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**Research Article**

# A New Characterization of Tzitzeica Curves in Euclidean 4-Space

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**Abstract:** In this study, we are interested in Tzitzeica curves (Tz-curves) in Euclidean 4-space $\mathbb{E}^4$. Tz-curve condition for Euclidean 4-space are determined as three types for three hyperplanes and some examples are given.

**Keywords:** Tzitzeica condition, Tzitzeica curve, hyperplane, Frenet frame.

## 1. Introduction

Gheorgha Tzitzeica, Romanian mathematician (1872-1939), introduced a class of surfaces [11], nowadays called Tzitzeica surfaces in 1907 and a class of curves [12], called Tzitzeica curves in 1911. A Tzitzeica curve in $\mathbb{E}^3$ is a spatial curve $x = x(s)$ with the Frenet frame $\{T, N_1, N_2\}$ and curvatures $\{k_1, k_2\}$, for which the ratio of its torsion $k_2$ and the square of the distance $d_{osc}$ from the origin to the osculating plane at an arbitrary point $x(s)$ of the curve is constant, i.e., a Tzitzeica curve in $\mathbb{E}^3$ is a curve satisfying the condition (Tzitzeica condition)

$$\frac{k_2}{{d_{osc}}^2} = a, \tag{1}$$

where $d_{osc} = \langle N_2, x\rangle$ and $a \neq 0$ is a real constant, $N_2$ is the binormal vector field of $x$.

A Tzitzeica surface in $\mathbb{E}^3$ is a spatial surface $M$ given with the parametrization $X(u,v)$, for which the ratio of its Gaussian curvature $K$ and the distance $d_{tan}$ from the origin to the tangent plane at any arbitrary point of the surface is constant, i.e., a Tzitzeica surface in $\mathbb{E}^3$ is a surface satisfying the condition (Tzitzeica condition)

$$\frac{K}{{d_{tan}}^4} = a_1 \tag{2}$$

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for a constant $a_1 \neq 0$. The orthogonal distance from the origin to the tangent plane is defined by

$$d_{tan} = \langle X, N \rangle, \tag{3}$$

where $X$ is the position vector of surface and $N$ is unit normal vector field of the surface.

In [1] the authors gave the connections between Tzitzeica curve and Tzitzeica surface in Minkowski 3-space and the original ones from the Euclidean 3-space. Besides, the asymptotic lines of a Tzitzeica surface with the negative Gaussian curvature are Tzitzeica curves [3]. In [3], the authors determined the elliptic and hyperbolic cylindrical curves satisfying Tzitzeica condition in Euclidean space. In [**? ?** ], hyperbolic and elliptic cylindrical curves verifying Tzitzeica condition were adapted to Minkowski 3-space, respectively.

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Let us denote $T(s) = x^{'}(s)$ and call $T(s)$ a unit tangent vector of $x$ at $s$. We denote the first Serret-Frenet curvature of $x$ by $k_1(s) = \|x^{''}(s)\|$. If $k_1(s) \neq 0$, then the unit principal normal vector $N_1(s)$ of the curve $x$ at $s$ is given by $T^{'}(s) = k_1(s)N_1(s)$. If $k_2(s) \neq 0$, then the unit second principal normal vector $N_2(s)$ of the curve $x$ at $s$ is given by $N_1{}^{'}(s) + k_1(s)T(s) = k_2(s)N_2(s)$, where $k_2$ is the second Serret-Frenet curvature of $x$. $N_2{}^{'}(s) + k_2(s)N_1(s) = k_3(s)N_3(s)$, where $k_3$ is the third Serret-Frenet curvature of $x$. Then, we have the Serret-Frenet formulae [5]:

$$\begin{aligned} T^{'}(s) &= k_1(s)N_1(s), \\ N_1{}^{'}(s) &= -k_1(s)T(s) + k_2(s)N_2(s), \\ N_2{}^{'}(s) &= -k_2(s)N_1(s) + k_3(s)N_3(s), \\ N_3{}^{'}(s) &= -k_3(s)N_2(s). \end{aligned} \tag{4}$$

If the Serret-Frenet curvatures $k_1(s), k_2(s)$ and $k_3(s)$ of $x$ are constant functions then $x$ is called a screw line or a helix [4]. Since these curves are the traces of 1-parameter family of the groups of Euclidean transformations, Klein and Lie called them W-curves [8]. If the tangent vector $T$ of the curve $x$ makes a constant angle with a unit vector $U$ of $\mathbb{E}^4$ then this curve is called a general helix (or inclined curve) in $\mathbb{E}^4$ [9].

Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. Position vector of $x = x(s)$ satisfies parametric equation

$$x(s) = m_0(s)T(s) + m_1(s)N_1(s) + m_2(s)N_2(s) + m_3(s)N_3(s), \tag{5}$$

where

$$m_0(s) = \langle x(s), T(s), \rangle \qquad m_1(s) = \langle x(s), N_1(s), \rangle$$
$$m_2(s) = \langle x(s), N_2(s), \rangle \qquad m_3(s) = \langle x(s), N_3(s). \rangle \tag{6}$$

By taking the derivative of (5) with respect to arclength parameter s and using Serret-Frenet equations (4) , we obtain

$$\begin{aligned} T(s) = x'(s) &= m_0{}'(s)T(s) + m_0(s)T'(s) + m_1{}'(s)N_1(s) + m_1(s)N_1{}'(s) + m_2{}'(s)N_2(s) \\ &+ m_2(s)N_2{}'(s) + m_3{}'(s)N_3(s) + m_3(s)N_3{}'(s) \\ &= (m_0{}'(s) - m_1(s)k_1(s))T(s) + (m_0(s)k_1(s) + m_1{}'(s) - m_2(s)k_2(s))N_1(s) \\ &+ (m_1(s)k_2(s) + m_2{}'(s) - m_3(s)k_3(s))N_2(s) + (m_2(s)k_3(s) + m_3'(s))N_3(s). \end{aligned}$$

It follows that

$$\begin{aligned} m_0' - k_1 m_1 &= 1, \\ m_1' + k_1 m_0 - k_2 m_2 &= 0, \\ m_2' + k_2 m_1 - k_3 m_3 &= 0, \\ m_3' + k_3 m_2 &= 0. \end{aligned} \tag{7}$$

We consider Tzitzeica curves in Euclidean 4-space $\mathbb{E}^4$ whose position vector $x = x(s)$ satisfies the parametric equation (5). We determine Tz-curve condition for Euclidean 4-space $\mathbb{E}^4$ as three types for three hyperplanes and give some examples. Besides, we express Tzitzeica curve conditions in terms of their curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$.

## 2. A Characterization of Tzitzeica Curves in Euclidean 4-Space

**Definition 2.1** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. A first type Tzitzeica curve $x = x(s)$, for which the ratio of its second Frenet curvature $k_2$ and the square of the distance $d_{\{T,N_1,\ N_3\}}$ from the origin to the hyperplane spanned by $\{T, N_1,\ N_3\}$ at an arbitrary point $x(s)$ of the curve is constant, i.e.,*

$$\frac{k_2}{d^2_{\{T,N_1,\ N_3\}}} = a_1, \tag{8}$$

*where*

$$d_{\{T,N_1,\ N_3\}} = \langle x, N_2 \rangle \tag{9}$$

*and $a_1 \neq 0$ is a real constant.*

**Definition 2.2** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. A second type Tzitzeica curve $x = x(s)$, for which the ratio of its first Frenet curvature $k_1$ and the square of the distance $d_{\{T,N_2,\ N_3\}}$ from the origin to the hyperplane spanned by $\{T, N_2,\ N_3\}$ at an arbitrary point $x(s)$ of the curve is constant, i.e.,*

$$\frac{k_1}{d^2_{\{T,N_2,\ N_3\}}} = a_2, \tag{10}$$

*where*

$$d_{\{T,N_2,\ N_3\}} = \langle x, N_1 \rangle \tag{11}$$

*and $a_2 \neq 0$ is a real constant.*

**Definition 2.3** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in Euclidean 4-space $\mathbb{E}^4$. A third type Tzitzeica curve $x = x(s)$, for which the ratio of its second Frenet curvature $k_3$ and the square of the distance $d_{\{T,N_1,\ N_2\}}$ from the origin to the hyperplane spanned by $\{T, N_1,\ N_2\}$ at an arbitrary point $x(s)$ of the curve is constant, i.e.,*

$$\frac{k_3}{d^2_{\{T,N_1,\ N_2\}}} = a_3, \tag{12}$$

*where*

$$d_{\{T,N_1,\ N_2\}} = \langle x, N_3 \rangle \tag{13}$$

*and $a_3 \neq 0$ is a real constant.*

**Theorem 2.4** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). $x$ is first type Tzitzeica curve if and only if the equation*

$$k_2' m_2 + 2k_2^2 m_1 - 2k_2 k_3 m_3 = 0 \tag{14}$$

*holds.*

**Proof** Let $x$ be the first type Tzitzeica curve. By taking the derivative of (8) with respect to arc length parameter $s$ and using (4) and (6), we get (14). The opposite of the proof is clear. □

**Proposition 2.5** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the* *parametrization* (5). *Then*

$$
\begin{aligned}
m_0 &= 0, \\
m_1 &= \frac{-1}{k_1}, \\
m_2 &= \frac{k_1'}{k_2 k_1^2}, \\
m_3 &= \frac{k_1''}{k_1^2 k_2 k_3} - \frac{2{k_1'}^2}{k_1^3 k_2 k_3} - \frac{k_1' k_2'}{k_1^2 k_2^2 k_3} - \frac{k_2}{k_1 k_3}
\end{aligned}
\tag{15}
$$

*hold.*

**Proof** Let $x$ be a unit speed spherical curve. Then, $\langle x, x\rangle = r^2$. By taking the derivative of this expression, we get

$$\langle x, T\rangle = 0 = m_0. \tag{16}$$

By taking the derivative of (16) and using (4) and (6), we get

$$\langle x, N_1\rangle = \frac{-1}{k_1} = m_1. \tag{17}$$

Again, by taking the derivative of (17) and using (4), (16) and (6), we get

$$\langle x, N_2\rangle = \frac{{k_1}'}{k_2 {k_1}^2} = m_2. \tag{18}$$

Similarly, by taking the derivative of (18) and using (4), (17) and (6), we get

$$\langle x, N_3\rangle = \frac{{k_1}''}{{k_1}^2 k_2 k_3} - \frac{2{k_1}'^2}{{k_1}^3 k_2 k_3} - \frac{{k_1}'{k_2}'}{{k_1}^2 {k_2}^2 k_3} - \frac{k_2}{k_1 k_3} = m_3.$$

□

**Theorem 2.6** *Let* $x : I \subset \mathbb{R} \to \mathbb{E}^4$ *be a unit speed spherical curve in* $\mathbb{E}^4$ *given with the parametrization* (5). $x$ *is first type Tzitzeica curve if and only if the equations*

$$3k_1 k_1' k_2' - 2k_1 k_1'' k_2 + 4{k_1'}^2 k_2 = 0 \tag{19}$$

*and* $k_2 = c.\left[\left(\frac{-1}{k_1}\right)'\right]^{\frac{2}{3}}$ *hold, where* $c$ *is integral constant.*

**Proof** Let $x$ be a first type Tzitzeica curve. Then, substituing (15) into (14) and arranging the expression, we get (19). From the solution of (19), we get $k_2 = c.\left[\left(\frac{-1}{k_1}\right)'\right]^{\frac{2}{3}}$. The opposite of the proof is clear. □

**Corollary 2.7** *Let $x$ be a first type spherical Tzitzeica curve. If $k_2$ is constant, then we get $k_1 = \frac{c_2}{c_1+s}$.*

**Proof** If $k_2$ is constant, equation $k_1 k_1'' - 2{k_1'}^2 = 0$ is obtained from (19). If this equation is solved, then we get $k_1 = \frac{c_2}{c_1+s}$. □

**Theorem 2.8** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). $x$ is second type Tzitzeica curve if and only if the equation*

$$k_1' m_1 + 2k_1^2 m_0 - 2k_1 k_2 m_2 = 0 \tag{20}$$

*holds.*

**Proof** Let $x$ be the second type Tzitzeica curve. By taking the derivative of (10) with respect to arc length parameter $s$ and using (4) and (6), we get (20). The opposite of the proof is clear. □

**Proposition 2.9** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the parametrization (5). $x$ is second type Tzitzeica curve if and only if $k_1 = c$, where $c$ is a constant.*

**Proof** Let $x$ be the second type spherical Tzitzeica curve. Substituing (15) into (20), we get $3\frac{k_1'}{k_1} = 0$. Which means that, $k_1 = c$ (constant). □

**Theorem 2.10** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$ given with the parametrization (5). $x$ is third type Tzitzeica curve if and only if the equation*

$$k_3' m_3 + 2k_3^2 m_2 = 0 \tag{21}$$

*holds.*

**Proof** Let $x$ be the third type Tzitzeica curve. By taking the derivative of (12) with respect to arc length parameter $s$ and using (4) and (6), we get (21). The opposite of the proof is clear. □

**Proposition 2.11** *Let $x : I \subset \mathbb{R} \to \mathbb{E}^4$ be a unit speed spherical curve in $\mathbb{E}^4$ given with the parametrization (5). $x$ is third type Tzitzeica curve if and only if the equation*

$$k_3' \left( k_1'' - 2\frac{{k_1'}^2}{k_1} - \frac{k_1 k_2'}{k_2} - k_1 k_2^2 \right) + 2k_1' k_3^3 = 0 \tag{22}$$

*holds.*

**Proof** Let $x$ be third type spherical Tzitzeica curve. Then, substituing (15) into (21) and arranging the expression, we get (22). The opposite of the proof is clear. □

**Corollary 2.12** *Let $x$ be third type spherical Tzitzeica curve. If $k_1$ and $k_2$ are non-zero constants, then $x$ is a $W$-curve.*

**Example 2.13** *Let $x = x(s)$ be regular $W$-curve in $\mathbb{E}^4$ given with the parametrization*

$$x(s) = (a\cos(cs), a\sin(cs), b\cos(ds), b\sin(ds)) \tag{23}$$

*is a second type and third type Tzitzeica curve, where $0 \le s \le 2\pi$, $a, b, c, d$ real constants and $c > 0$, $d > 0$.*

*Then, $x$ without loss of generality, let $x$ be unit speed curve, i.e., $a^2c^2 + b^2d^2 = 1$. If $c = d$, then $x$ is a circle, otherwise $(c \neq d)$ $x$ is a curve in $\mathbb{E}^4$.*

*The Frenet curvatures $k_1, k_3$ and the Frenet vector fields $N_1, N_3$ of the curve $x$ can be given by*

$$k_1 = \sqrt{a^2c^4 + b^2d^4}, \tag{24}$$

$$k_3 = \frac{cd}{\sqrt{a^2c^4 + b^2d^4}}, \tag{25}$$

$$N_1 = \frac{1}{k_1}\left[-ac^2\cos(cs), -ac^2\sin(cs), -bd^2\cos(ds), -bd^2\sin(ds)\right], \tag{26}$$

$$N_3 = \frac{1}{k_1}\left[bd^2\cos(cs), bd^2\sin(cs), -ac^2\cos(ds), -ac^2\sin(ds)\right] \tag{27}$$

*[2]. By the use of (23) and (26) at (11), we get*

$$d_{\{T,N_2,\ N_3\}} = \frac{-1}{\sqrt{a^2c^4 + b^2d^4}}. \tag{28}$$

*Substituting (24) and (28) into (10), we get $a_2 = \left(a^2c^4 + b^2d^4\right)^{\frac{3}{2}}$, which means that $a_2$ is constant and $x$ is a second type Tzitzeica curve.*

*Further by the use of (23) and (27) at (13), we obtain*

$$d_{\{T,N_1,\ N_2\}} = \frac{ab\left(d^2 - c^2\right)}{\sqrt{a^2c^4 + b^2d^4}}. \tag{29}$$

*Substituing (25) and (29) into (12), we get $a_3 = \frac{cd\sqrt{a^2c^4+b^2d^4}}{a^2b^2(d^2-c^2)^2}$, which means that $a_3$ is constant and $x$ is a third type Tzitzeica curve.*

*Then, the projection of $W$-curve with the parametrization (23) on $x_4 = 0$ coordinate hyperplane in $\mathbb{E}^4$ is $x(s) = (\cos\left(s\sqrt{10}\right), \sin\left(s\sqrt{10}\right), \cos\left(3s\sqrt{10}\right))$ if we take $a = 1,\ b = 1,\ c = 1\sqrt{10}, d = 3\sqrt{10}$.*

*We can plot this $W$-curve with maple command with (plots):*

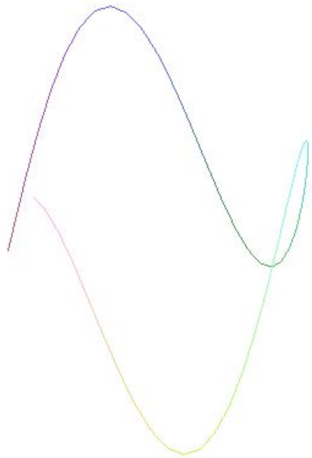

Figure 1: Second type and third type Tzitzeica curves, m=0, n=5*pi

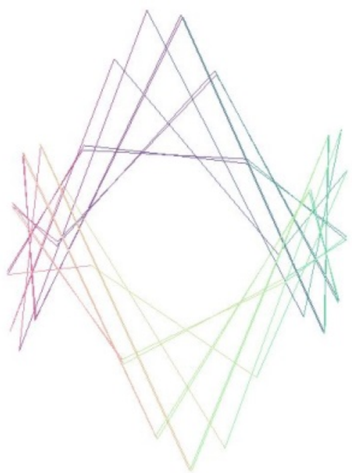

Figure 2: Second type and third type Tzitzeica curves, m=0, n=50*pi

*spacecurve([cos(t/sqrt(10)),sin(t/sqrt(10)),cos(3*t/sqrt(10))], t=m.n, grid=[30,30]*

**Example 2.14** *Let $x = x(s)$ be a helix on the unit 3-sphere $S^3(1)$ embedded in $\mathbb{E}^4$ given with the parametrization*

$$x(s) = (\cos\theta\cos(as), \cos\theta\sin(as), \sin\theta\cos(bs), \sin\theta\sin(bs)), \tag{30}$$

*where $a^2\cos^2\theta + b^2\sin^2\theta = 1$ and ${x_1}^2 + {x_2}^2 = \cos^2\theta, {x_3}^2 + {x_4}^2 = \sin^2\theta$. Then, $x$ is a second type and third type Tzitzeica curve.*

*The Frenet curvatures $k_1, k_3$ and the Frenet vector fields $N_1, N_3$ of the curve $x$ can be given by*

$$k_1 = \sqrt{a^4\cos^2\theta + b^4\sin^2\theta}, \tag{31}$$

$$k_3 = \frac{ab}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}, \tag{32}$$

$$N_1 = \frac{\left(-a^2\cos\theta\cos(as), -a^2\cos\theta\sin(as), -b^2\sin\theta\cos(bs), -b^2\sin\theta\sin(bs)\right)}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}} \tag{33}$$

$$N_3 = \frac{(b^2\sin\theta\cos(as), b^2\sin\theta\sin(as), -a^2\cos\theta\cos(bs), -a^2\cos\theta\sin(bs))}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}, \tag{34}$$

*[10]. By the use of (30) and (33) at (11), we get*

$$d_{\{T,N_2,\ N_3\}} = \frac{-1}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}. \tag{35}$$

*Substituting (31) and (35) into (10), we get* $a_2 = \left(\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}\right)^3$, *which means that* $a_2$ *is constant and* $x$ *is a second type Tzitzeica curve.*

*Further, by the use of (30) and (34) at (13), we obtain*

$$d_{\{T,N_1,\ N_2\}} = \frac{\cos\theta\sin\theta\left(b^2 - a^2\right)}{\sqrt{a^4\cos^2\theta + b^4\sin^2\theta}}. \tag{36}$$

*Substituting (35) and (36) into (12), we get* $a_3 = \frac{ab\left(a^4\cos^2\theta + b^4\sin^2\theta\right)^{\frac{1}{2}}}{\cos^2\theta\sin^2\theta\left(b^2-a^2\right)^2}$, *which means that* $a_3$ *is constant and* $x$ *is a third type Tzitzeica curve.*

### Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

### Authors Contributions

Author [Emrah Tunç]: Collected the data, contributed to research method or evaluation of data, wrote the manuscript (%50).

Author [Bengü Bayram]: Thought and designed the research/problem, contributed to completing the research and solving the problem (%50).

### Conflicts of Interest

The authors declare no conflict of interest.

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