TURKISH JOURNAL OF SCIENCE **KeD PUBLISHING**
VOLUME I, ISSUE I, 28-40 **http://dergipark.gov.tr/tjos**

s −Konveks ve s −Konkav Fonksiyonlar İçin Kesirli İntegraller Yardımıyla Hermite-Hadamard Tipli Eşitsizlikler

Hermite-Hadamard type inequalities for s −convex and s −concave functions via fractional integrals

M. Emin ÖZDEMİR[1], Merve AVCI-ARDIÇ[2,*] and Havva KAVURMACI-ÖNALAN[3]

[1]*Department of Mathematics Education, Uludağ University, Bursa, Turkey*
[2]*Department of Mathematics, Adıyaman University, Adıyaman, Turkey*
[3]*Department of Mathematics Education, Yüzüncü Yıl University, Van, Turkey*
**Sorumlu yazar / Corresponding Author: merveavci@ymail.com*

Geliş Tarihi / Received Date: 12 November 2016
Kabul Tarihi / Accepted Date: 27 December 2016

Öz: Kesirli integraller için yeni bir integral özdeşliği tanımlandı. Bu özdeşlik yardımıyla Riemann-Liouville kesirli integralleri için bazı yeni Hermite-Hadamard tipli eşitsizlikler geliştirildi. Elde edilen sonuçların Avcı vd. [4, Appl. Math. Comput., 217 (2011) 5171-5176] adlı makalede ispat edilen sonuçlarla ilişkili olduğu belirlendi.

Anahtar Kelimeler — s-konveks fonksiyon, Hölder eşitsizliği, Power-Mean eşitsizliği, Riemann-Liouville kesirli integral, Euler Gama fonksiyonu, Euler Beta fonksiyonu.

Abstract: New identity for fractional integrals have been defined. By using this identity, some new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral have been developed. It has been determined that the results are related to the results of Avcı et al., proved in [4, published in Appl. Math. Comput., 217 (2011) 5171-5176].

Keywords — s −convex function, Hölder inequality, Power-mean inequality, Riemann Liouville fractional integral, Euler Gamma function, Euler Beta function.

1.Introduction

The following double inequality, named Hermite-Hadamard inequality, is one of the best known results in the literature.

Theorem 1. *Let $f:I\subseteq \mathbb{R}\to\mathbb{R}$ be a convex function on an interval I of real numbers and $a,b\in I$ with $a<b$. Then the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}.$$

The above double inequality is reversed if f is concave.

In [6], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense.

Definition 1. *A function* $f : \mathbb{R}^+ \to \mathbb{R}$, *where* $\mathbb{R}^+ = [0,\infty)$, *is said to be* $s-$ *convex in the second sense if*

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in 0,\infty)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, $s-$ convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [7], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$ convex functions in the second sense as following.

Theorem 2. *Suppose that f:* $[0,\infty) \to [0,\infty)$ *is an* $s-$ *convex functions in the second sense, where* $s \in (0,1)$, *and let a,b*$\in[0,\infty)$ $a < b$. *If* $f \in L^1[a,b]$, *then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

The constant $k = \dfrac{1}{s+1}$ is the best possible in the second inequality in (1.1).

In [5], Kavurmacı *et al*. proved the following identity.

Lemma 1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *where* $a,b \in I$ *with* $a < b$. *If* $f' \in L[a,b]$, *then the following equality holds:*

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du$$

$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt.$$

In [4], Avcı *et al*. obtained the following results by using the above Lemma.

Theorem 3. *Let f:I*$\subset[0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a,b \in I$ *with* $a < b < \infty$. *If* $|f'|$ *is* $s-$ *convex on* $[a,b]$ *for some fixed* $s \in (0,1]$, *then*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.2}$$

$$\leq \frac{1}{(s+1)(s+2)}\left|f'(x)\right|\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]+\frac{1}{(s+2)}\left[\frac{(x-a)^2}{b-a}\left|f'(a)\right|+\frac{(b-x)^2}{b-a}\left|f'(b)\right|\right].$$

Theorem 4. *Let* $f:I\subset[0,\infty)\rightarrow\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b<\infty$. *If* $\left|f'\right|^q$ *is* $s-convex$ *on* $[a,b]$ *for some fixed* $s\in(0,1]$, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.3}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right]^{\frac{1}{q}}+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}.$$

Theorem 5. *Suppose that all the assumptions of Theorem 4 are satisfied. Then*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.4}$$

$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|f'(a)\right|^q\frac{1}{s+2}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|f'(b)\right|^q\frac{1}{s+2}\right)^{\frac{1}{q}}.$$

Theorem 6. *Let* $f:I\subset[0,\infty)\rightarrow\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $\left|f'\right|^q$ *is* $s-concave$ *on* $[a,b]$ *for* $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)du\right| \tag{1.5}$$

$$\leq \frac{2^{\frac{s-1}{q}}}{(1+p)^{\frac{1}{p}}(b-a)}\left\{(x-a)^2\left|f'\left(\frac{x+a}{2}\right)\right|+(b-x)^2\left|f'\left(\frac{x+b}{2}\right)\right|\right\}.$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 2. *Let* $f \in L_1[a,b]$. *The Riemann-Liouville integrals* $J_{a^+}^{\alpha}(f)$ *and* $J_{b^-}^{\alpha}(f)$ *of order* $\alpha > 0$ *with* $a \geq 0$ *are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ b > x$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du.$ Here $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x).$

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties of this operator can be found in [1]-[3].

The main aim of this paper is to establish Hermite-Hadamard type inequalities for $s-$convex and $s-$concave functions in the second sense via Riemann-Liouville fractional integral.

2. Hermite-Hadamard type inequalities for fractional integrals

In order to prove our main results we need the following Lemma.

Lemma 2. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *the interior of* I, *where* $a, b \in I$ *with* $a < b$. *If* $f' \in L[a,b]$, *then for all* $x \in [a,b]$ *and* $\alpha > 0$ *we have:*

$$\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right]$$

$$= \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 (t^{\alpha}-1) f'(tx+(1-t)a)dt + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 (1-t^{\alpha}) f'(tx+(1-t)b)dt$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du.$

Proof. By integration by parts, we can state

$$\int_0^1 (t^{\alpha}-1) f'(tx+(1-t)a)dt \tag{2.1}$$

$$= \left(t^{\alpha}-1\right)\frac{f\left(tx+(1-t)a\right)}{x-a}\Big|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f\left(tx+(1-t)a\right)}{x-a}dt$$

$$= \frac{f(a)}{x-a} - \frac{\alpha}{x-a}\int_a^x \left(\frac{u-a}{x-a}\right)^{\alpha-1}\frac{f(u)}{x-a}du$$

$$= \frac{f(a)}{x-a} - \frac{\alpha\Gamma(\alpha)}{(x-a)^{\alpha+1}} J_{x^-}^{\alpha} f(a)$$

and

$$\int_0^1 \left(1-t^{\alpha}\right) f'\left(tx+(1-t)b\right)dt \tag{2.2}$$

$$= \left(1-t^{\alpha}\right)\frac{f\left(tx+(1-t)b\right)}{x-b}\Big|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f\left(tx+(1-t)b\right)}{x-b}dt$$

$$= \frac{f(b)}{b-x} - \frac{\alpha}{b-x}\int_x^b \left(\frac{u-b}{x-b}\right)^{\alpha-1}\frac{f(u)}{x-b}du$$

$$= \frac{f(b)}{b-x} - \frac{\alpha\Gamma(\alpha)}{(b-x)^{\alpha+1}} J_{x^+}^{\alpha} f(b).$$

Multiplying the both sides of (2.1) and (2.2) by $\frac{(x-a)^{\alpha+1}}{b-a}$ and $\frac{(b-x)^{\alpha+1}}{b-a}$, respectively, and then adding the resulting identities we obtain the desired result.

Theorem 7. *Let $f: I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|$ is $s-$convex on $[a,b]$ for some fixed $s\in(0,1]$ and $x \in [a,b]$ then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\left|\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right]\right|$$

$$\leq \frac{\alpha}{(s+1)(\alpha+s+1)}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\left|f'(x)\right|$$

$$+\left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left[\frac{(x-a)^{\alpha+1}\left|f'(a)\right| + (b-x)^{\alpha+1}\left|f'(b)\right|}{b-a}\right]$$

where Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the $s-$ convexity of $\left|f'\right|$, we have

$$\left|\frac{(x-a)^{\alpha} f(a)+(b-x)^{\alpha} f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a)+J_{x^+}^{\alpha} f(b)\right]\right|$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left|t^{\alpha}-1\right|\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left|1-t^{\alpha}\right|\left|f'(tx+(1-t)b)\right|dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left(1-t^{\alpha}\right)\left[t^s\left|f'(x)\right|+(1-t)^s\left|f'(a)\right|\right]dt$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left(1-t^{\alpha}\right)\left[t^s\left|f'(x)\right|+(1-t)^s\left|f'(b)\right|\right]dt$$

$$=\frac{(x-a)^{\alpha+1}}{b-a}\left\{\int_0^1\left(1-t^{\alpha}\right)t^s\left|f'(x)\right|dt+\int_0^1\left(1-t^{\alpha}\right)(1-t)^s\left|f'(a)\right|dt\right\}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\int_0^1\left(1-t^{\alpha}\right)t^s\left|f'(x)\right|dt+\int_0^1\left(1-t^{\alpha}\right)(1-t)^s\left|f'(b)\right|dt\right\}$$

$$=\frac{\alpha}{(s+1)(\alpha+s+1)}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right]\left|f'(x)\right|$$

$$+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left[\frac{(x-a)^{\alpha+1}\left|f'(a)\right|+(b-x)^{\alpha+1}\left|f'(b)\right|}{b-a}\right].$$

We have used the facts that

$$\int_0^1\left(1-t^{\alpha}\right)t^s dt=\frac{\alpha}{(s+1)(\alpha+s+1)}$$

and

$$\int_0^1\left(1-t^{\alpha}\right)(1-t)^s dt=\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]$$

where β is Euler Beta function defined by

$$\beta(x,y)=\int_0^1 t^x(1-t)^y dt,\quad x,y>0$$

and

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The proof is completed.

Remark 1. *In Theorem 7, if we choose* $\alpha=1$, *we get the inequality in (1.2).*

Theorem 8. *Let* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^{q}$ *is* $s-$*convex on* $[a,b]$ *for some fixed* $s\in(0,1]$, $p,q>1$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{|f'(x)|^{q}+|f'(a)|^{q}}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{|f'(x)|^{q}+|f'(b)|^{q}}{s+1}\right)^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the Hölder inequality we have

$$\left|\frac{(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right|$$

$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1|t^{\alpha}-1||f'(tx+(1-t)a)|dt+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1|1-t^{\alpha}||f'(tx+(1-t)b)|dt$$

$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\left\{\left(\int_0^1(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^{q}dt\right)^{\frac{1}{q}}\right\}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\left(\int_0^1(1-t^{\alpha})^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^{q}dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f'\right|^q$ is $s-$convex on $[a,b]$, we get

$$\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt \le \frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1},$$

$$\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt \le \frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}$$

and by simple computation

$$\int_0^1 (1-t^\alpha)^p dt = \frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}.$$

Hence we have

$$\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right]\right|$$

$$\le \frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)}\right)^{\frac{1}{p}}\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

which completes the proof.

Remark 2. *In Theorem 8, if we choose* $\alpha=1$, *we get the inequality in (1.3).*

Corollary 1. *In Theorem 8, if we choose* $x=\frac{a+b}{2}$, *we obtain the following inequality:*

$$\left|(b-a)^{\alpha-1}\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{\frac{a+b}{2}^-}^\alpha f(a)+J_{\frac{a+b}{2}^+}^\alpha f(b)\right]\right|$$

$$\leq \left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)} \right)^{\frac{1}{p}} \frac{(b-a)^{\alpha}}{2^{\alpha+1}}$$

$$\times \left\{ \left(\frac{\left| f'\left(\frac{a+b}{2}\right) \right|^q + \left| f'(a) \right|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{\left| f'\left(\frac{a+b}{2}\right) \right|^q + \left| f'(b) \right|^q}{s+1} \right)^{\frac{1}{q}} \right\}.$$

Theorem 9. *Let $f:I\subset[0,\infty)\rightarrow \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left| f' \right|^q$ is $s-$convex on $[a,b]$ for some fixed $s\in(0,1]$, $q\geq 1$, $x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(b) \right] \right|$$

$$\leq \left(\frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \frac{(x-a)^{\alpha+1}}{b-a} \left(\frac{\alpha}{(s+1)(\alpha+s+1)} \left| f'(x) \right|^q + \left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)} \right] \left| f'(a) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \frac{(b-x)^{\alpha+1}}{b-a} \left(\frac{\alpha}{(s+1)(\alpha+s+1)} \left| f'(x) \right|^q + \left[\frac{1}{s+1} - \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)} \right] \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right\}$$

where $\alpha > 0$ and Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the power-mean inequality we have

$$\left| \frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J^{\alpha}_{x^-} f(a) + J^{\alpha}_{x^+} f(b) \right] \right| \tag{2.3}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \left| t^{\alpha} - 1 \right| \left| f'(tx+(1-t)a) \right| dt + \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \left| 1-t^{\alpha} \right| \left| f'(tx+(1-t)b) \right| dt$$

$$\le \frac{(x-a)^{\alpha+1}}{b-a}\left\{\left(\int_0^1 (1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t^\alpha)\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$+\frac{(b-x)^{\alpha+1}}{b-a}\left\{\left(\int_0^1 (1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t^\alpha)\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f'\right|^q$ is $s-$convex on $[a,b]$, we get

$$\int_0^1 (1-t^\alpha)\left|f'(tx+(1-t)a)\right|^q dt \tag{2.4}$$

$$\le \int_0^1 (1-t^\alpha)\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]$$

$$=\frac{\alpha}{(s+1)(\alpha+s+1)}\left|f'(x)\right|^q+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left|f'(a)\right|^q$$

and

$$\int_0^1 (1-t^\alpha)\left|f'(tx+(1-t)b)\right|^q dt \tag{2.5}$$

$$\le \int_0^1 (1-t^\alpha)\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]$$

$$=\frac{\alpha}{(s+1)(\alpha+s+1)}\left|f'(x)\right|^q+\left[\frac{1}{s+1}-\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\left|f'(a)\right|^q.$$

If we use (2.4) and (2.5) in (2.3), we obtain the desired result.

Remark 3. *In Theorem 9, if we choose* $\alpha=1$, *we get the inequality in (1.4).*

Theorem 10. *Let* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $\left|f'\right|^q$ *is* $s-$*concave on* $[a,b]$ *for some fixed* $s\in(0,1]$, $q>1$, $x\in[a,b]$, *then the following inequality for fractional integrals holds:*

$$\left|\frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a}-\frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right]\right|$$

$$\leq \left(\frac{\Gamma(1+p)\Gamma\left(1+\frac{1}{\alpha}\right)}{\Gamma\left(1+p+\frac{1}{\alpha}\right)} \right)^{\frac{1}{p}} \frac{2^{\frac{s-1}{q}}}{b-a} \times \left\{ (x-a)^{\alpha+1} \left| f'\left(\frac{x+a}{2}\right) \right| + (b-x)^{\alpha+1} \left| f'\left(\frac{x+b}{2}\right) \right| \right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$ and Γ is Euler Gamma function.

Proof. From Lemma 2, property of the modulus and using the Hölder inequality we have

$$\left| \frac{(x-a)^\alpha f(a)+(b-x)^\alpha f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^\alpha f(a)+J_{x^+}^\alpha f(b)\right] \right| \tag{2.6}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 |t^\alpha-1| \left| f'(tx+(1-t)a) \right| dt + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 |1-t^\alpha| \left| f'(tx+(1-t)b) \right| dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left\{ \left(\int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(tx+(1-t)a) \right|^q dt \right)^{\frac{1}{q}} \right\}$$

$$+ \frac{(b-x)^{\alpha+1}}{b-a}\left\{ \left(\int_0^1 (1-t^\alpha)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(tx+(1-t)b) \right|^q dt \right)^{\frac{1}{q}} \right\}.$$

Since $|f'|^q$ is $s-$concave on $[a,b]$, using the inequality (1.1), we have

$$\int_0^1 \left| f'(tx+(1-t)a) \right|^q dt \leq 2^{s-1} \left| f'\left(\frac{x+a}{2}\right) \right|^q \tag{2.7}$$

and

$$\int_0^1 \left| f'(tx+(1-t)b) \right|^q dt \leq 2^{s-1} \left| f'\left(\frac{x+b}{2}\right) \right|^q. \tag{2.8}$$

From (2.6)-(2.8), we complete the proof.

Remark 4. *In Theorem 10, if we choose* $\alpha=1$, *we get the inequality in (1.5).*

3. *Applications for P.D.F.'s*

Let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ with the cumulative distribution function $F(x)=Pr(X\leq x)=\int_a^b f(t)dt$.

Proposition 1. *With the assumptions of Theorem 7 with* $\alpha=1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{1}{(s+1)(s+2)}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left|F'(x)\right|+\frac{1}{s+2}\left[\frac{(x-a)^2\left|F'(a)\right|+(b-x)^2\left|F'(b)\right|}{b-a}\right]$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X where

$$E(X)=\int_a^b t\,dF(t)=b-\int_a^b F(t)dt.$$

Proof. If we write the inequality in Theorem 7 with $\alpha=1$ for F, we get the desired result.

Proposition 2. *With the assumptions of Theorem 8 with* $\alpha=1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{(x-a)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|F'(x)\right|^q+\left|F'(a)\right|^q}{s+1}\right]^{\frac{1}{q}}+\frac{(b-x)^2}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{\left|F'(x)\right|^q+\left|F'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X.

Proof. If we write the inequality in Theorem 8 with $\alpha=1$ for F, we get the desired result.

Proposition 3. *With the assumptions of Theorem 9 with* $\alpha=1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq\frac{(x-a)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|F'(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|F'(a)\right|^q\frac{1}{s+2}\right]^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left|F'(x)\right|^q\frac{1}{(s+1)(s+2)}+\left|F'(b)\right|^q\frac{1}{s+2}\right]^{\frac{1}{q}}$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X.

Proof. If we write the inequality in Theorem 9 with $\alpha = 1$ for F, we get the desired result.

Proposition 4. *With the assumptions of Theorem 10 with* $\alpha = 1$, *we have the inequality*

$$\left|\frac{(b-x)F(b)+(x-a)F(a)}{b-a}-\frac{b-E(X)}{b-a}\right|$$

$$\leq \frac{2^{\frac{s-1}{q}}}{(1+p)^{\frac{1}{p}}(b-a)}\left\{(x-a)^2\left|F'(\frac{x+a}{2})\right|+(b-x)^2\left|F'(\frac{x+b}{2})\right|\right\}$$

for all $x\in[a,b]$ and $E(X)$ is the expectation of X.

Proof. If we write the inequality in Theorem 10 with $\alpha = 1$ for F, we get the desired result.

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