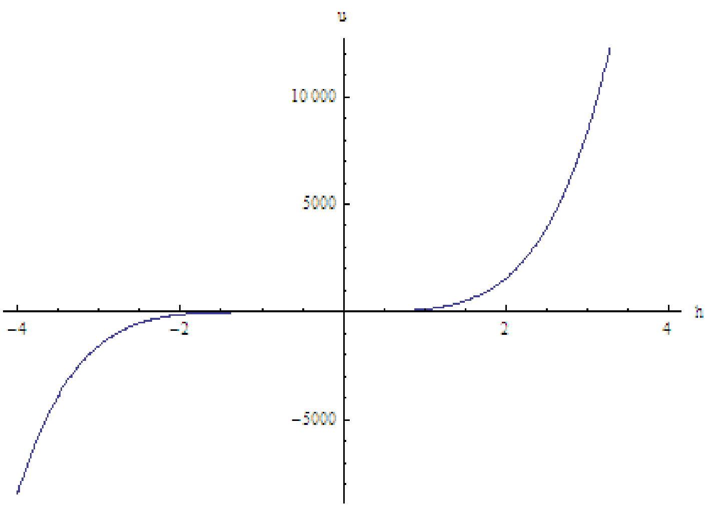


Figure 3.3: Graph of $\hbar$ curves of Burgers' equation for $\phi_6(x,t)$

Example 3.2. We focus on the following Burgers' equation

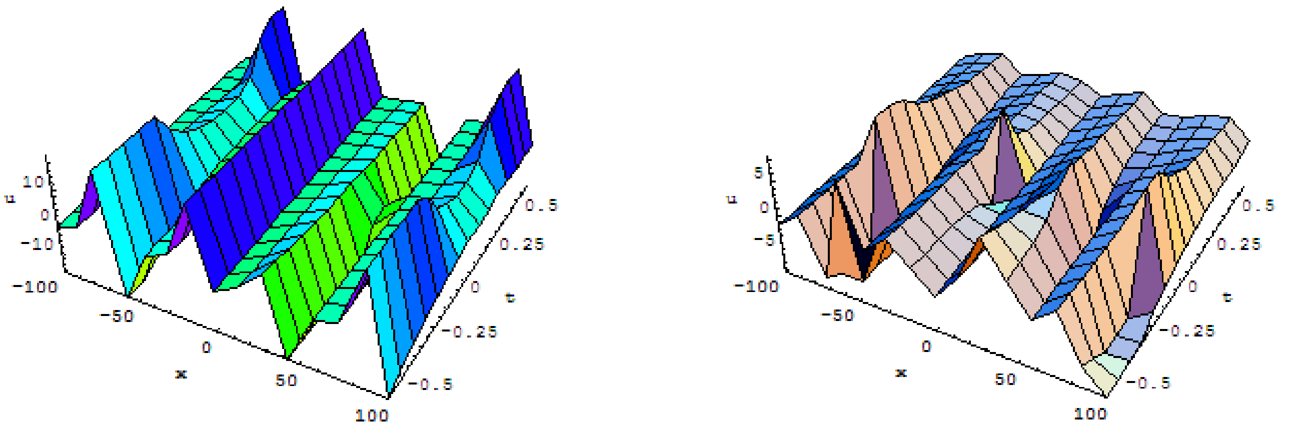
$$u_t+uu_x=u_{xx}, \tag{3.6}$$
$$u(x,0)=\frac{-2\cos x}{1+\sin x}.$$

By means of the process explained before, five components of the series were attained of which $u(x,t)$ was evaluated to have the following expansion

$$\begin{aligned} u(x,t)=&\frac{\text{-2cosx}}{\text{1+sinx}}-2\frac{\hbar t\text{cosx}}{(\text{1+sinx})^2}+\frac{\hbar t\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[-2-\hbar(2+t)+(-2+\hbar(-2+t))\sin x\right]}{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^5}\\ &+\frac{\hbar t\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[-3\left(6+2\hbar(t+6)+\hbar^2\left(t^2+2t+6\right)\right)+\left[6-6\hbar(t-2)+\hbar^2\left(t^2-6t+6\right)\right]\cos 2x\right]}{6\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^7}\\ &+\frac{8\left[-3-6\hbar+\hbar^2\left(t^2-3\right)\right]\sin x]}{6\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^7}+\ldots \end{aligned} \tag{3.7}$$

Then, the exact solution of the equation (3.6) has the form

$$u(x,t) = \frac{-2e^{-t}\cos x}{1+e^{-t}\sin x}.$$



(a) Exact solution function's graph of the equation (3.6)

(b) (3.7) approximate solution for $u(x,t)$ as a graphical sketch $(\hbar = -1)$

Figure 3.4: Exact and approximate solution function of Burgers' equation by means of HAM

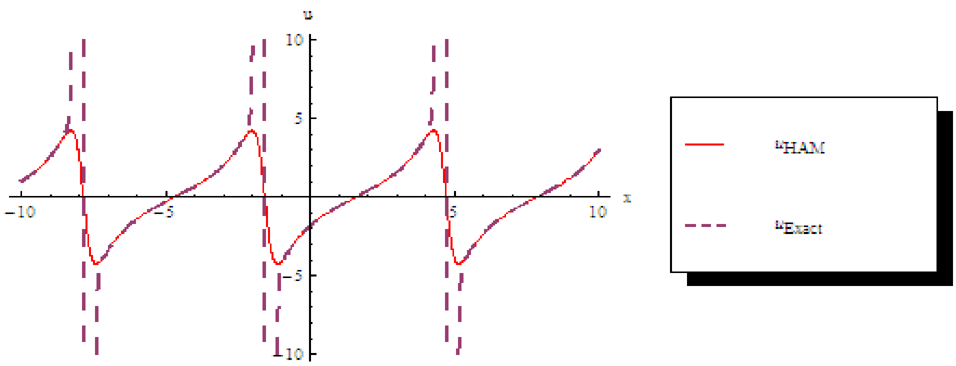


Figure 3.5: Comparison between HAM and exact solution when $\hbar = -1$

In this part of the manuscript, approximate solutions of the equation acquired by HAM are compared with the exact solution. The results presented in Figure 3.4 and 3.5, respectively, clearly show the good accuracy of the HAM with exact solution and the good agreement between HAM and exact solution. In addition, we get the following results:

The detailed results are demonstrated in Table 3-4. In Table 3, we compared the five step approximation of HAM with the exact solution. We see that numerical approximation by means of HAM show good agreement with the exact solution $u(x,t)$.

t/x	0.1	0.2	0.3	0.4	0.5
0.1	8.75091×10^{-8}	2.71323×10^{-6}	0.0000199575	0.0000814457	0.000240664
0.2	9.64863×10^{-8}	3.0651×10^{-6}	0.0000230798	0.0000963385	0.000290944
0.3	5.46078×10^{-8}	1.78415×10^{-6}	0.0000137918	0.0000590014	0.000182335
0.4	1.5428×10^{-8}	5.54887×10^{-7}	4.65233×10^{-6}	0.0000213409	0.0000700775
0.5	1.04609×10^{-8}	2.72352×10^{-7}	1.6068×10^{-6}	4.88699×10^{-6}	9.36752×10^{-6}

Table 3: Numerical solution's absolute error according to the exact solution by using HAM when $\hbar = -1$

t	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar=-1$	$\hbar=-0.9$	$\hbar=-0.8$
0.1	0.0126481	0.00176267	0.000240664	0.000112127	0.000394338
0.2	0.00480997	0.000128467	0.000290944	0.000147172	0.000500502
0.3	0.0012364	0.00071677	0.000182335	0.000135249	0.000559188
0.4	0.000334458	0.000802543	0.0000700775	0.0000985342	0.000572464
0.5	0.000957229	0.000712871	9.36752×10^{-6}	0.0000559068	0.000551245

Table 4: Absolute error of Burgers' equation's solution for different values of auxiliary parameter $\hbar$

In Table 3, it is seen that numerical solutions also obtained for five components by means of HAM are very convergent to the exact solution. Furthermore, in Figure 3.6, curve of the auxiliary parameter $\hbar$ is attained for series solution of the equation. As seen in this figure, interval of the convergence is approximately $-1.2\leq\hbar\leq-0.8$ for the series solution. The best value of the approximation on this interval can be found by giving values of $\hbar$. This situation is given in Table 4. $\hbar=-1$ is the best approximation to the exact solution as seen in the mentioned table.

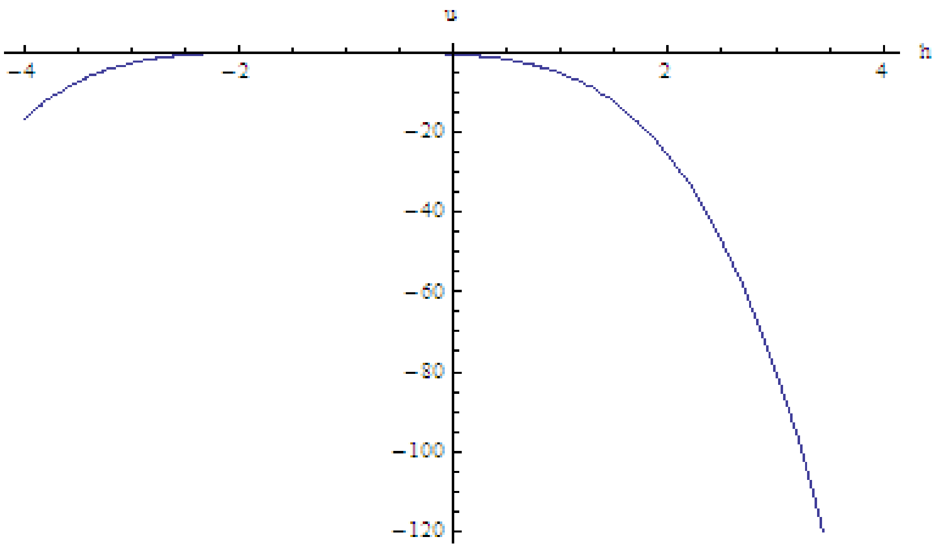


Figure 3.6: Graph of $\hbar$ curves of equation (3.6) for $\phi_5(x,t)$

4. Conclusion

In the present research, we investigate the numerical solution of Burgers' equations acquired by HAM. The application of the proposed technique has also been discussed. Besides, we draw two-three dimensional graphs and tables of this equation by use of a ready-made package program.
From the obtained results, it has been deduced that HAM is highly effective, credible and strong in the sense that finding analytical solutions. Thus the validity and flexibility of the proposed method are verified via all of these successful applications. The study demonstrates that the HAM algorithm is productive and can be used for many other complicated nonlinear partial differential equations in mathematical physics.

Acknowledgements

The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

There are no competing interests.

Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

References

[1] S. J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Champan & Hall/CRC Press, Boca Raton, 2003.
[2] A. M. Wazwaz, Balkema Publishers, *Partial Differential Equations: Methods and Applications*, The Netherlands, 2002.
[3] S. J. Liao, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput., **147**(2) (2004), 499-513.
[4] J. H. He, *Comparison of Homotopy perturbation method and Homotopy analysis method*, Appl. Math. Comput., **156**(2) (2004), 527-539.
[5] S. J. Liao, *Comparison between the Homotopy analysis method and Homotopy perturbation method*, Appl. Math. Comput., **169**(2) (2005), 1186-1194.
[6] M. Naim, Y. Sabbar, A. Zeb, *Stability characterization of a fractional-order viral system with the non-cytolytic immune assumption*, Mathematical Modelling and Numerical Simulation with Applications, **2**(3) (2022), 164-176.
[7] W. Wu, S. J. Liao, *Solving solitary waves with discontinuity by means of the Homotopy analysis method*, Chaos, Solitons & Fractals, **26** (2005), 177-185.
[8] Z. Hammouch, M. Yavuz, N. Özdemir, *Numerical solutions and synchronization of a variable-order fractional chaotic system*, Mathematical Modelling and Numerical Simulation with Applications, **1**(1) (2021), 11-23.
[9] S. Abbasbandy, *The application of Homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation*, Phys. Lett. A, **361** (2007), 478-483.
[10] H. M. Baskonus, J. L. García Guirao, A. Kumar, F. S. Vidal Causanilles, G. Rodriguez Bermudez, *Instability modulation properties of the (2 + 1)-dimensional Kundu-Mukherjee-Naskar model in travelling wave solutions*, Mod. Phys. Lett. B, **35**(13) (2021), 2150217.
[11] B. Zogheib, E. Tohidi, H. M. Baskonus, C. Cattani, *Method of lines for multi-dimensional coupled viscous Burgers' equations via nodal Jacobi spectral collocation method*, Phys. Scr., **96** (2021), 124011.
[12] J. Nee, J. Duan, *Limit set of trajectories of the coupled viscous Burgers' equations*, Appl. Math. Lett., **11**(1) (1998), 57-61.
[13] S. J. Liao, The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems, Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, 1992.
[14] A. M. Lyapunov, *General Problem of the Stability of Motion (English translation)*, Taylor & Francis, London, 1992.
[15] P. Veeresha, M. Yavuz, C. Baishya, *A computational approach for shallow water forced Korteweg-De Vries equation on critical flow over a hole with three fractional operators*, Int. J. Optim. Control: Theor. Appl., **11**(3) (2021), 52-67.
[16] Md. Fayz-Al-Asad, T. Oreyeni, M. Yavuz, P. O. Olanrewaju, *Analytic simulation of MHD boundary layer flow of a chemically reacting upper-convected Maxwell fluid past a vertical surface subjected to double stratifications with variable properties*, Eur. Phys. J. Plus, **137**(7) (2022), 1-11.
[17] M. Yavuz, *Novel solution methods for initial boundary value problems of fractional order with conformable differentiation*, An International Journal of Optimization and Control: Theories & Applications (IJOCTA), **8**(1) (2018), 1-7.
[18] P. Veeresha, *A numerical approach to the coupled atmospheric ocean model using a fractional operator*, Mathematical Modelling and Numerical Simulation with Applications, **1**(1) (2021), 1-10.
[19] S. Pak, *Solitary wave solutions for the RLW equation by He's semi inverse method*, International Journal of Nonlinear Sciences and Numerical Simulation, **10**(4) (2009), 505-508.
[20] M. Yavuz, *European option pricing models described by fractional operators with classical and generalized Mittag-Leffler kernels*, Numer. Methods Partial Differ. Equ., **38**(3) (2020), 434-456.
[21] M. Yavuz, T. A. Sulaiman, F. Usta, H. Bulut, *Analysis and numerical computations of the fractional regularized long-wave equation with damping term*, Math. Methods Appl. Sci., **44**(9) (2020), 7538-7555.
[22] A. Yokuş, M. Yavuz, *Novel comparison of numerical and analytical methods for fractional Burger–Fisher equation*, Discrete Contin. Dyn. Syst. -S, **14**(7) (2021), 2591-2606.