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Koordinatlarda h −Konveks İki Fonksiyonun Çarpımı İçin Bazı Hermite-Hadamard Tipli Eşitsizlikler Üzerine

On Some Hadamard-Type Inequalities for Product of Two h −Convex Functions On the Co-ordinates

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Öz: Bu çalışmada, koordinatlarda h-konveks fonksiyonların çarpımı için Hadamard tipli eşitsizlikler oluşturulmuştur. Elde edilen sonuçlar literatürde bazı iyi bilinen sonuçları genelleştirmiştir.

Anahtar Kelimeler — Koordinatlar, Hadamard eşitsizliği, h-konveks fonksiyonlar.

Abstract: In this paper, Hadamard-type inequalities for product of h-convex functions on the co-ordinates on the rectangle from the plane are established. Obtained results generalize the corresponding to some well-known results given before now.

Keywords — co-ordinates, Hadamard's inequality, h-convex functions

1.Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard's inequality for convex mapping. For particular choice of the function f in (1.1) yields some classical inequalities of means.

Definition 1. *(See [11]) A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to Godunova-Levin function or f is said to belong to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and for $\alpha \in (0,1)$ we have the inequality:*

$$f(\alpha x + (1-\alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}.$$

The class $Q(I)$ was firstly described in [11] by Godunova-Levin. Some further properties of it can be found in [10], [15] and [16]. Among others, it is noted that non-negative monotone and non-negative convex functions belongs to this class of functions. In [6], Breckner introduced $s-$convex functions as a generalization of convex functions. In [7], he proved the important fact that the set-valued map is $s-$convex only if associated support function is $s-$convex. A number of properties and connections with $s-$convexity in the first sense are discussed in paper [12]. It is clear that $s-$convexity is merely convexity for $s=1$.

Definition 2. *(See [6]) Let $s\in(0,1]$ be fixed real number. A function $f:[0,\infty)\to[0,\infty)$ is said to be $s-$convex in the second sense, or that f belongs to the class K_s^2, if*

$$f(\alpha x+(1-\alpha)y)\le\alpha^s f(x)+(1-\alpha)^s f(y)$$

for all $x,y\in[0,\infty)$ and $\alpha\in[0,1]$.

Definition 3. *(See [10]) A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be $P-$function or that f is said to belong to the class $P(I)$ if f is non-negative and for all $x,y\in I$ and $\alpha\in[0,1]$, if*

$$f(\alpha x+(1-\alpha)y)\le f(x)+f(y).$$

In [9], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which holds for $s-$convex function in the second sense:

Theorem 1. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an $s-$convex function in the second sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L_1([a,b])$ then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k=\dfrac{1}{s+1}$ is the best possible in the second inequality in (1.2).

In [9], Dragomir and Fitzpatrick also proved the following Hadamard-type inequality which holds for $s-$convex functions in the first sense:

Theorem 2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an $s-$convex function in the first sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$. If $f\in L_1([a,b])$ then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+sf(b)}{s+1} \tag{1.3}$$

The above inequalities are sharp.

A modification for convex functions which is also known as co-ordinated convex(concave) functions was introduced by Dragomir in [8] as following:

Let us now consider a bidimensional interval $\Delta = [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le \alpha f(x,y)+(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$. If the inequality reversed then f is said to be concave on Δ. A function $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ are convex where defined for all $x\in a,b]$, $y\in c,d]$.

A formal definition for co-ordinated convex functions may be stated as follow [see [23]]:

Definition 4. *A function* $f:\Delta\to\mathbb{R}$ *is said to be convex on the co-ordinates on* Δ *if the following inequality:*

$$f(tx+(1-t)y,su+(1-s)w)\le tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w)$$

holds for all $t,s\in 0,1]$ and $(x,u),(x,w),(y,u),(y,w)\in\Delta$.

Clearly, every convex mapping $f:\Delta\to\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex. In [8], Dragomir established the following inequalities of Hadamard's type for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 3. *Suppose* $f:\Delta\to\mathbb{R}$ *is convex function on the co-ordinates on* Δ. *Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.4}$$

$$\le\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}$$

In [1], Alomari and Darus proved the following inequalities of Hadamard-type as above for $s-$ convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 4. *Suppose* $f:\Delta\to\mathbb{R}$ *is* $s-$*convex function (in the second sense) on the co-ordinates on* Δ. *Then one has the inequalities:*

$$4^{s-1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.5}$$

$$\leq\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2}$$

Also in [4] (see also [5]), Alomari and Darus established the following inequalities of Hadamard-type similar to (1.5) for $s-$convex functions in the first sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 5. *Suppose* $f:\Delta\to\mathbb{R}$ *is* $s-$*convex function on the co-ordinates on* Δ *in the first sense. Then one has the inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.6}$$

$$\leq\frac{f(a,c)+sf(b,c)+sf(a,d)+s^2f(b,d)}{(s+1)^2}$$

The above inequalities are sharp.

For refinements, counterparts, generalizations and new Hadamard-type inequalities see the papers [1, 2, 3, 4, 5, 8, 9, 10, 12, 21, 22, 23, 24].

In [17], Pachpatte established two Hadamard-type inequalities for product of convex functions. An analogous results for $s-$convex functions is due to Kırmacı *et al.* [13].

Theorem 6. *Let* $f,g:[a,b]\subset R\to[0,\infty)$ *be convex functions on* $[a,b]$, $a<b$. *Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b) \tag{1.7}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{1.8}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Theorem 7. *Let* $f,g:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$ $a,b\in[a,b]$, $a<b$, *be functions such that* g *and* fg *are in* $L_1([a,b])$. *If* f *is convex and non-negative on* $[a,b]$ *and if* g *is* s*-convex on* $[a,b]$ *for some* $s\in(0,1)$. *Then*

$$2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \tag{1.9}$$

$$\leq \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b)$$

and

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq \frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b) \tag{1.10}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

The class of $h-$convex functions was introduced by S. Varosanec in [19] (see [19] for further properties of $h-$convex functions).

Definition 5. *Let* $h:J\subseteq\mathbb{R}\to\mathbb{R}$, *where* $(0,1)\subseteq J$, *be a positive function. A function* $f:I\subseteq\mathbb{R}\to\mathbb{R}$ *is said to be* $h-$*convex or that* f *is said to belong to the class* $SX(h,I)$, *if* f *is non-negative and for all* $x,y\in I$ *and* $\alpha\in(0,1)$, *we have*

$$f(\alpha x+(1-\alpha)y)\leq h(\alpha)f(x)+h(1-\alpha)f(y)$$

if the inequality is reversed then f is said to be $h-$concave and we say that f belongs to the class $SV(h,I)$.

Remark 1. *Obviously, if* $h(\alpha)=\alpha$, *then all the non-negative convex functions belong to the class* $SX(h,I)$ *and all non-negative concave functions belong to the class* $SV(h,I)$. *Also note that if* $h(\alpha)=\frac{1}{\alpha}$, *then* $SX(h,I)=Q(I)$; *if* $h(\alpha)=1$, *then* $SX(h,I)\supseteq P(I)$; *and if* $h(\alpha)=\alpha^s$, *where* $s\in(0,1)$, *then* $SX(h,I)\supseteq K_s^2$.

In [18], Sarıkaya *et al.* established the following inequalities of Hadamard's type for product of $h-$ convex functions.

Theorem 8. *Let* $f\in SX(h_1,I)$, $g\in SX(h_2,I)$, $a,b\in I$, $a<b$, *be functions such that* $fg\in L_1([a,b])$ *and* $h_1h_2\in L_1([0,1])$, *then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt \tag{1.11}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 9. *Let* $f \in SX(h_1, I)$, $g \in SX(h_2, I)$, $a,b \in I$, $a < b$, *be functions such that* $fg \in L_1([a,b])$ *and* $h_1h_2 \in L_1([0,1])$, *then*

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \tag{1.12}$$

$$\le M(a,b)\int_0^1 h_1(t)h_2(1-t)dt + N(a,b)\int_0^1 h_1(t)h_2(t)dt$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

In [20], Sarıkaya *et al.* established the following inequality of Hadamard's type which involving $h-$ convex functions:

Theorem 10. *Let* $f \in SX(h,I)$, $a,b \in I$ *with* $a < b$, $f \in L_1([a,b])$ *and* $g:[a,b] \to \mathbb{R}$ *is non-negative, integrable and symmetric about* $\frac{a+b}{2}$. *Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b \left(h\left(\frac{b-x}{b-a}\right) + h\left(\frac{x-a}{b-a}\right)\right)g(x)dx. \tag{1.13}$$

In [14], authors proved the following results for product of two convex functions on the co-ordinates on rectagle from the plane $\mathbb{R}^2$.

Theorem 11. *Let* $f,g:[a,b] \subseteq \mathbb{R} \to \mathbb{R}$ *be convex functions on the co-ordinates on* Δ *with* $a < b, c < d$. *Then*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \le \frac{1}{9}L(a,b,c,d) + \frac{1}{18}M(a,b,c,d) + \frac{1}{36}N(a,b,c,d) \tag{1.14}$$

where

$$L(a,b,c,d) = f(a,c)g(a,c) + f(b,c)g(b,c) + f(a,d)g(a,d) + f(b,d)g(b,d)$$

$$M(a,b,c,d) = f(a,c)g(a,d) + f(a,d)g(a,c) + f(b,c)g(b,d) + f(b,d)g(b,c)$$

$$+ f(b,c)g(a,c) + f(b,d)g(a,d) + f(a,c)g(b,c) + f(a,d)g(b,d)$$

$$N(a,b,c,d) = f(b,c)g(a,d) + f(b,d)g(a,c) + f(a,c)g(b,d) + f(a,d)g(b,c)$$

Theorem 12. *Let* $f,g:[a,b]\subseteq \mathrm{R}\to \mathrm{R}$ *be convex functions on the co-ordinates on* Δ *with* $a<b, c<d$. *Then*

$$4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \tag{1.15}$$

$$+\frac{5}{36}L(a,b,c,d)+\frac{7}{36}M(a,b,c,d)+\frac{2}{9}N(a,b,c,d)$$

where $L(a,b,c,d), M(a,b,c,d)$, and $N(a,b,c,d)$ as in Theorem 10.

Similar to definition of co-ordinated convex functions Latif and Alomari gave the notion of $h-$ convexity of a function f on a rectangle from the plane R^2 and $h-$ convexity on the co-ordinates on a rectangle from the plane R^2 in [23], as follows:

Definition 6. *(See [23]) Let us consider a bidimensional interval* $\Delta=[a,b]\times[c,d]$ *in* R^2 *with* $a<b$ *and* $c<d$. *Let* $h:J\subseteq \mathrm{R}\to \mathrm{R}$, *where* $(0,1)\subseteq J$, *be a positive function. A mapping* $f:\Delta\to R$ *is said to be* $h-$ *convex on* Δ, *if* f *is non-negative and if the following inequality:*

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le h(\alpha)f(x,y)+h(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in (0,1)$. Let us denote this class of functions by $SX(h,\Delta)$. The function f is said to be $h-$ concave if the inequality reversed. We denote this class of functions by $SV(h,\Delta)$.

A function $f:\Delta\to \mathrm{R}$ is said to be $h-$ convex on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to \mathrm{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to \mathrm{R}$, $f_x(v)=f(x,v)$ are $h-$ convex where defined for all $x\in a,b]$, $y\in c,d]$. A formal definition of $h-$ convex functions may also be stated as follows:

Definition 7. *(See [23]) A function* $f:\Delta\to \mathrm{R}$ *is said to be* $h-$ *convex on the co-ordinates on* Δ, *if the following inequality:*

$$f(tx+(1-t)y,su+(1-s)w)\le h(t)h(s)f(x,u)+h(t)h(1-s)f(x,w)$$

$$+h(s)h(1-t)f(y,u)+h(1-t)h(1-s)f(y,w)$$

holds for all $t,s\in[0,1]$ and $(x,u),(x,w),(y,u),(y,w)\in\Delta$.

Lemma 1. *(See [23]) Every* $h-$*convex mapping* $f:\Delta\to\mathbb{R}$ *is* $h-$*convex on the co-ordinates, but the converse is not generally true.*

The main purpose of the present paper is to establish new Hadamard-type inequalities like those given above in the Theorem 11-12, but now for product of two $h-$convex functions on the co-ordinates on rectangle from the plane $\mathbb{R}^2$.

2. *Main Results*

In this section we establish some Hadamard's type inequalities for product of two $h-$convex functions on the co-ordinates on rectangle from the plane. In the sequel of the paper h_1 and h_2 are positive functions defined on J, where $(0,1)\subseteq J\subseteq\mathbb{R}$ and f and g are non-negative functions defined on $\Delta=[a,b]\times[c,d]$.

Theorem 13. *Let* $f,g:\Delta=[a,b]\times[c,d]\to R$ *where* $a<b$ *and* $c<d$, *be functions such that* $fg\in L^2(\Delta)$, $h_1h_2\in L_1[0,1])$. *If* f *is* h_1-*convex on the co-ordinates on* Δ *and if* g *is* h_2-*convex on the co-ordinates on* Δ, *then*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.1}$$

$$\leq p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d)$$

where $L(a,b,c,d)$, $M(a,b,c,d)$, $N(a,b,c,d)$ as in Theorem 10 and $p=\int_0^1 h_1(t)h_2(t)dt$ and $q=\int_0^1 h_1(t)h_2(1-t)dt$.

Proof. Since $f,g:\Delta=[a,b]\times[c,d]\to R$ be functions such that $fg\in L^2(\Delta)$ and f is h_1-convex on the co-ordinates on Δ and g is h_2-convex on the co-ordinates on Δ, therefore the partial mappings

$$f_y:[a,b]\to\mathbb{R},\ f_y(x)=f(x,y)$$

$$g_y:[a,b]\to\mathbb{R},\ g_y(x)=g(x,y)$$

and

$$f_x:[c,d]\to\mathbb{R},\ f_x(y)=f(x,y)$$

$$g_x:[c,d]\to\mathbb{R},\ g_x(y)=g(x,y)$$

are h_1-, h_2-convex on $[a,b]$ and $[c,d]$, respectively, for all $x \in [a,b]$ and $y \in [c,d]$, . Now by applying (1.11) to $f_x(y)g_x(y)$ on $[c,d]$ we get

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \le p\left[f_x(c)g_x(c)+f_x(d)g_x(d)\right]+q\left[f_x(c)g_x(d)+f_x(d)g_x(c)\right].$$

That is

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \le p\left[f(x,c)g(x,c)+f(x,d)g(x,d)\right]+q\left[f(x,c)g(x,d)+f(x,d)g(x,c)\right].$$

Integrating over $[a,b]$ and dividing both sides by $b-a$, we have

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.2}$$

$$\le p\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx\right]$$

$$+q\left[\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx\right].$$

Now by applying (1.11) to each integral on R.H.S of (2.2) again, we get

$$\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx \le p\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right]+q\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right]$$

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx \le p\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right]+q\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right]$$

$$\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx \le p\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]+q\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]$$

$$\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx \le p\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right]+q\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right].$$

On substitution of these inequalities in (2.2) yields

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$\le p^2\left[f(a,c)g(a,c)+f(b,c)g(b,c)\right]+pq\left[f(a,c)g(b,c)+f(b,c)g(a,c)\right]$$

$$+p^2\left[f(a,d)g(a,d)+f(b,d)g(b,d)\right]+pq\left[f(a,d)g(b,d)+f(b,d)g(a,d)\right]$$

$$+pq\left[f(a,c)g(a,d)+f(b,c)g(b,d)\right]+q^2\left[f(a,c)g(b,d)+f(b,c)g(a,d)\right]$$

$$+pq\left[f(a,d)g(a,c)+f(b,d)g(b,c)\right]+q^2\left[f(a,d)g(b,c)+f(b,d)g(a,c)\right]$$

$$=p^2L(a,b,c,d)+pqM(a,b,c,d)+q^2N(a,b,c,d).$$

This completes the proof.

Remark 2. *If we take* $h_1(t)=h_2(t)=t$, *then inequality (2.1) reduces to the inequality (1.14).*

Theorem 14. *Let* $f,g:\Delta=[a,b]\times[c,d]\to R$ *where* $a<b$ *and* $c<d$, *be functions such that* $fg\in L^2(\Delta)$, $h_1h_2\in L_1[0,1])$. *If* f *is* h_1-*convex on the co-ordinates on* Δ *and if* g *is* h_2-*convex on the co-ordinates on* Δ, *then*

$$\frac{1}{4h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.3}$$

$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

$$+(q^2+2pq)L(a,b,c,d)+(p^2+pq+q^2)M(a,b,c,d)+(p^2+2pq)N(a,b,c,d)$$

where $L(a,b,c,d),M(a,b,c,d),$ and $N(a,b,c,d)$ as in Theorem 10 and $p=\int_0^1 h_1(t)h_2(t)dt$ and $q=\int_0^1 h_1(t)h_2(1-t)dt$.

Proof. Now applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$, we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.4}$$

$$\leq\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx$$

$$+q\left[f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]$$

$$+p\left[f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]$$

and

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.5}$$

$$\leq \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$+q\left[f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]$$

$$+p\left[f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].$$

Adding (2.4) and (2.5) and multiplying both sides by $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}$, we get

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.6}$$

$$\leq \frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx$$

$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$+q\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\right]$$

$$+p\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\right]$$

$$+q\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\right]$$

$$+p\left[\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\right].$$

Applying (1.12) to each term within the brackets, we have

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy$$

$$+q[f(a,c)g(a,c)+f(a,d)g(a,d)]+p[f(a,c)g(a,d)+f(a,d)g(a,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+q[f(b,c)g(b,c)+f(b,d)g(b,d)]+p[f(b,c)g(b,d)+f(b,d)g(b,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(a,\frac{c+d}{2}\right)g\left(b,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy$$

$$+q[f(a,c)g(b,c)+f(a,d)g(b,d)]+p[f(a,c)g(b,d)+f(a,d)g(b,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(b,\frac{c+d}{2}\right)g\left(a,\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+q[f(b,c)g(a,c)+f(b,d)g(a,d)]+p[f(b,c)g(a,d)+f(b,d)g(a,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},c\right)\leq\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx$$

$$+q[f(a,c)g(a,c)+f(b,c)g(b,c)]+p[f(a,c)g(b,c)+f(b,c)g(a,c)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},d\right)\leq\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+q[f(a,d)g(a,d)+f(b,d)g(b,d)]+p[f(a,d)g(b,d)+f(b,d)g(a,d)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},c\right)g\left(\frac{a+b}{2},d\right)\leq\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx$$

$$+q[f(a,c)g(a,d)+f(b,c)g(b,d)]+p[f(a,c)g(b,d)+f(b,c)g(a,d)]$$

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},d\right)g\left(\frac{a+b}{2},c\right)\leq\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

$$+q[f(a,d)g(a,c)+f(b,d)g(b,c)]+p[f(a,d)g(b,c)+f(b,d)g(a,c)].$$

Substituting these inequalities in (2.6) and simplifying we have;

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{2.7}$$

$$\leq\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx$$

$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$+q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

$$+2q^2L(a,b,c,d)+2pqM(a,b,c,d)+p^2N(a,b,c,d)$$

Now by applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)$, integrating over $[c,d]$ and dividing both sides by $d-c$, we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy \tag{2.8}$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dxdy$$

$$+q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

Now again by applying (1.12) to $\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)$, integrating over $[a,b]$ and dividing both sides by $b-a$, we get

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx \tag{2.9}$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx.$$

Adding (2.8) and (2.9), we have

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{b-a}\int_c^d f\left(x,\frac{c+d}{2}\right)g\left(x,\frac{c+d}{2}\right)dx \tag{2.10}$$

$$+\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})}\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)g\left(\frac{a+b}{2},y\right)dy$$

$$\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$+q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

Therefore from (2.7) and (2.10), we get

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq \frac{2}{(b-a)(d-c)}\int_a^b f(x,y)g(x,y)dxdy$$

$$+2q\frac{1}{d-c}\int_c^d f(a,y)g(a,y)dy+2q\frac{1}{d-c}\int_c^d f(b,y)g(b,y)dy$$

$$+2p\frac{1}{d-c}\int_c^d f(a,y)g(b,y)dy+2p\frac{1}{d-c}\int_c^d f(b,y)g(a,y)dy$$

$$+2q\frac{1}{b-a}\int_a^b f(x,c)g(x,c)dx+2q\frac{1}{b-a}\int_a^b f(x,d)g(x,d)dx$$

$$+2p\frac{1}{b-a}\int_a^b f(x,c)g(x,d)dx+2p\frac{1}{b-a}\int_a^b f(x,d)g(x,c)dx$$

$$+2q^2L(a,b,c,d)+2pqM(a,b,c,d)+2p^2N(a,b,c,d)$$

By using (1.11) to each of the above integral and simplifying, we get

$$\frac{1}{2\left[h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$+(2q^2+4pq)L(a,b,c,d)+(2p^2+2pq+2q^2)M(a,b,c,d)+(2p^2+4pq)N(a,b,c,d)$$

Dividing both sides by 2;

$$\frac{1}{\left[2h_1(\frac{1}{2})h_2(\frac{1}{2})\right]^2} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)g\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\leq \frac{1}{(b-a)(d-c)}\int_a^b f(x,y)g(x,y)dxdy$$

$$+(q^2+2pq)L(a,b,c,d)+(p^2+pq+q^2)M(a,b,c,d)+(p^2+2pq)N(a,b,c,d)$$

This completes the proof of the theorem.

Remark 3. *If we take* $h_1(t)=h_2(t)=t$, *then inequality (2.3) reduces to the inequality (1.15).*

Theorem 15. *Suppose that all the assumptions of Theorem 12 are satisfied, if* g_x *and* g_y *are symmetric about* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$, *respectively, with* $h_1=h_2=h$, *then one has the inequality;*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$\leq \frac{1}{4(b-a)}\int_a^b\int_c^d [f(x,c)+f(x,d)]\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dydx$$

$$+\frac{1}{4(d-c)}\int_c^d\int_a^b [f(a,y)+f(b,y)]\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x,y)dxdy$$

Proof. Since the partial mappings f_x and g_x are $h-$convex, by applying to the inequality (1.13), we can write

$$\frac{1}{d-c}\int_c^d f_x(y)g_x(y)dy \leq \frac{f_x(c)+f_x(d)}{2}\int_c^d\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g_x(y)dy.$$

That is;

$$\frac{1}{d-c}\int_c^d f(x,y)g(x,y)dy \leq \frac{f(x,c)+f(x,d)}{2}\int_c^d\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dy.$$

Integrating the result with respect to x on $[a,b]$ and dividing both sides of inequality, we get;

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.11}$$

$$\leq \frac{1}{2(b-a)}\int_a^b\int_c^d \left[f(x,c)+f(x,d)\right]\left(h\left(\frac{d-y}{d-c}\right)+h\left(\frac{y-c}{d-c}\right)\right)g(x,y)dydx.$$

By a similar argument f_y and g_y are $h-$convex, by applying to the inequality (1.13), we get;

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{2.12}$$

$$\leq \frac{1}{2(d-c)}\int_c^d\int_a^b \left[f(a,y)+f(b,y)\right]\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x,y)dxdy.$$

Summing (2.11) and (2.12), we obtain the required result.

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