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Notes about Anti-paraHermitian Metric Connections

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Abstract. In the present paper firstly, we introduce classes of anti-paraKähler-Codazzi manifolds and we discuss the problem of integrability for almost paracomplex structures on these manifolds. Secondly, we introduce new classes of anti-paraHermitian manifolds associated with these anti-paraHermitian metric connections with torsion, we look for the conditions in which they become anti-paraKähler manifolds or anti-paraKähler-Codazzi manifolds.

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1. Introduction

An almost product structure φ on a manifold M is a $(1,1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M,φ) is called an almost product manifold.

A linear connection ∇ on (M,φ) such that $\nabla\varphi = 0$ is said an almost product connection. There exists an almost product connection on every almost product manifold [2].

An almost paracomplex manifold is an almost product manifold (M,φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [1].

The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y],$$

for any vector fields X and Y on M.

A paracomplex structure is an integrable almost paracomplex structure. On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection such that $\nabla\varphi = 0$ [5,6]

Let (M^{2m},φ) be an almost paracomplex manifold. A Riemannian metric g is said anti-paraHermitian metric with respect to the paracomplex structure φ if

$$g(\varphi X, \varphi Y) = g(X,Y),$$

or equivalently (purity condition), (B-metric) [6]

$$g(\varphi X, Y) = g(X, \varphi Y),$$

for any vector fields X and Y on M.

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Also note that

$$G(X,Y) = g(\varphi X, Y),$$

is a bilinear, symmetric tensor field of type $(0,2)$ on (M,φ) and pure with respect to the paracomplex structure φ, which is called the twin anti-paraHermitian metric of g, and it plays a role similar to the Kähler form in Hermitian Geometry. Some properties of twin Norden Riemannian metric are investigated in [4, 6].

If (M^{2m},φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m},φ,g) is said to be an almost anti-paraHermitian manifold (almost B-manifold) [6]. Moreover, (M^{2m},φ,g) is said to be anti-paraKähler manifold (B-manifold) [6] if φ is parallel with respect to the Levi-Civita connection ∇ of g i.e. $\nabla\varphi = 0$.

A Tachibana operator ϕ_φ applied to the anti-paraHermitian metric (pure metric) g is given by

$$(\phi_\varphi g)(X,Y,Z) \quad = \quad \varphi X(g(Y,Z)) - X(g(\varphi Y,Z)) + g((L_Y\varphi)X,Z) + g((L_Z\varphi)X,Y),$$

for any vector fields X, Y and Z on M [11].

In an almost anti-paraHermitian manifold, an anti-paraHermitian metric g is called paraholomorphic if

$$(\phi_\varphi g)(X,Y,Z) = 0, \tag{1.1}$$

for any vector fields X, Y and Z on M [6].

If (M,g,φ) is an anti-paraKähler manifold with paraholomorphic anti-paraHermitian, we say that (M,g,φ) is a paraholomorphic anti-paraKähler manifold.

As the anti-paraKähler condition ($\nabla\varphi = 0$) is equivalent to paraholomorphicity condition of the anti-paraHermitian metric g, $(\phi_\varphi g) = 0$ [5, 6].

Salimov and his collaborators studied several aspects related to the topic of anti-hermetic metric communication, which is considered one of the new works in the topic, of which we mention the following: the Curvature properties of anti-Kähler-Codazzi manifolds, isotropic property of anti-Kähler-Codazzi manifolds, classes of anti-Hermitian manifolds and the purity condition of torsion tensor of anti-Hermitian metric connection see [7–10], as well as in this direction Ida and Manea have presented some properties of para-Norden metric connections [3].

2. Anti-paraKähler-Codazzi Manifolds

Let (M,g,φ) be an almost anti-paraHermitian manifold. If the twin anti-paraHermitian metric G satisfies the Codazzi equation

$$(\nabla_X G)(Y,Z) - (\nabla_Y G)(X,Z) = 0, \tag{2.1}$$

for any vector fields X, Y and Z on M, then the triple (M,g,φ) is called an anti-paraKähler-Codazzi manifold. For complex version, is called an anti-Kähler-Codazzi manifold, see [7]. Since

$$\begin{aligned}(\nabla_X G)(Y,Z) &= XG(Y,Z) - G(\nabla_X Y,Z) - G(Y,\nabla_X Z)\\ &= Xg(\varphi Y,Z) - g(\varphi\nabla_X Y,Z) - g(\varphi Y,\nabla_X Z)\\ &= g(\nabla_X(\varphi Y),Z) + g(\varphi Y,\nabla_X Z) - g(\varphi\nabla_X Y,Z) - g(\varphi Y,\nabla_X Z)\\ &= g((\nabla_X\varphi)Y,Z),\end{aligned} \tag{2.2}$$

the equation (2.1) is equivalent to

$$(\nabla_X\varphi)Y - (\nabla_Y\varphi)X = 0, \tag{2.3}$$

for any vector fields X and Y on M.

Remark 2.1.
(i) It is clear that any anti-paraKähler manifold ($\nabla\varphi = 0$) is anti-paraKähler-Codazzi (2.3).
(ii) The converse statement is not true, i.e., the condion ($\nabla\varphi = 0$) is not true for anti-paraKähler-Codazzi manifolds in general.

Theorem 2.2. *Let (M,g,φ) be an almost anti-paraHermitian manifold. If (M,g,φ) is an anti-paraKähler-Codazzi manifold, then the almost paracomplex structure φ is integrable.*

Proof. Using $[X,Y] = \nabla_X Y - \nabla_Y X$, $(\nabla_X\varphi)Y = \nabla_X(\varphi Y) - \varphi(\nabla_X Y)$ and (2.3), we find

$$\begin{aligned}
N_\varphi(X,Y) &= [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] + [X,Y] \\
&= \nabla_{(\varphi X)}(\varphi Y) - \nabla_{(\varphi Y)}(\varphi X) - \varphi(\nabla_{(\varphi X)}Y - \nabla_Y(\varphi X)) - \varphi(\nabla_X(\varphi Y) - \nabla_{(\varphi Y)}X) + \nabla_X Y - \nabla_Y X \\
&= (\nabla_{(\varphi X)}\varphi)(\varphi Y) + \varphi(\nabla_{(\varphi X)}Y) - (\nabla_{(\varphi Y)}\varphi)(\varphi X) - \varphi(\nabla_{(\varphi Y)}X) - \varphi(\nabla_{(\varphi X)}Y) + \varphi(\nabla_Y(\varphi X)) - \varphi(\nabla_X(\varphi Y)) \\
&\quad + \varphi(\nabla_{(\varphi Y)}X) + \nabla_X Y - \nabla_Y X \\
&= (\nabla_{(\varphi X)}\varphi)(\varphi Y) - (\nabla_{(\varphi Y)}\varphi)(\varphi X) + \varphi(\nabla_Y(\varphi X)) - \nabla_Y X - \varphi(\nabla_X(\varphi Y)) + \nabla_X Y \\
&= (\nabla_{(\varphi X)}\varphi)(\varphi Y) - (\nabla_{(\varphi Y)}\varphi)(\varphi X) + \varphi(\nabla_Y(\varphi X) - \varphi(\nabla_Y X)) - \varphi(\nabla_X(\varphi Y) - \varphi(\nabla_X Y)) \\
&= (\nabla_{(\varphi X)}\varphi)(\varphi Y) - (\nabla_{(\varphi Y)}\varphi)(\varphi X) + \varphi((\nabla_Y\varphi)X - (\nabla_X\varphi)Y) \\
&= 0.
\end{aligned}$$

□

3. Anti-paraHermitian Metric Connections with Torsion

Given an anti-paraHermitian manifold (M,g,φ), the Levi-Civita connection of g is the only torsion-free connection parallelizing the metric g. But there are many other connections with torsion parallelizing the metric g. We call these connections anti-paraHermitian metric connections.

3.1. Let $\overline{\nabla}$ be an arbitrary linear connection on M. We define a tensor field S of type $(1,2)$ on M by

$$\overline{\nabla}_X Y = \nabla_X Y + S(X,Y), \tag{3.1}$$

for any vector fields X and Y on M, where ∇ is the Levi-Civita connection of g, then the torsion tensor $\overline{T}$ of connection $\overline{\nabla}$ is given by

$$\begin{aligned}
\overline{T}(X,Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \\
&= \nabla_X Y + S(X,Y) - \nabla_Y X - S(Y,X) - [X,Y] \\
&= S(X,Y) - S(Y,X). \qquad (3.2)
\end{aligned}$$

Remark 3.1. We will denote again by $\overline{T}$ (resp. S) the $(0,3)$-tensor field obtained by contracting with the metric g, defined by $\overline{T}(X,Y,Z) = g(\overline{T}(X,Y),Z)$ (resp. $S(X,Y,Z) = g(S(X,Y),Z)$),
for all vector fields X,Y and Z on M.

Lemma 3.2. *Let (M,g,φ) be an almost anti-paraHermitian manifold. A connection $\overline{\nabla}$ is metric connection of g if and only if*

$$S(X,Y,Z) + S(X,Z,Y) = 0, \tag{3.3}$$

for all vector fields X,Y and Z on M.

Proof. We compute the covariant derivative,

$$\begin{aligned}
(\overline{\nabla}_X g)(Y,Z) &= Xg(Y,Z) - g(\overline{\nabla}_X Y,Z) - g(Y,\overline{\nabla}_X Z) \\
&= Xg(Y,Z) - g(\nabla_X Y + S(X,Y),Z) - g(Y,\nabla_X Z + S(X,Z)) \\
&= (\nabla_X g)(Y,Z) - g(S(X,Y),Z) - g(Y,S(X,Z)) \\
&= -S(X,Y,Z) - S(X,Z,Y),
\end{aligned}$$

then, we get

$$\overline{\nabla} g = 0 \quad \Leftrightarrow \quad S(X,Y,Z) + S(X,Z,Y) = 0.$$

□

Remark 3.3. Using (3.2), we have

$$\begin{aligned}
\overline{T}(X,Y,Z) &= S(X,Y,Z) - S(Y,X,Z), \\
\overline{T}(Z,X,Y) &= S(Z,X,Y) - S(X,Z,Y), \\
\overline{T}(Z,Y,X) &= S(Z,Y,X) - S(Y,Z,X).
\end{aligned}$$

From (3.3), we get

$$S(X,Y,Z) = \frac{1}{2}(\overline{T}(X,Y,Z) + \overline{T}(Z,X,Y) + \overline{T}(Z,Y,X)). \tag{3.4}$$

Lemma 3.4. *Let (M, g, φ) be an almost anti-paraHermitian manifold. A connection $\overline{\nabla}$ is metric connection of G if and only if*

$$S(X,Y,\varphi Z) + S(X,Z,\varphi Y) = g((\nabla_X \varphi)Y, Z), \tag{3.5}$$

for all vector fields X, Y and Z on M.

Proof. We compute the covariant derivative of G and by (2.2), we find

$$\begin{aligned}
(\overline{\nabla}_X G)(Y,Z) &= XG(Y,Z) - G(\overline{\nabla}_X Y, Z) - G(Y, \overline{\nabla}_X Z) \\
&= XG(Y,Z) - G(\nabla_X Y + S(X,Y), Z) - G(Y, \nabla_X Z + S(X,Z)) \\
&= (\nabla_X G)(Y,Z) - G(S(X,Y),Z) - G(Y, S(X,Z)) \\
&= g((\nabla_X \varphi)Y, Z) - g(\varphi S(X,Y), Z) - g(\varphi Y, S(X,Z)) \\
&= g((\nabla_X \varphi)Y, Z) - S(X,Y,\varphi Z) - S(X,Z,\varphi Y),
\end{aligned}$$

then, we get

$$\overline{\nabla} G = 0 \quad \Leftrightarrow \quad S(X,Y,\varphi Z) + S(X,Z,\varphi Y) = g((\nabla_X \varphi)Y, Z).$$

□

We study two cases:
First case: We put

$$S(X,Y,\varphi Z) = S(X,Z,\varphi Y),$$

from (3.5), we have

$$2g(S(X,Y),\varphi Z) = g((\nabla_X \varphi)Y, Z),$$

then,

$$2g(S(X,Y),\varphi Z) = g(\varphi(\nabla_X \varphi)Y, \varphi Z).$$

Hence, we find

$$S(X,Y) = \frac{1}{2}\varphi((\nabla_X \varphi)Y). \tag{3.6}$$

Using (3.1) and (3.6) we get

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\varphi((\nabla_X \varphi)Y) \tag{3.7}$$

and

$$\overline{T}(X,Y) = \frac{1}{2}\varphi((\nabla_X \varphi)Y - (\nabla_Y \varphi)X). \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}
S(X,Y,Z) + S(X,Z,Y) &= g(S(X,Y),Z) + g(S(X,Z),Y) \\
&= \frac{1}{2}g(\varphi((\nabla_X \varphi)Y), Z) + \frac{1}{2}g(\varphi((\nabla_X \varphi)Z), Y) \\
&= \frac{1}{2}g(\nabla_X(\varphi Y) - \varphi(\nabla_X Y), \varphi Z) + \frac{1}{2}g(\nabla_X(\varphi Z) - \varphi(\nabla_X Z), \varphi Y) \\
&= \frac{1}{2}g(\nabla_X(\varphi Y), \varphi Z) - \frac{1}{2}g(\nabla_X Y, Z) + \frac{1}{2}g(\nabla_X(\varphi Z), \varphi Y) - \frac{1}{2}g(\nabla_X Z, Y) \\
&= \frac{1}{2}Xg(\varphi Y, \varphi Z) - \frac{1}{2}Xg(Y,Z) = 0,
\end{aligned}$$

and

$$\begin{aligned} S(X,Y,\varphi Z)+S(X,Z,\varphi Y) &= g(S(X,Y),\varphi Z)+g(S(X,Z),\varphi Y)\\ &= \frac{1}{2}g((\nabla_X\varphi)Y,Z)+\frac{1}{2}g((\nabla_X\varphi)Z,Y)\\ &= \frac{1}{2}g((\nabla_X\varphi)Y,Z)+\frac{1}{2}g((\nabla_X\varphi)Y,Z)\\ &= g((\nabla_X\varphi)Y,Z). \end{aligned}$$

In an almost anti-paraHermitian manifold, by (3.6) the tensor S satisfies the equation (3.3) (i.e. $\overline{\nabla}g=0$) and the equation (3.5) (i.e. $\overline{\nabla}G=0$). Using (3.7), the connection $\overline{\nabla}$ is anti-paraHermitian metric connection.

We compute the covariant derivative of φ

$$\begin{aligned} (\overline{\nabla}_X\varphi)Y &= \overline{\nabla}_X(\varphi Y)-\varphi(\overline{\nabla}_X Y)\\ &= \nabla_X(\varphi Y)+\frac{1}{2}\varphi(\nabla_X\varphi)(\varphi Y)-\varphi(\nabla_X Y+\frac{1}{2}\varphi(\nabla_X\varphi)Y)\\ &= \nabla_X(\varphi Y)-\varphi(\nabla_X Y)+\frac{1}{2}\varphi((\nabla_X\varphi)(\varphi Y))-\frac{1}{2}(\nabla_X\varphi)Y\\ &= (\nabla_X\varphi)Y+\frac{1}{2}\varphi(\nabla_X Y-\varphi(\nabla_X(\varphi Y)))-\frac{1}{2}(\nabla_X\varphi)Y\\ &= (\nabla_X\varphi)Y-\frac{1}{2}(\nabla_X\varphi)Y-\frac{1}{2}(\nabla_X\varphi)Y\\ &= 0, \end{aligned}$$

then, we get $\overline{\nabla}\varphi=0$. Hence, we have the following theorem

Theorem 3.5. *Let (M,g,φ) be an almost anti-parHermitian manifold, there is a unique connection $\overline{\nabla}$ with torsion $\overline{T}$, that is parallelizing (compatible with) both the anti-paraHermitian metric g, the twin anti-paraHermitian metric G and the paracomplex structure φ i.e., $\overline{\nabla}g=0$, $\overline{\nabla}G=0$ and $\overline{\nabla}\varphi=0$. Moreover, $\overline{\nabla}$ and $\overline{T}$ are explicitly given by*

$$\overline{\nabla}_X Y = \nabla_X Y+\frac{1}{2}\varphi((\nabla_X\varphi)Y),$$

and

$$\overline{T}(X,Y) = \frac{1}{2}\varphi((\nabla_X\varphi)Y-(\nabla_Y\varphi)X),$$

where ∇ is the Levi-Civita connection of g.

Remark 3.6. Using (3.4) and (3.6), we get

$$g((\nabla_X\varphi)Y,\varphi Z) = \overline{T}(X,Y,Z)+\overline{T}(Z,X,Y)+\overline{T}(Z,Y,X).$$

Now let $\overline{\nabla}$ be a unique connection with torsion $\overline{T}$ which parallelizing (compatible with) both the anti-paraHermitian metric g, the paracomplex structure φ (see Theorem 3.5). From (1.1) we have

$$\begin{aligned} \phi_\varphi g(X,Y,Z) &= (\varphi X)g(Y,Z)-Xg(\varphi Y,Z)+g((L_Y\varphi)X,Z)+g(Y,(L_Z\varphi)X)\\ &= (\varphi X)g(Y,Z)-Xg(\varphi Y,Z)+g(L_Y(\varphi X),Z)-g(\varphi(L_Y X),Z)+g(Y,L_Z(\varphi X))-g(Y,\varphi(L_Z X)). \end{aligned}$$

Using $L_Y X=[Y,X]=\overline{\nabla}_Y X-\overline{\nabla}_X Y-\overline{T}(Y,X)$, we have

$$\begin{aligned} \phi_\varphi g(X,Y,Z) &= (\varphi X)g(Y,Z)-Xg(\varphi Y,Z)+g(\overline{\nabla}_Y(\varphi X),Z)-g(\overline{\nabla}_{\varphi X}Y,Z)-g(\overline{T}(Y,\varphi X),Z)-g(\overline{\nabla}_Y X,\varphi Z)\\ &\quad +g(\overline{\nabla}_X Y,\varphi Z)+g(\overline{T}(Y,X),\varphi Z)+g(Y,\overline{\nabla}_Z(\varphi X))-g(Y,\overline{\nabla}_{\varphi X}Z)-g(Y,\overline{T}(Z,\varphi X))\\ &\quad -g(\varphi Y,\overline{\nabla}_Z X)+g(\varphi Y,\overline{\nabla}_X Z)+g(\varphi Y,\overline{T}(Z,X)). \end{aligned}$$

Since $\overline{\nabla}g=0$ and $\overline{\nabla}\varphi=0$, we get

$$\begin{aligned} \phi_\varphi g(X,Y,Z) &= (\overline{\nabla}_{\varphi X}g)(Y,Z)+(\overline{\nabla}_X g)(\varphi Y,Z)+g((\overline{\nabla}_Y\varphi)X,Z)+g(Y,(\overline{\nabla}_Z\varphi)X)+\overline{T}(Y,X,\varphi Z)-\overline{T}(Y,\varphi X,Z)\\ &\quad +\overline{T}(Z,X,\varphi Y)-\overline{T}(Z,\varphi X,Y)\\ &= \overline{T}(Y,X,\varphi Z)-\overline{T}(Y,\varphi X,Z)+\overline{T}(Z,X,\varphi Y)-\overline{T}(Z,\varphi X,Y). \end{aligned}$$

From here we see that, if a torsion tensor $\overline{T}$ of $\overline{\nabla}$ is pure with respect to the last arguments, i.e. $\overline{T}(X, \varphi Y, Z) = \overline{T}(X, Y, \varphi Z)$, then the given anti-paraHermitian metric g is paraholomorphic, i.e. $\phi_\varphi g = 0$. On the other hand, the anti-paraKähler (B-manifold) condition ($\nabla\varphi = 0$) is equivalent to paraholomorphicity condition of the anti-paraHermitian metric g, $(\phi_\varphi g) = 0$ [6], hence the triple (M, g, φ) is anti-paraKähler manifold. Thus, we have the following theorem.

Theorem 3.7. *Let (M, g, φ) be an anti-paraHermitian manifold and let $\overline{\nabla}$ be an anti-paraHermitian connection with torsion $\overline{T}$ which parallelizing (compatible with) both the anti-paraHermitian metric g. If the torsion tensor $\overline{T}$ is pure with respect to the last arguments $\overline{T}(X, \varphi Y, Z) = \overline{T}(X, Y, \varphi Z)$, then the given anti-paraHermitian manifold is an anti-paraKähler.*

If (M, g, φ) is an anti-paraKähler-Codazzi manifold. Then, from (2.3) and (3.8) we get that $\overline{T} = 0$ and from (3.4) we find $S = 0$, then $\overline{\nabla} = \nabla$. Hence, we have the following theorem

Theorem 3.8. *Let (M, g, φ) be an anti-paraKähler-Codazzi manifold, then the anti-paraHermitian metric connection $\overline{\nabla}$ coincides with the Levi-Civita connection ∇ of g.*

Second case: we put

$$S(X, Y, \varphi Z) = S(Z, Y, \varphi X), \tag{3.9}$$

from (3.5), we have

$$\begin{aligned} S(X, Y, \varphi Z) + S(X, Z, \varphi Y) &= g((\nabla_X\varphi)Y, Z), \\ S(Y, Z, \varphi X) + S(Y, X, \varphi Z) &= g((\nabla_Y\varphi)Z, X), \\ S(Z, X, \varphi Y) + S(Z, Y, \varphi X) &= g((\nabla_Z\varphi)X, Y), \end{aligned}$$

using (3.9), we get

$$2g(S(X, Y), \varphi Z) = g((\nabla_X\varphi)Y, Z) - g((\nabla_Y\varphi)Z, X) + g((\nabla_Z\varphi)X, Y). \tag{3.10}$$

We distinguish several cases:

i) Since in an almost anti-paraHermitian manifold (almost B-manifold) [6] the operator $\phi_\varphi g$ reduces to form

$$(\phi_\varphi g)(Y, Z, X) = -g((\nabla_Y\varphi)Z, X) + g((\nabla_Z\varphi)X, Y) + g((\nabla_X\varphi)Y, Z),$$

and the anti-paraKähler condition ($\nabla\varphi = 0$) is equivalent to paraholomorphicity condition of the anti-paraHermitian metric g, $(\phi_\varphi g) = 0$, from (3.10) we get $S = 0$, $\overline{\nabla} = \nabla$ and the tensor S ($S = 0$) satisfies the equation (3.3). Hence, we have the following theorem

Theorem 3.9. *Let (M, g, φ) be an anti-paraKähler manifold, then the anti-paraHermitian metric connection $\overline{\nabla}$ coincides with the Levi-Civita connection ∇ of g.*

ii) If (M, g, φ) is an anti-paraKähler-Codazzi manifold, then from (2.3) and (3.10), since $g((\nabla_Z\varphi)X, Y) = g((\nabla_Z\varphi)Y, X)$ [6], we get

$$2g(S(X, Y), \varphi Z) = g((\nabla_X\varphi)Y, Z),$$

then,

$$2g(S(X, Y), \varphi Z) = g(\varphi(\nabla_X\varphi)Y, \varphi Z).$$

Hence, we find

$$S(X, Y) = \frac{1}{2}\varphi((\nabla_X\varphi)Y). \tag{3.11}$$

Using (3.1) and (3.11) we get

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\varphi((\nabla_X\varphi)Y), \tag{3.12}$$

and

$$\overline{T}(X, Y) = \frac{1}{2}\varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X).$$

On the other hand, we have

$$S(X,Y,Z)+S(X,Z,Y)=0 \text{ and } S(X,Y,\varphi Z)+S(X,Z,\varphi Y)=g((\nabla_X\varphi)Y,Z).$$

Then, in an anti-paraKähler-Codazzi manifold, by (3.11) the tensor S satisfies the equation (3.3) (i.e. $\overline{\nabla}g=0$) and the equation (3.5) (i.e. $\overline{\nabla}G=0$). Using (3.12), the connection $\overline{\nabla}$ is anti-paraHermitian metric connection. Hence, we have the following theorem.

Theorem 3.10. *Let (M,g,φ) be an anti-paraKähler-Codazzi manifold, then the anti-paraHermitian metric connection $\overline{\nabla}$ is explicitly given by*

$$\overline{\nabla}_X Y=\nabla_X Y+\frac{1}{2}\varphi((\nabla_X\varphi)Y),$$

where ∇ is the Levi-Civita connection of g.

iii) Let (M,g,φ) be an almost anti-paraHermitian manifold, if

$$S(X,Y)=-\varphi((\nabla_X\varphi)Y),$$

then

$$\overline{\nabla}_X Y=\nabla_X Y-\varphi((\nabla_X\varphi)Y),$$

and

$$\begin{aligned}
S(X,Y,Z)+S(X,Z,Y) &= g(S(X,Y),Z)+g(S(X,Z),Y)\\
&= -g(\varphi((\nabla_X\varphi)Y),Z)-g(\varphi((\nabla_X\varphi)Z),Y)\\
&= -g(\nabla_X(\varphi Y)-\varphi(\nabla_X Y),\varphi Z)-g(\nabla_X(\varphi Z)-\varphi(\nabla_X Z),\varphi Y)\\
&= -g(\nabla_X(\varphi Y),\varphi Z)+g(\nabla_X Y,Z)-g(\nabla_X(\varphi Z),\varphi Y)+g(\nabla_X Z,Y)\\
&= -Xg(\varphi Y,\varphi Z)+Xg(Y,Z)=0.
\end{aligned}$$

Hence, we have the following theorem

Theorem 3.11. *Let (M,g,φ) be an almost anti-parHermitian manifold, there is a unique connection $\overline{\nabla}$ with torsion $\overline{T}$, that is parallelizing (compatible with) the anti-paraHermitian metric g i.e., $\overline{\nabla}g=0$. Moreover, $\overline{\nabla}$ and $\overline{T}$ are explicitly given by*

$$\overline{\nabla}_X Y = \nabla_X Y-\varphi((\nabla_X\varphi)Y),$$

and

$$\overline{T}(X,Y) = -\varphi((\nabla_X\varphi)Y-(\nabla_Y\varphi)X),$$

where ∇ is the Levi-Civita connection of g.

3.2. Let $\widehat{\nabla}$ be an arbitrary linear connection on M. We define a tensor field P of type $(1,2)$ on M by

$$\widehat{\nabla}_X Y=\nabla_Y X+P(X,Y), \tag{3.13}$$

for any vector fields X and Y on M, where ∇ is the Levi-Civita connection of g. By virtue of $[X,Y]=\nabla_X Y-\nabla_Y X$ the equation (3.13) becomes

$$\widehat{\nabla}_X Y=\nabla_X Y-[X,Y]+P(X,Y).$$

If we put $S(X,Y)=-[X,Y]+P(X,Y)$, the connection $\widehat{\nabla}$ coincides with the connection $\overline{\nabla}$ defined by (3.1). Then in this case, the study of properties of the connection $\widehat{\nabla}$ coincides to properties of $\overline{\nabla}$.

3.3. Let $\widetilde{\nabla}$ be an arbitrary linear connection on M. We define a tensor field Q of type $(1,2)$ on M by

$$\widetilde{\nabla}_X Y=\varphi\nabla_X Y+Q(X,Y), \tag{3.14}$$

for any vector fields X and Y on M, where ∇ is the Levi-Civita connection of g. We can write (3.14) as follows

$$\widetilde{\nabla}_X Y=\nabla_X Y+(\varphi-I)\varphi\nabla_X Y+Q(X,Y).$$

If we put $S(X,Y)=(\varphi-I)\varphi\nabla_X Y+Q(X,Y)$, the connection $\widetilde{\nabla}$ coincides with the connection $\overline{\nabla}$ defined by (3.1). Then, in this case, the study of properties of the connection $\widetilde{\nabla}$ coincides to properties of $\overline{\nabla}$.

4. Example of Anti-paraHermitian Metrics

Example 4.1. Let $\mathbb{R}^2$ be endowed with the Riemannian metric g and the structure paracomplex φ in polar coordinate defined by

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix},$$

and

$$\varphi = \begin{pmatrix} \sin 2\theta & r\cos 2\theta \\ \frac{1}{r}\cos 2\theta & -\sin 2\theta \end{pmatrix}.$$

We easily verify that

$$\varphi^2 = I,$$

and

$$g(\varphi X, \varphi Y) = X^r Y^r + r^2 X^\theta Y^\theta = g(X, Y),$$

for any vector fields $X = X^r\partial_r + X^\theta\partial_\theta$, $Y = Y^r\partial_r + Y^\theta\partial_\theta$, i.e. $(\mathbb{R}^2, g, \varphi)$ is an almost anti-paraHermitian manifold.
The Levi-Civita connection ∇ of g is given by

$$\nabla_{\partial_r}\partial_r = 0,\ \nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta,\ \nabla_{\partial_\theta}\partial_\theta = -r\partial_r.$$

We also check that,

$$(\nabla_{\partial_r}\varphi)\partial_r = (\nabla_{\partial_r}\varphi)\partial_\theta = (\nabla_{\partial_\theta}\varphi)\partial_r = (\nabla_{\partial_\theta}\varphi)\partial_\theta = 0,$$

i.e. φ is parallel with respect to the Levi-Civita connection ∇ of g, hence $(\mathbb{R}^2, g, \varphi)$ is anti-paraKähler manifold. Then, from the Theorem 3.5 and Theorem 3.9, the anti-paraHermitian metric connection $\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\varphi((\nabla_X\varphi)Y)$ coincides with the Levi-Civita connection ∇ of g.

Example 4.2. We consider the hyperbolic space H^4 endowed with the anti-paraHermitian structure (φ, h) defined by

$$H^4 = \{(x, y, z, t) \in \mathbb{R}^2 :\ t > 0\},$$

$$g = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & \frac{1}{t} & 0 \\ 0 & 0 & 0 & \frac{1}{t} \end{pmatrix} \textit{and } \varphi = \begin{pmatrix} 0 & \frac{1}{t} & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We easily verify that

$$\varphi^2 = I$$

and

$$g(\varphi X, \varphi Y) = g(X, Y),$$

for any vector fields X and Y.
The Levi-Civita connection ∇ of h is given by

$$\nabla_{\partial_x}\partial_t = \nabla_{\partial_t}\partial_x = \frac{1}{2t}\partial_x,\ \nabla_{\partial_y}\partial_t = \nabla_{\partial_t}\partial_y = \frac{-1}{2t}\partial_y,\ \nabla_{\partial_z}\partial_t = \nabla_{\partial_t}\partial_z = \frac{-1}{2t}\partial_z,\ \nabla_{\partial_x}\partial_x = \frac{-t}{2}\partial_t,$$

$$\nabla_{\partial_y}\partial_y = \nabla_{\partial_z}\partial_z = -\nabla_{\partial_t}\partial_t = \frac{1}{2t}\partial_t,\ \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \nabla_{\partial_x}\partial_z = \nabla_{\partial_z}\partial_x = \nabla_{\partial_y}\partial_z = \nabla_{\partial_z}\partial_y = 0.$$

Note that

$$(\nabla_{\partial_x}\varphi)\partial_Y = \frac{-1}{2}\partial_t \neq 0,$$

i.e φ is not parallel with respect to the Levi-Civita connection ∇ of g, hence (H^4, φ, h) is not anti-paraKähler manifold. Then, from the Theorem 3.5, the anti-paraHermitian metric connection $\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\varphi((\nabla_X\varphi)Y)$ is given by

$$\overline{\nabla}_{\partial_x}\partial_x = \frac{-t}{4}\partial_t,\ \overline{\nabla}_{\partial_x}\partial_y = \frac{-1}{4}\partial_z,\ \overline{\nabla}_{\partial_x}\partial_z = \frac{1}{4}\partial_y,\ \overline{\nabla}_{\partial_x}\partial_t = \frac{1}{4t}\partial_x,\ \overline{\nabla}_{\partial_y}\partial_y = \frac{1}{4t}\partial_t,\ \overline{\nabla}_{\partial_y}\partial_z = \frac{-1}{4t^2}\partial_x,\ \overline{\nabla}_{\partial_y}\partial_t = \frac{-1}{4t}\partial_y,$$

$$\overline{\nabla}_{\partial_t}\partial_z = \frac{-1}{2t}\partial_z,\ \overline{\nabla}_{\partial_t}\partial_t = \frac{-1}{2t}\partial_t,\ \overline{\nabla}_{\partial_y}\partial_x = \overline{\nabla}_{\partial_z}\partial_x = \overline{\nabla}_{\partial_z}\partial_y = \overline{\nabla}_{\partial_z}\partial_z = \overline{\nabla}_{\partial_z}\partial_t = \overline{\nabla}_{\partial_t}\partial_x = \overline{\nabla}_{\partial_t}\partial_y = 0.$$

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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