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Qualitative Study of a Discrete-Time Harvested Fishery Model in the Presence of Toxicity

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Abstract

This paper analyses a discrete-time Michaelis-Menten type harvested fishery model in the presence of toxicity. Boundary and interior (positive) fixed points are examined. Using an iteration scheme and the comparison principle of difference equations, we determined the sufficient condition for global stability of the interior fixed point. It is shown that the sufficient criterion for Neimark-Sacker bifurcation and flip bifurcation can be established. It is observed that the system behaves in a chaotic manner when a specific set of system parameters is selected, which are controlled by a hybrid control method. Examples are cited to illustrate our conclusions.

1. Introduction

In ecological modelling, harvesting is considered as a crucial factor that creates the attention among researchers due to its importance in resource management from the biological and economic point of views. The effect of harvesting on population is a basic problem in fishery theory. How harvesting influences population dynamics depends not only stock structure and ecological parameters but also on fishing strategies. The over-exploitation weakens the conservation of populations and also creates problem for fishery. So optimal harvesting problem should be taken into account within the models describing population dynamics. Optimal harvesting and mathematical models are investigated in [1]. Our main concern on fishery model as fish are one of the most valuable source of protein and many people depend on it and it is one of the most renewable resources in ecological system [2,3]. Fish populations facing extinction not only for over fishing, but also on other factors such as competition and toxic materials. Industrial waste is a form of toxicity in aquatic ecosystems. In case of open access fishery, harvest by fishermen are unregulated. Under these conditions, there may be possibility of extinction of species. Different types of interactions are observed in fisheries systems. For the objectives of bioeconomic modelling, the most important are biological, harvest and market interactions [4]. In particular, the biological interactions indicate predator-prey, competition between them. The interactions between the fish populations is also significant. However, the impact of toxicity among the fish species emitted by each of them and emerge from factories, agricultural land etc. become problems of major environmental concern. On this issue, several works are done through mathematical models [5–10]. All these studies are mainly confined into one or two species without considering aquatic environment. It creates among researchers to examine the effects of toxicant released by the marine biological species. The toxin emitted by one species not only affects that species, but also affect the growth of the other species.
Maynard-Smith [11] considered the impact of toxic material in a two species Lotka-Volterra system, taking into account that each species produce a chemical toxic to the other but only when the other is present. Kar and Chaudhuri [6] modified the system studied in [11] to a two competing fish species which are commercially exploited. They proposed and analysed the following model:

$$\begin{aligned} \frac{dx}{dt} &= x(k_1-\alpha_1 x-\beta_{12}y-\gamma_1 xy-q_1E), \\ \frac{dy}{dt} &= y(k_2-\alpha_2 y-\beta_{21}x-\gamma_2 xy-q_2E) \end{aligned} \tag{1.1}$$

where $x(t), y(t)$ are the densities of two competing fish species at time t, and k_1, k_2 are intrinsic growth rates, α_1, α_2 are the intra specific competition rates respectively. The constants β_{12}, β_{21} are the relative rate of inter specific competition. γ_1, γ_2 are the coefficients of toxicity.

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E denotes the harvesting effort. q_1, q_2 are the catchability coefficients of the two species. Similar type of system (1.1) is investigated in [12]. In studying harvesting phenomena, non-linear Michaelis-Menten harvesting is more realistic in fisheries modelling from the biological and economic point of views. The conventional catch-per-unit effort harvesting faces different unrealistic and insignificant characteristics. It is customary to take catch-per-unit effort harvesting h in the form $h = qEx$ where q stands for the catchability coefficient of harvested population x and E denotes the harvesting effort used during the farming process. It is clear that when harvesting effort E is constant, harvesting activity $h \to \infty$ as harvested population $x \to \infty$ or when harvested population x is constant and fixed, harvesting activity $h \to \infty$ as the harvesting effort $E \to \infty$. This hypothesis is unrealistic in case of random fishing due to unbounded linear increase of h with E and x [13]. In fact, Michaelis-Menten harvesting can eliminate this unrealistic situation by considering the harvesting term in the form $\frac{qEx}{d_1E+d_2x}$. It can be noted that for fixed effort E, $h \to \frac{q}{d_1}$ when $x \to \infty$ or for fixed harvested population x, $h \to \frac{qx}{d_1}$ when $E \to \infty$.
The above studies address into continuous capture system. Though, we know that fish distribution is inhomogeneous and it is more appropriate to consider the discrete system's capture which in turn maintains the ecological balance and save time and produce more economic revenue for fishermen. The dynamical behaviour of discrete time system is more complex than those obtained in continuous systems [14–16]. Even discrete time models can show chaotic dynamics [14, 15]. Hening [17] analysed the long-term behaviour of interacting populations in a discrete time stochastic system that can be controlled through harvesting. Ding et al. [18] investigated discrete time harvesting model of fish populations and they derived the necessary and sufficient conditions and the characterizations of the harvesting strategies.
The main aim of this work is to investigate the discrete version of system (1.1) as well as selective non-linear harvesting $\frac{qEx}{d_1E+d_2x}$ in the first equation of system (1.1) where q is the catchability coefficient of the first species and d_1, d_2 are the degree of competition in the harvesting business and handling time respectively. As we consider selective harvesting so $q_2Ey = 0$.
In this paper, we propose a discrete-time two competing fish species where each species release chemical toxic to another. We study the existence and stability of different fixed points. After then, we identify the system parameters that give Neimark-Sacker and flip bifurcation. Chaos control of the system will be examined. Finally, we examine the global stability of the interior fixed point of the method of iteration scheme.
The paper is formatted as follows. In Section 2, we present a discrete version of system (1.1). The dynamical behaviour of different fixed points is described in Section 3. Chaos control is shown in Section 4. Global stability criterion of interior fixed point is presented in Section 5. In Section 6, the dynamical behaviour of the system is demonstrated when values of parameters are changed. A brief discussion is given in Section 7.

2. Discrete Model

Now, we present the following discrete version of system (1.1):

$$\begin{aligned} x_{n+1} &= x_n \exp(k_1 - \alpha_1 x_n - \beta_{12} y_n - \gamma_1 x_n y_n - \frac{qE}{d_1E + d_2 x_n}), \\ y_{n+1} &= y_n \exp(k_2 - \alpha_2 y_n - \beta_{21} x_n - \gamma_2 x_n y_n) \end{aligned} \tag{2.1}$$

where x_n and y_n represent population densities of two competing fish species at n-generation respectively.

3. Fixed Points and Their Nature

In this section, we determine the fixed points and their dynamics. Evidently, system (1.1) has at most four non-negative fixed points $E_0 = (0,0)$. If $q < k_1 d_1$ then the fixed point $E_1 = (\bar{x}, 0)$ exists uniquely where

$$\bar{x} = \frac{k_1 d_2 - \alpha_1 d_1 + \sqrt{(k_1 d_2 - \alpha_1 d_1)^2 + 4\alpha_1 d_2 E(k_1 d_1 - q)}}{2\alpha_1 d_2}.$$

If $q > k_1 d_1, k_1 d_2 > \alpha_1 d_1$ and $(k_1 d_2 - \alpha_1 d_1)^2 + 4\alpha_1 d_2 E(k_1 d_1 - q) > 0$ then multiple fixed points exist $E_{1\pm} = (x_\pm, 0)$ where

$$x_\pm = \frac{k_1 d_2 - \alpha_1 d_1 \pm \sqrt{(k_1 d_2 - \alpha_1 d_1)^2 + 4\alpha_1 d_2 E(k_1 d_1 - q)}}{2\alpha_1 d_2} \quad \text{and} \quad E_2 = (0, \frac{k_2}{\alpha_2}).$$

There exists interior fixed point $E^* = (x^*, y^*)$ where x^* is a positive root of the equation

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0 \tag{3.1}$$

with

$$\begin{aligned} a_0 &= d_2(\gamma_1 \beta_{21} - \alpha_1 \gamma_2), \\ a_1 &= \frac{1}{3}[d_1 E(\gamma_1 \beta_{21} - \alpha_1 \gamma_2) + d_2(k_1 \gamma_2 - \alpha_1 \alpha_2 + \beta_{12}\beta_{21} - \gamma_1 k_2)], \\ a_2 &= \frac{1}{3}[d_1 E(k_1 \gamma_2 - \alpha_1 \alpha_2 + \beta_{12}\beta_{21} - \gamma_1 k_2) - d_2 \beta_{12} k_2 - qE\gamma_2 + d_2 k_1 \alpha_2], \\ a_3 &= E(d_1 k_1 \alpha_2 - d_1 \beta_{12} k_2 - q\alpha_2) \end{aligned} \tag{3.2}$$

and $y^* = \frac{k_2 - \beta_{21} x^*}{\alpha_2}$ provided $k_2 > \beta_{21} x^*$. Define

$$G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3, H = a_0 a_2 - a_1^2.$$

Structure of the interior fixed points

Theorem 3.1. *Eq. (3.1) has*

(a) a unique positive root x^* if $G^2+H^3>0, a_0$ and a_3 are of opposite signs.
(b) two positive roots x_1^*, x_2^* if $G^2+H^3<0, a_0$ and a_3 are of the same signs and a_1 and a_2 are of opposite signs.
(c) three positive roots $x_{11}^*, x_{12}^*, x_{13}^*$ if $G^2+H^3<0, a_0, a_2>0$ (or <0) and $a_1, a_3<0$ (or >0).

Proof. The positive fixed point of system (2.1) satisfies the equations

$$k_1-\alpha_1 x-\beta_{12} y-\gamma_1 xy-\frac{qE}{d_1E+d_2x}=0, \tag{3.3}$$

$$k_2-\alpha_2 y-\beta_{21} x-\gamma_2 xy=0. \tag{3.4}$$

From the Eq. (3.4), we get $y=\frac{k_2-\beta_{21}x}{\alpha_2+\gamma_2 x}$. Substituting the value of y to the equation (3.3) is precisely equation (3.1).
(a) Assumptions of the theorem implies that (3.1) has one real root and two imaginary roots. Since a_0 and a_3 are of opposite signs so (3.1) has a unique positive root.
(b)Assumptions of the theorem implies that (3.1) has three real roots and one of them is negative. Consequently, (3.1) has two positive roots.
(c) Assumptions of the theorem implies that (3.1) has three real roots and there is no negative roots. Consequently, (3.1) has three positive roots. This completes the proof.
To determine the nature of the fixed points, we compute the Jacobian matrix at each fixed point. The Jacobian matrix at an arbitrary fixed point (x,y) is given by

$$J(x,y)=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{3.5}$$

where

$$\begin{aligned}
m_{11} &= \{1-(\alpha_1+\gamma_1 y-\frac{d_2qE}{(d_1E+d_2x)^2})x\}\exp(k_1-\alpha_1 x-\beta_{12}y-\gamma_1 xy-\frac{qE}{d_1E+d_2x}),\\
m_{12} &= -(\beta_{12}+\gamma_1 x)x\exp(k_1-\alpha_1 x-\beta_{12}y-\gamma_1 xy-\frac{qE}{d_1E+d_2x}),\\
m_{21} &= -(\beta_{21}+\gamma_2 y)y\exp(k_2-\alpha_2 y-\beta_{21}x-\gamma_2 xy),\\
m_{22} &= \{1-(\alpha_2+\gamma_2 x)y\}\exp(k_2-\alpha_2 y-\beta_{21}x-\gamma_2 xy).
\end{aligned}$$

We first present the results which will be required to investigate the nature of fixed points.

Lemma 3.2. *([19]) Let the characteristic equation of J is $F(\lambda)=\lambda^2+p\lambda+q=0$. Suppose λ_1 and λ_2 are two roots of $F(\lambda)=0$. Then there are the following definitions.*

1. *If $|\lambda_1|<1$ and $|\lambda_2|<1$ then the fixed point is called a sink and is locally asymptotically stable.*
2. *If $|\lambda_1|>1$ and $|\lambda_2|>1$ then the fixed point is called a source and unstable.*
3. *If $|\lambda_1|>1$ and $|\lambda_2|<1$ then the fixed point is called a saddle.*
4. *If $|\lambda_1|=1$ or $|\lambda_2|=1$ then the fixed point is called non-hyperbolic.*

Lemma 3.3. *([19]) Let $F(\lambda)=\lambda^2+p\lambda+q$ where p and q are constants. Suppose $F(1)>0$ and λ_1 and λ_2 are two roots of $F(\lambda)=0$. Then*

1. *$|\lambda_1|<1$ and $|\lambda_2|<1$ if and only if $F(-1)>0$ and $q<1$,*
2. *$|\lambda_1|>1$ and $|\lambda_2|>1$ if and only if $F(-1)>0$ and $q>1$,*
3. *$|\lambda_1|<1$ and $|\lambda_2|>1$ if and only if $F(-1)<0$,*
4. *$\lambda_1=-1$ and $|\lambda_2|\neq 1$ if and only if $F(-1)=0$ and $p\neq 0,2$,*
5. *λ_1 and λ_2 are the conjugate complex roots and $|\lambda_1|=|\lambda_2|=1$ if and only if $p^2-4q<0$ and $q=1$.*

Theorem 3.4. *For all positive parameters, system (2.1) has the fixed point $E_0=(0,0)$ then E_0 is:*

1. *source if $k_1>\frac{q}{d_1}$ and hence unstable.*
2. *saddle if $k_1<\frac{q}{d_1}$.*
3. *non-hyperbolic if $k_1=\frac{q}{d_1}$.*

Proof. The Jacobian matrix of system (2.1) at E_0 is

$$J(E_0)=\begin{pmatrix} \exp(k_1-\frac{q}{d_1}) & 0 \\ 0 & \exp k_2 \end{pmatrix} \tag{3.6}$$

Here the eigenvalues of $J(E_0)$ are $\lambda_1=\exp(k_1-\frac{q}{d_1})>1$ if $k_1>\frac{q}{d_1}$ and $\lambda_1<1$ if $k_1<\frac{q}{d_1}$. If $k_1=\frac{q}{d_1}$ then $\lambda_1=1$ and $\lambda_2=\exp>1$ since $k_2>0$. Hence E_0 is a source when $k_1>\frac{q}{d_1}$ and hence unstable. E_0 is a saddle when $k_1<\frac{q}{d_1}$. Lastly, E_0 is non-hyperbolic when $k_1=\frac{q}{d_1}$. This completes the proof.

Theorem 3.5. *Assume that $q<k_1d_1$. The fixed point $E_1=(\bar{x},0)$. E_1 is*

1. *sink if $\frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2}<\alpha_1\bar{x}<q+\frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2}$ and $k_2<\beta_{21}\bar{x}$.*
2. *saddle if one of the following conditions hold:*

   (a) $\alpha_1\bar{x} > q + \frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2}$ *and* $k_2 < \beta_{21}\bar{x}$.
   (b) $\frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2} < \alpha_1\bar{x} < q + \frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2}$ *and* $k_2 > \beta_{21}\bar{x}$.

3. *source if* $\alpha_1\bar{x} > q + \frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2}$ *and* $k_2 > \beta_{21}\bar{x}$*, then* E_1 *is unstable.*
4. *non-hyperbolic if* $1 + \frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2} = \alpha_1\bar{x}$ *or* $k_2 = \beta_{21}\bar{x}$.

Proof. The Jacobian matrix of system (2.1) at E_1 is

$$J(E_1) = \begin{pmatrix} 1-(\alpha_1 - \frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2})\bar{x} & -(\beta_{12}+\gamma_1\bar{x})\bar{x} \\ 0 & \exp(k_2-\beta_{21}\bar{x}) \end{pmatrix} \tag{3.7}$$

The eigenvalues of $J(E_1)$ are $\lambda_1 = 1-(\alpha_1 - \frac{d_2qE\bar{x}}{(d_1E+d_2\bar{x})^2})\bar{x}, \lambda_2 = \exp(k_2-\beta_{21}\bar{x})$. Similar to the proof of Theorem 3.4, the above results can be easily derived.

Remark. In case of multiple fixed points $E_\pm$, we can obtain similar type of conditions as in Theorem 3.5, where $\bar{x}$ is replaced by $x_\pm$ for determining the nature of $E_{1\pm}$.

Theorem 3.6. *System (2.1) always has the fixed point* $E_2 = (0, \frac{k_2}{\alpha_2})$. E_2 *is*

1. *sink if* $k_2 < 2$ *and* $k_1 < \frac{d_1\beta_{12}k_2+q\alpha_2}{\alpha_2 d_1}$.
2. *saddle if one of the following conditions hold:*
   (a) $k_2 > 2$ *and* $k_1 < \frac{d_1\beta_{12}k_2+q\alpha_2}{\alpha_2 d_1}$.
   (b) $k_2 < 2$ *and* $k_1 > \frac{d_1\beta_{12}k_2+q\alpha_2}{\alpha_2 d_1}$.
3. *source if* $k_2 > 2$ *and* $k_1 > \frac{d_1\beta_{12}k_2+q\alpha_2}{\alpha_2 d_1}$*, then* E_1 *is unstable.*
4. *non-hyperbolic if* $k_2 = 2$ *or* $k_1 = \frac{d_1\beta_{12}k_2+q\alpha_2}{\alpha_2 d_1}$.

Proof. The proof is similar to the proof of Theorem 3.4 and is omitted here.

Theorem 3.7. *Assume that the conditions of Theorem 3.1 be hold and also suppose that* $a > 0$*. Then the fixed point* E^* *is*

1. *sink if* $a < b \leq 2$ *or* $b > 2$ *and* $2b-4 < a < b$,
2. *source if* $b \leq 2$ *and* $a > b$ *or* $b > 2$ *and* $a > max\{b, 2b-4\}$,
3. *saddle if* $b > 2$ *and* $a < 2b-4$,
4. *non-hyperbolic if* $a < 2b-4$*, where*

$$a = x^*y^*\{(\alpha_1+\gamma_1y^* - \frac{d_2qE}{(d_1E+d_2x^*)^2})(\alpha_2+\gamma_2x^*) - (\beta_{12}+\gamma_1x^*)(\beta_{21}+\gamma_2y^*)\} \tag{3.8}$$

$$b = (\alpha_1+\gamma_1y^* - \frac{d_2qE}{(d_1E+d_2x^*)^2})x^* + (\alpha_2+\gamma_2x^*)y^* \tag{3.9}$$

Proof. The Jacobian matrix at E^* is

$$J(E^*) = \begin{pmatrix} 1-(\alpha_1+\gamma_1y^* - \frac{d_2qE}{(d_1E+d_2x^*)^2})x^* & -(\beta_{12}+\gamma_1x^*)x^* \\ -(\beta_{21}+\gamma_2y^*)y^* & 1-(\alpha_2+\gamma_2x^*)y^* \end{pmatrix} \tag{3.10}$$

so the characteristic equation of the above matrix can be written as

$$F(\lambda) = \lambda^2 + p\lambda + q = 0 \tag{3.11}$$

where $p = -2+b$ and $q = 1-b+a$. After simple calculation, we get

$$\begin{aligned} F(1) &= 1+p+q = a, \\ F(-1) &= 1-p+q = 4-2b+a, \\ q-1 &= a-b. \end{aligned}$$

Now $F(1) > 0$ if $a > 0$. $F(-1) > 0$ if $b \leq 2$ and $a > 0$ or $b > 2$ and $a > 2b-4$, $F(-1) < 0$ if $b > 2$ and $a < 2b-4$, $q-1 < 0$ if $a < b$. According to Lemma 3.2 and 3.3, E^* is a sink and it is stable, if the conclusion (1) of Theorem 3.7 holds. Next, if the other conditions of Theorem 3.7 hold separately, E^* is a source, saddle and non-hyperbolic, respectively, at which E^* is unstable. This completes the proof.

3.1. Bifurcation around the interior fixed point

System (2.1) has at most an unique fixed point E^*, hence the system does not admit fold bifurcation. So we are interested in examining the Neimark-Sacker bifurcation and flip bifurcation.

Theorem 3.8. *The fixed point* E^* *changes from the stable state to Neimark-Sacker bifurcation if the following conditions are satisfied:* $a = b$ *and* $b < 4$ *where* a *and* b *are defined in (3.8) and (3.9).*

Proof. If the Jacobian matrix $J(E^*)$ has two complex conjugate eigenvalues with modulus 1, Neimark-Sacker bifurcation appears [20]. This requires that $\det(J(E^*)) = q = 1$ and $-2 < \mathrm{tr}(J(E^*)) = -p < 2$. Replacing p and q, we have $a = b$ and $b < 4$. This completes the proof.

Theorem 3.9. *The fixed point* E^* *changes from the stable state to flip bifurcation if the following conditions are satisfied:*

$$a+4 = 2b.$$

Proof. System (1.1) admits flip bifurcation when a single eigenvalue -1. Thus the condition for flip bifurcation can be written in the form $1-p+q = 0$. Replacing Replacing p and q, we have $a+4 = 2b$. This completes the proof.

4. Chaos Control

In discrete dynamical system, one can observe chaotic behaviour for certain choices of the system parameters and controlling chaos is an important issue. There are different methods for controlling chaos. We use mainly use hybrid control technique [21] to stabilize a chaotic orbit at an unstable fixed point of system (2.1). Define the following controlled system with respect to (2.1):

$$\begin{aligned} x_{n+1} &= \rho x_n \exp(k_1 - \alpha_1 x_n - \beta_{12} y_n - \gamma_1 x_n y_n - \frac{qE}{d_1 E + d_2 x_n}) + (1-\rho) x_n, \\ y_{n+1} &= \rho y_n \exp(k_2 - \alpha_2 y_n - \beta_{21} x_n - \gamma_2 x_n y_n) + (1-\rho) y_n \end{aligned} \tag{4.1}$$

where $0 < \rho < 1$ is taken as a control parameter. The Jacobian matrix of controlled system (4.1) evaluated at E^* is given by

$$J(x^*, y^*) = \begin{pmatrix} 1 - \rho x^*(\alpha_1 + \gamma_1 y^* - \frac{qEd_2}{(d_1 E + d_2 x^*)^2}) & -\rho x^*(\beta_{12} + \gamma_1 x^*) \\ -\rho y^*(\beta_{21} + \gamma_2 y^*) & 1 - \rho y^*(\alpha_2 + \gamma_2 x^*) \end{pmatrix} \tag{4.2}$$

The fixed point E^* of the controlled system (4.1) is locally asymptotically stable if all the roots of the characteristic polynomial of (4.2) lie in an unit open disk.

5. Global Stability

In this section, we will utilize the process of iteration scheme and the comparison principle of difference equation to investigate the global stability of the positive fixed point of system (2.1). To establish global stability result, we require the following lemmas.

Lemma 5.1. *([19]) Let $f(u) = u\exp(\delta - \eta u)$, where δ and η are positive constants. Then $f(u)$ is nondecreasing for $u \in (0, \frac{1}{\eta}]$.*

Lemma 5.2. *([19]) Assume that the sequence u_n satisfies $u_{n+1} = u_n \exp(\delta - \eta u_n), n = 1, 2, 3, \ldots$ where δ and η are positive constants and $u_0 > 0$. Then;*

1. *If $\delta < 2$, then $\lim_{n\to\infty} u_n = \frac{\delta}{\eta}$.*
2. *If $\delta \leq 1$, then $u_n \leq \frac{1}{\eta}, n = 2, 3, \ldots$*

Lemma 5.3. *([22]) Suppose that functions $f, g : \mathbb{Z}_+ \times [0, \infty)$ satisfy $f(n,x) \leq g(n,x)$ $(f(n,x) \geq g(n,x))$ for $n \in \mathbb{Z}_+$ and $g(n,x)$ is nondecreasing with respect to x. If u_n are the nonnegative solutions of the difference equations*

$$\begin{aligned} x_{n+1} &= f(n, x_n), \\ u_{n+1} &= g(n, u_n) \end{aligned}$$

respectively, and $x_0 \leq u_0$ $(x_0 \geq u_0)$ then $x_n \leq u_n$ $(x_n \geq u_n)$ for all $n \geq 0$.

Theorem 5.4. *Assume that $\frac{k_2 d_1(\beta_{12}\alpha_1 + \gamma_1 k_1) + q\alpha_1\alpha_2}{d_1\alpha_1\alpha_2} < k_1 \leq 1$ and $\frac{k_1(\beta_{21}\alpha_2 + \gamma_2 k_2)}{\alpha_1\alpha_2} < k_2 \leq 1$ then the fixed point $E^*(x^*, y^*)$ of system (2.1) is globally asymptotically stable.*

Proof. Assume that (x_n, y_n) is any solution of system (2.1) with initial values $x_0 > 0, y_0 > 0$. Let

$$\begin{aligned} U_1 &= \text{limsup}_{n\to\infty} x_n,\ V_1 = \text{liminf}_{n\to\infty} x_n, \\ U_2 &= \text{limsup}_{n\to\infty} y_n,\ V_2 = \text{liminf}_{n\to\infty} y_n. \end{aligned}$$

In the following, we will prove that $U_1 = V_1 = x^*, U_2 = V_2 = y^*$.
First we show that $U_1 \leq M_1^x, U_2 \leq M_1^y$. From the first equation of system (2.1), we get

$$x_{n+1} \leq x_n \exp(k_1 - \alpha_1 x_n), \quad n = 0, 1, 2, \ldots$$

Considering the auxiliary equation

$$u_{n+1} = u_n \exp(k_1 - \alpha_1 u_n) \tag{5.1}$$

by Lemma5.2 (ii), because of $k_1 \leq 1$, we get $u_n \leq \frac{1}{\alpha_1}$ for all $n \geq 2$. By Lemma 5.1, we obtain $f(u) = u\exp(k_1 - \alpha_1 u)$ is nondecreasing for $u \in (0, \frac{1}{\alpha_1}]$. Thus from Lemma 5.3, we get $x_n \leq u_n$ for all $n \geq 2$, where u_n is the solution of Eq. (5.1) with initial value $u_2 = x_2$. By Lemma 5.2 (i), we get

$$U_1 = \text{limsup}_{n\to\infty} x_n \leq \lim_{n\to\infty} u_n = \frac{k_1}{\alpha_1}.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $n_1 > 2$ such that if $n \geq n_1$, then

$$x_n \leq \frac{k_1}{\alpha_1} + \varepsilon = M_1^x.$$

Similarly, from the second equation of system (2.1), we obtain,

$$U_2 = \text{limsup}_{n\to\infty} y_n \leq \lim_{n\to\infty} u_n = \frac{k_2}{\alpha_2} \quad \text{as} \quad k_2 \leq 1.$$

Hence, for any sufficiently small $\varepsilon > 0$, there exists a $n_2 > n_1$ such that if $n \geq n_2$, then

$$y_n \leq \frac{k_2}{\alpha_2} + \varepsilon = M_1^y.$$

Next we show that $V_1 \geq N_1^x$ and $V_2 \geq N_1^y$. From the first equation of system (2.1), we have

$$x_{n+1} \geq x_n \exp(k_1 - \alpha_1 x_n - \beta_{12} M_1^y - \gamma_1 M_1^x M_1^y - \frac{q}{d_1})$$

Consider the auxiliary equation

$$u_{n+1} = u_n \exp(k_1 - \frac{q}{d_1} - \alpha_1 u_n - \beta_{12} M_1^y - \gamma_1 M_1^x M_1^y). \tag{5.2}$$

Since we have $k_1 - \frac{q}{d_1} - \beta_{12} M_1^y - \gamma_1 M_1^x M_1^y < 1$, by Lemma 5.2 (ii), we have, $u_n \leq \frac{1}{\alpha_1}$ for $n \geq n_2$. By Lemma 5.1, we obtain $f(u) = u\exp(k_1 - \frac{q}{d_1} - \beta_{12} M_1^y - \gamma_1 M_1^x M_1 y - \alpha_1 u)$ is nondecreasing for $u \in (0, \frac{1}{\alpha_1}]$. Thus from Lemma 5.3, we get $x_n \geq u_n$ for all $n \geq n_2$. By Lemma 5.2 (i), we get

$$V_1 = \liminf_{n\to\infty} x_n \geq \lim_{n\to\infty} u_n = \frac{d_1(k_1 - \beta_{12} M_1^y - \gamma_1 M_1^x M_1^y) - q}{d_1 \alpha_1}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_3 > n_2$ such that for $n \geq n_3$,

$$x_n \geq \frac{d_1(k_1 - \beta_{12} M_1^y - \gamma_1 M_1^x M_1^y) - q}{d_1 \alpha_1} - \varepsilon = N_1^x.$$

From the second equation of system (2.1), we have

$$y_{n+1} \geq y_n \exp(k_2 - \alpha_2 y_n - \beta_{21} M_1^x - \gamma_2 M_1^x M_1^y).$$

Since we have $0 < k_2 - \beta_{21} M_1^x - \gamma_2 M_1^x M_1^y < 1$, a similar argument as above, we can get

$$V_2 = \liminf_{n\to\infty} y_n = \frac{k_2 - \beta_{21} M_1^x - \gamma_2 M_1^x M_1^y}{\alpha_2}.$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_4 > n_3$ such that for $n \geq n_4$,

$$y_n \geq \frac{k_2 - \beta_{21} M_1^x - \gamma_2 M_1^x M_1^y}{\alpha_2} - \varepsilon = N_1^y.$$

Now we show that $U_1 \leq M_2^x, U_2 \leq M_2^y$ where $M_2^x \leq M_1^x, \quad M_2^y \leq M_1^y$ respectively. From the first equation of system (2.1) for $n > n_4$, we get

$$x_{n+1} \leq x_n \exp(k_1 - \alpha_1 x_n - \beta_{12} N_1^y - \gamma_1 N_1^x N_1^y - \frac{qE}{d_1 E + d_2 M_1^x}).$$

Since $M_1^x > N_1^x$ and $M_1^y > N_1^y$, we get

$$k_1 - \frac{q}{d_1} - \beta_{12} M_1^y - \gamma_1 M_1^x M_1^y < k_1 - \beta_{12} N_1^y - \gamma_1 N_1^x N_1^y - \frac{qE}{d_1 E + d_2 M_1^x} \leq k_1 \leq 1.$$

Using the similar argument as in above, we can get

$$U_1 = \limsup_{n\to\infty} x_n \leq \frac{1}{\alpha_1}[k_1 - \beta_{12} N_1^y - \gamma_1 N_1^x N_1^y].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_5 > n_4$ such that for $n \geq n_5$,

$$x_n \leq \frac{1}{\alpha_1}[k_1 - \beta_{12} N_1^y - \gamma_1 N_1^x N_1^y - \frac{qE}{d_1 E + d_2 M_1^x}] + \frac{\varepsilon}{2} = M_2^x \leq M_1^x.$$

Similarly, from the second equation of system (2.1) for $n > n_5$, we get

$$y_{n+1} \leq y_n \exp[k_2 - \alpha_2 y_n - \beta_{21} N_1^x - \gamma_2 N_1^x N_1^y],$$

since

$$k_2 - \beta_{21} M_1^x - \gamma_2 M_2^x M_2^y < k_2 - \beta_{21} N_1^x - \gamma_2 N_1^x N_1^y \leq k_2 \leq 1.$$

Using the similar argument as in above, we can get

$$U_2 = \limsup_{n\to\infty} y_n \leq \frac{1}{\alpha_2}[k_2 - \beta_{21} N_1^x - \gamma_2 N_1^x N_1^y],$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_6 > n_5$ such that for $n \geq n_6$,

$$y_n \leq \frac{1}{\alpha_2}[k_2 - \beta_{21} N_1^x - \gamma_2 N_1^x N_1^y] + \frac{\varepsilon}{2} = M_2^y \leq M_1^y.$$

Now we show that $V_1 \geq N_2^x$ and $V_2 \geq N_2^y$. Further, from the first of system (2.1) for $n > n_6$, we get

$$x_{n+1} \geq x_n \exp[k_1 - \frac{qE}{d_1E + d_2N_1^x} - \alpha_1 x_n - \beta_{12}M_2^y - \gamma_1 M_2^x M_2^y].$$

Since $M_1^y \geq M_2^y, M_1^x \geq M_2^x$, we have

$$0 < k_1 - \frac{q}{d_1} - \beta_{12}M_1^y - \gamma_1 M_1^x M_1^y < k_1 - \frac{qE}{d_1E + d_2N_1^x} - \beta_{12}M_2^y - \gamma_1 M_2^x M_2^y.$$

Using a similar argument, we get

$$V_1 = \liminf_{n\to\infty} x_n \geq \frac{1}{\alpha_1}[k_1 - \frac{qE}{d_1E + d_2N_1^x} - \beta_{12}M_2^y - \gamma_1 M_2^x M_2^y].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_7 > n_6$ such that for $n \geq n_7$,

$$x_n \geq \frac{1}{\alpha_1}[k_1 - \frac{qE}{d_1E + d_2N_1^x} - \beta_{12}M_2^y - \gamma_1 M_2^x M_2^y] - \frac{\varepsilon}{2} = N_2^x.$$

Similarly, from the second equation of system (2.1) for $n > n_7$, we have

$$y_{n+1} \geq y_n \exp[k_2 - \alpha_2 y_n - \beta_{21}M_2^x - \gamma_2 M_2^x M_2^y].$$

Since

$$0 < k_2 - \beta_{21}M_1^x - \gamma_2 M_1^x M_1^y < k_2 - \beta_{21}M_2^x - \gamma_2 M_2^x M_2^y \leq k_2 \leq 1$$

we have

$$V_2 = \liminf_{n\to\infty} y_n \geq \frac{1}{\alpha_2}[k_2 - \beta_2 M_2^x - \gamma_2 M_2^x M_2^y].$$

Hence for any sufficiently small $\varepsilon > 0$, there exists $n_8 > n_7$ such that for $n \geq n_8$,

$$y_n \geq \frac{1}{\alpha_2}[k_2 - \beta_2 M_2^x - \gamma_2 M_2^x M_2^y] - \frac{\varepsilon}{2} = N_2^y.$$

Repeating the above process, we ultimately get four sequences $\{M_n^x\}, \{M_n^y\}, \{N_n^x\}, \{N_n^y\}$ such that for all $n \geq 2$,

$$\begin{aligned}
M_n^x &= \frac{1}{\alpha_1}[k_1 - \beta_{12}N_{n-1}^y - \gamma_1 N_{n-1}^x N_{n-1}^y - \frac{qE}{d_1E + d_2M_{n-1}^x}] + \frac{\varepsilon}{n}, \\
M_n^y &= \frac{1}{\alpha_2}[k_2 - \beta_{21}N_{n-1}^x - \gamma_2 N_{n-1}^x N_n^y] + \frac{\varepsilon}{n}, \\
N_n^x &= \frac{1}{\alpha_1}[k_1 - \beta_1 M_n^y - \gamma_1 M_n^x M_n^y - \frac{qE}{d_1E + d_2N_{n-1}^x}] - \frac{\varepsilon}{n}, \\
N_n^y &= \frac{1}{\alpha_2}[k_2 - \beta_{21}M_n^x - \gamma_2 M_n^x M_n^y] - \frac{\varepsilon}{n}.
\end{aligned} \tag{5.3}$$

Clearly, we have for any integer $n > 0$, $N_n^x \leq V_1 \leq U_1 \leq M_n^x$ and $N_n^y \leq V_2 \leq U_2 \leq M_n^y$.
In the following, we will prove that $\{M_n^x\}$ and $\{M_n^y\}$ are monotonically decreasing and $\{N_n^x\}$ and $\{N_n^y\}$ are monotonically increasing, with the help of mathematical induction. Firstly, when $n = 2$, it is clear that

$$M_2^x \leq M_1^x, M_2^y \leq M_1^y, N_2^x \geq N_1^x \quad \text{and} \quad N_2^y \geq N_1^y.$$

For $n = k (k \geq 2)$, we assume that

$$M_k^x \leq M_{k-1}^x, M_k^y \leq M_{k-1}^y, N_k^x \geq N_{k-1}^x \quad \text{and} \quad N_k^y \geq N_{k-1}^y.$$

Now

$$\begin{aligned}
M_{k+1}^x - M_k^x &= \frac{1}{\alpha_1}[k_1 - \beta_{12}N_k^y - \gamma_1 N_k^x N_k^y - \frac{qE}{d_1E + d_2M_k^x}] + \frac{\varepsilon}{k+1} - \frac{1}{\alpha_1}[k_1 - \beta_{12}N_{k-1}^y - \gamma_1 N_{k-1}^x N_{k-1}^y - \frac{qE}{d_1E + d_2M_{k-1}^x}] - \frac{\varepsilon}{k} \\
&= -\frac{\beta_{12}}{\alpha_1}[N_k^y - N_{k-1}^y] - \frac{\gamma_1}{\alpha_1}[N_k^x N_k^y - N_{k-1}^x N_{k-1}^y] + \frac{qEd_2(M_k^x - M_{k-1}^x)}{\alpha_1(d_1E + d_2M_k^x)(d_1E + d_2M_{k-1}^x)} - \frac{\varepsilon}{k(k+1)} \leq 0. \\
M_{k+1}^y - M_k^y &= \frac{1}{\alpha_2}[k_2 - \beta_{21}N_k^x - \gamma_2 N_k^x N_k^y] + \frac{\varepsilon}{k+1} - \frac{1}{\alpha_2}[k_2 - \beta_{21}N_{k-1}^x - \gamma_2 N_{k-1}^x N_{k-1}^y] - \frac{\varepsilon}{k} \\
&= -\frac{\beta_{21}}{\alpha_2}[N_k^x - N_{k-1}^x] - \frac{\gamma_2}{\alpha_2}[N_k^x N_k^y - N_{k-1}^x N_{k-1}^y] - \frac{\varepsilon}{k(k+1)} \leq 0 \\
N_{k+1}^x - N_k^x &= \frac{1}{\alpha_1}[k_1 - \beta_{12}M_{k+1}^y - \gamma_1 M_{k+1}^x M_{k+1}^y - \frac{qE}{d_1E + d_2N_k^x}] - \frac{\varepsilon}{k+1} - \frac{1}{\alpha_1}[k_1 - \beta_{12}M_k^y - \gamma_1 M_k^x M_k^y - \frac{qE}{d_1E + d_2N_{k-1}^x}] + \frac{\varepsilon}{k} \\
&= -\frac{\beta_{12}}{\alpha_1}[M_{k+1}^y - M_k^y] - \frac{\gamma_1}{\alpha_1}[M_{k+1}^x M_{k+1}^y - M_k^x M_k^y] + \frac{qEd_2(N_k^x - N_{k-1}^x)}{\alpha_1(d_1E + d_2N_k^x)(d_1E + d_2N_{k-1}^x)} + \frac{\varepsilon}{k(k+1)} \geq 0 \\
N_{k+1}^y - N_k^y &= \frac{1}{\alpha_2}[k_2 - \beta_{21}M_{k+1}^x - \gamma_2 M_{k+1}^x M_{k+1}^y] - \frac{\varepsilon}{k+1} - \frac{1}{\alpha_2}[k_2 - \beta_{21}M_k^x - \gamma_2 M_k^x M_k^y] + \frac{\varepsilon}{k} \\
&= -\frac{\beta_{21}}{\alpha_2}[M_{k+1}^x - M_k^x] - \frac{\gamma_2}{\alpha_2}[M_{k+1}^x M_{k+1}^y - M_k^x M_k^y] + \frac{\varepsilon}{k(k+1)} \geq 0.
\end{aligned}$$

This shows that $\{M_n^x\}$ and $\{M_n^y\}$ are monotonically decreasing and $\{N_n^x\}$ and $\{N_n^y\}$ are monotonically increasing. Therefore, by the criterion of monotonic bounded, we have established that every one of this four sequences has a limit.
Let $\lim_{n\to\infty}M_n^x = x_1, \lim_{n\to\infty}M_n^y = y_1, \lim_{n\to\infty}N_n^x = x_2, \lim_{n\to\infty}N_n^y = y_2$. Passing to the limit as $n \to \infty$ in (5.3), we get

$$\begin{aligned} x_1 &= \frac{1}{\alpha_1}[k_1 - \beta_{12}y_2 - \gamma_1 x_2 y_2 - \frac{qE}{d_1E + d_2x_1}], \\ y_1 &= \frac{1}{\alpha_2}[k_2 - \beta_{21}x_2 - \gamma_2 x_2 y_2], \\ x_2 &= \frac{1}{\alpha_1}[k_1 - \beta_{12}y_1 - \gamma_1 x_1 y_1 - \frac{qE}{d_1E + d_2x_2}], \\ y_2 &= \frac{1}{\alpha_2}[k_2 - \beta_{21}x_1 - \gamma_2 x_1 y_1]. \end{aligned} \tag{5.4}$$

It is clear that $x_1 = x_2$ and $x_2 = y_2$. Thus we obtain $x_1 = x_2 = x^*, y = y_2 = y^*$ as a solution of (5.3). Hence, the global asymptotic stability of $E^*(x^*, y^*)$ is obtained. This completes the proof of the theorem.

6. Numerical Simulation

In this section, we present some numerical simulation to illustrate the usefulness of the obtained results as well as for giving direction to find desirable bifurcations and chaos of the discrete time system (2.1).

Example 6.1. *Suppose $k_1 = 0.8, k_2 = 0.6, \alpha_1 = 1, \alpha_2 = 1, \beta_{12} = 0.1, \beta_{21} = 0.01, \gamma_1 = 1, \gamma_2 = 1, E = 4, q = 0.1, d_1 = 1, d_2 = 1$. It follows from Theorem 5.4 that the fixed point (0.478, 0.4027) is globally stable (see Figure 6.1a and 6.1b) for initial points (0.1,0.1) and (0.5,0.2) respectively.*

Example 6.2. *Suppose $k_1 = 2, k_2 = 2.8, \alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 1.1, \gamma_1 = 0.5, \gamma_2 = 1.4, E = 1, q = 0.1, d_1 = 1, d_2 = 1$ and the initial point ((0.5,0.5) for system (2.1). We draw the bifurcation diagram with respect to the parameter α_1 in the interval (0.75, 1.5). As α_1 increases, we observe a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos (see Figure 6.2).*

Example 6.3. *Suppose $k_1 = 2, k_2 = 2.8, \alpha_1 = 1, \alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 1.1, \gamma_1 = 0.5, E = 1, q = 0.1, d_1 = 1, d_2 = 1$ and the initial point (0.5,0.5) for system (2.1). We draw the bifurcation diagram with respect to the parameter γ_2 in the interval (0.7, 1.5). As γ_2 increases, we observe a transition phase from chaotic behaviour to stable state (see Figure 6.3).*

Example 6.4. *Suppose $k_1 = 1.1, \alpha_1 = 0.1\alpha_2 = 0.5, \beta_{12} = 0.1, \beta_{21} = 0.1, \gamma_1 = 0.5, \gamma_2 = 0.1, E = 1, q = 0.1, d_1 = 1, d_2 = 1$ and the initial point (0.5,0.5) for system (2.1). We draw the bifurcation diagram with respect to the parameter k_2 in the interval (1.5, 3). As k_2 increases, we observe a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos (see Figure 6.4).*

Example 6.5. *Suppose $k_1 = 2.2, k_2 = 3.2, \alpha_1 = 1, \alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 0.5, \gamma_1 = 0.01, \gamma_2 = 2, E = 1, d_1 = 1, d_2 = 1, q = 0.1$ and the initial point (0.1,0.1) for system (2.1) showing chaotic dynamics. The condition (3) of Theorem 3.7 is satisfied and the fixed point (1.664, 0.4905) is saddle in nature and hence unstable. Chaotic dynamics is observed (see Figure 6.5a). The chaotic system is controlled when we choose $\rho = 0.5$ for system (4.1) (see Figure 6.5b).*

Example 6.6. *Suppose $k_1 = 2.2, k_2 = 3.5, , \alpha_1 = 1\alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 2.1, \gamma_1 = 0.04, \gamma_2 = 0.62, q = 0.1, d_1 = 1, d_2 = 1$ and the initial point (0.5,0.5) for system (2.1). We draw the bifurcation diagram with respect to the parameter E in the interval (0.1,1.5). As E increases, we observe a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos (see Figure 6.6).*

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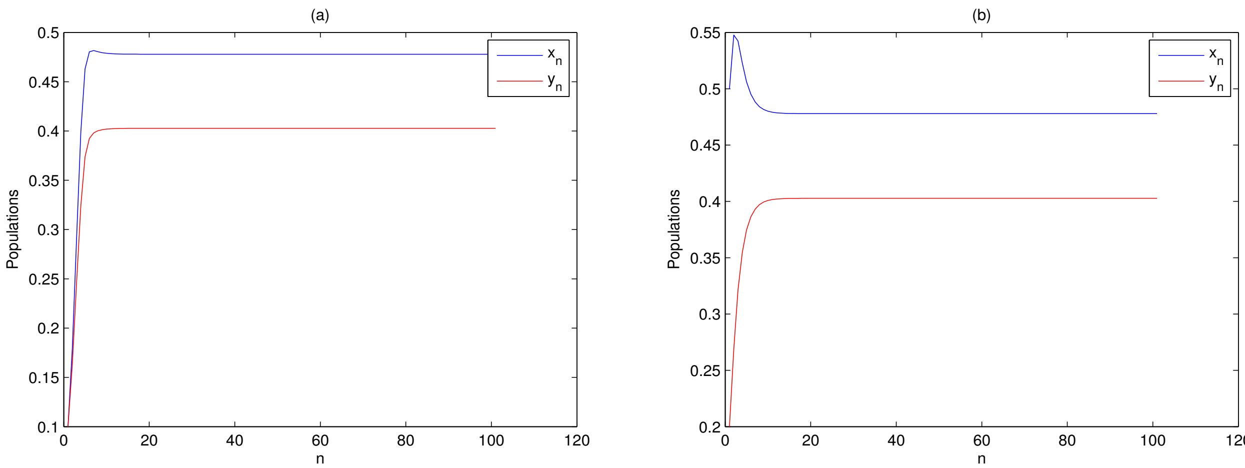

Figure 6.1: Time series plots of system (2.1) for $k_1 = 0.8, k_2 = 0.6, \alpha_1 = 1, \alpha_2 = 1, \beta_{12} = 0.1, \beta_{21} = 0.01, \gamma_1 = 1, \gamma_2 = 1, E = 4, q = 0.1, d_1 = 1, d_2 = 1$ with initial points (0.1,0.1) and (0.5,0.2) respectively

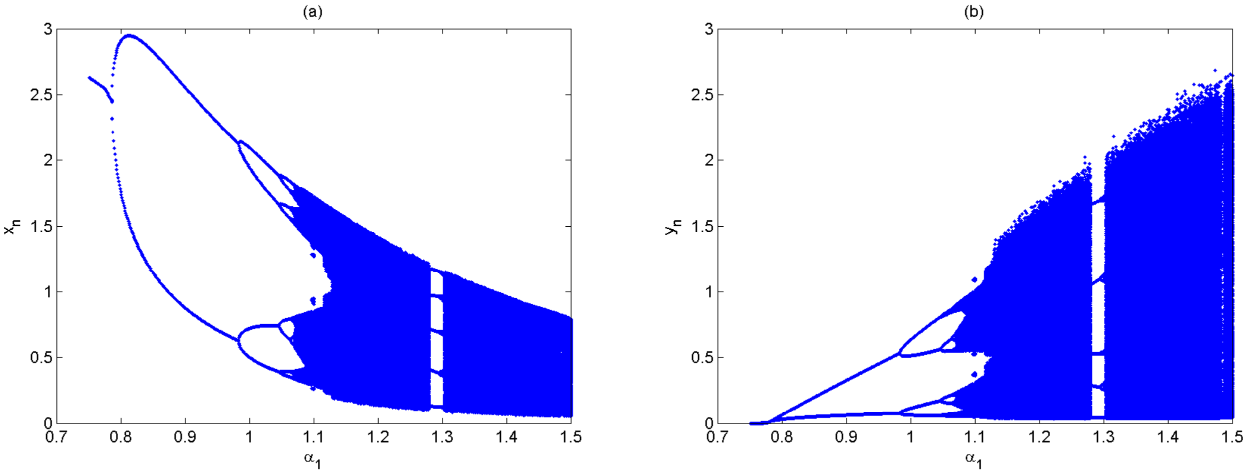

Figure 6.2: Bifurcation diagram for two competing fish species with α_1 of system (2.1) for fixed values $k_1 = 2, k_2 = 2.8, \alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 1.1$, $\gamma_1 = 0.5, \gamma_2 = 1.4, E = 1, q = 0.1, d_1 = 1, d_2 = 1$.

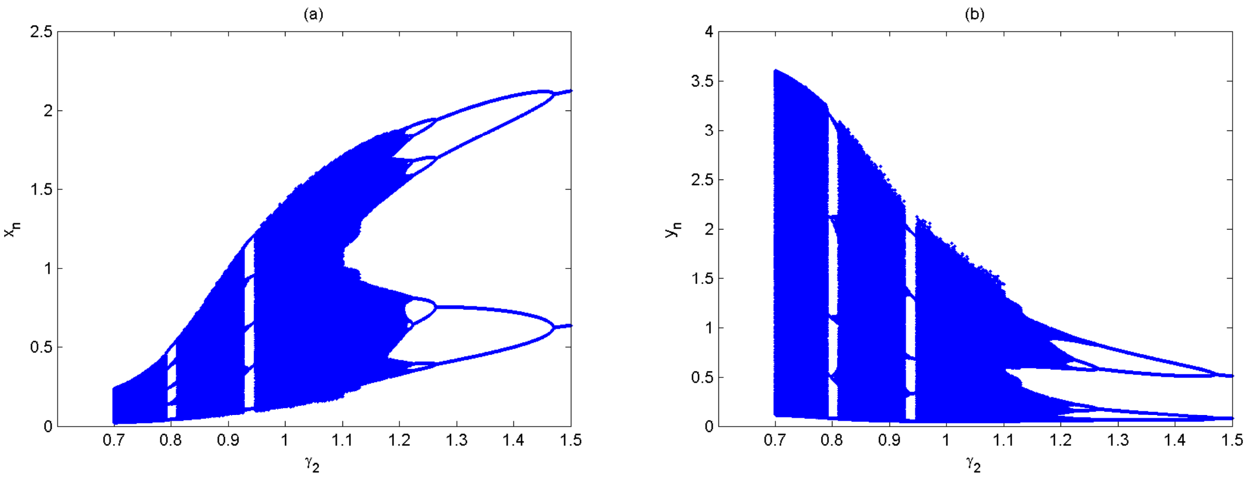

Figure 6.3: Bifurcation diagram for competing fish species γ_2 of system (1.1) for fixed values $k_1 = 2, k_2 = 2.8, \alpha_1 = 1, \alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 1.1$, $\gamma_1 = 0.5, E = 1, q = 0.1, d_1 = 1, d_2 = 1$.

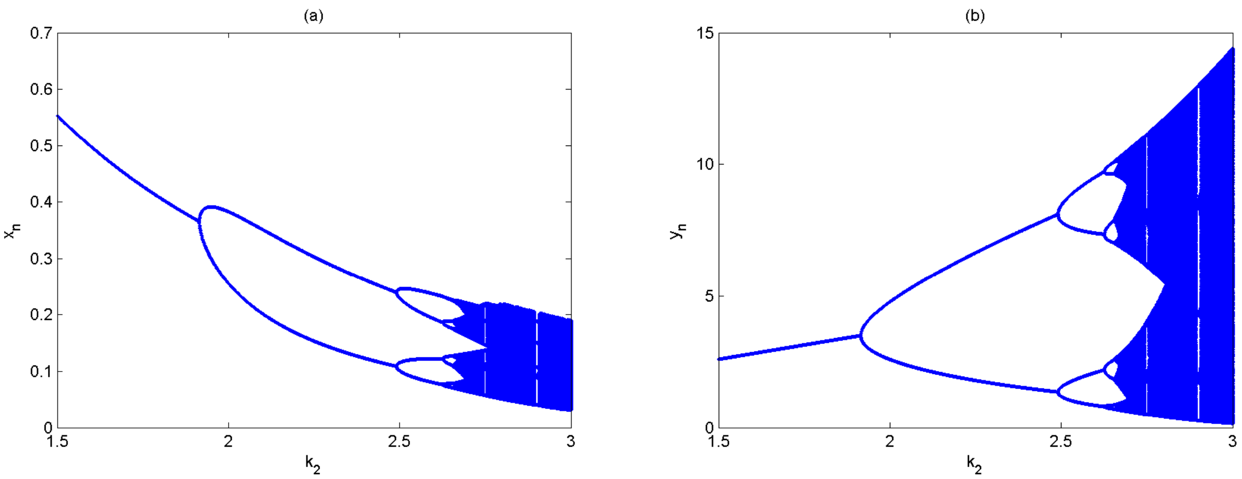

Figure 6.4: Bifurcation diagram for competing fish species with k_2 of system (2) for fixed values $k_1 = 1.1, \alpha_1 = 0.11, \alpha_2 = 0.5, \beta_{12} = 0.1, \beta_{21} = 0.1$, $\gamma_1 = 0.5, \gamma_2 = 1, E = 1, q = 0.1, d_1 = 1, d_2 = 1$.

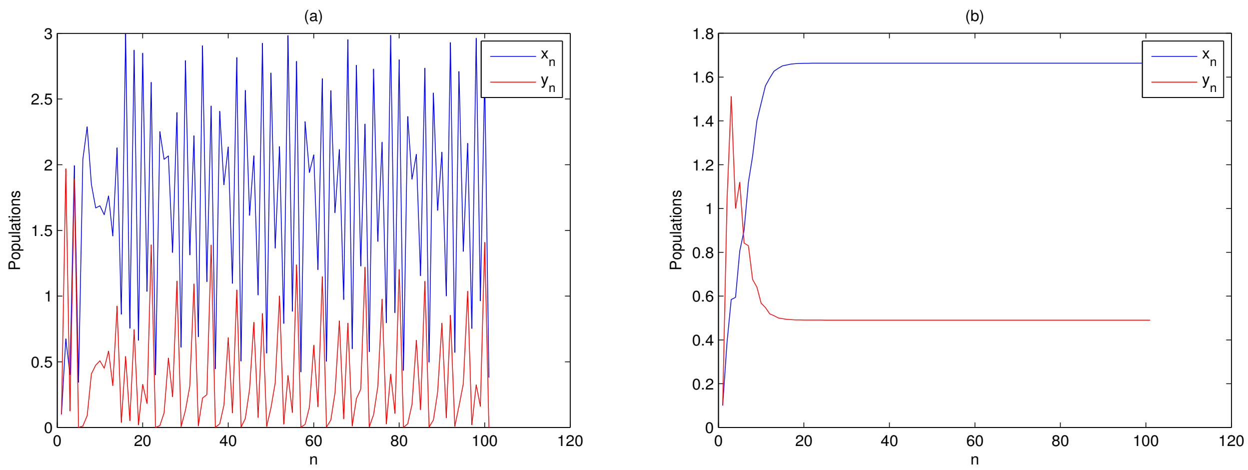

Figure 6.5: Time series plots for two competing fish species of system (2.1) fixed values $k_1 = 2.2, k_2 = 3.2, \alpha_1 = 1, \alpha_2 = 1.5, \beta_{12} = 1, \beta_{21} = 0.5,$ $\gamma_1 = 0.01, \gamma_2 = 2, E = 1, d_1 = 1, d_2 = 1, q = 0.1$ and for system (14) with $\rho = 0.5$.

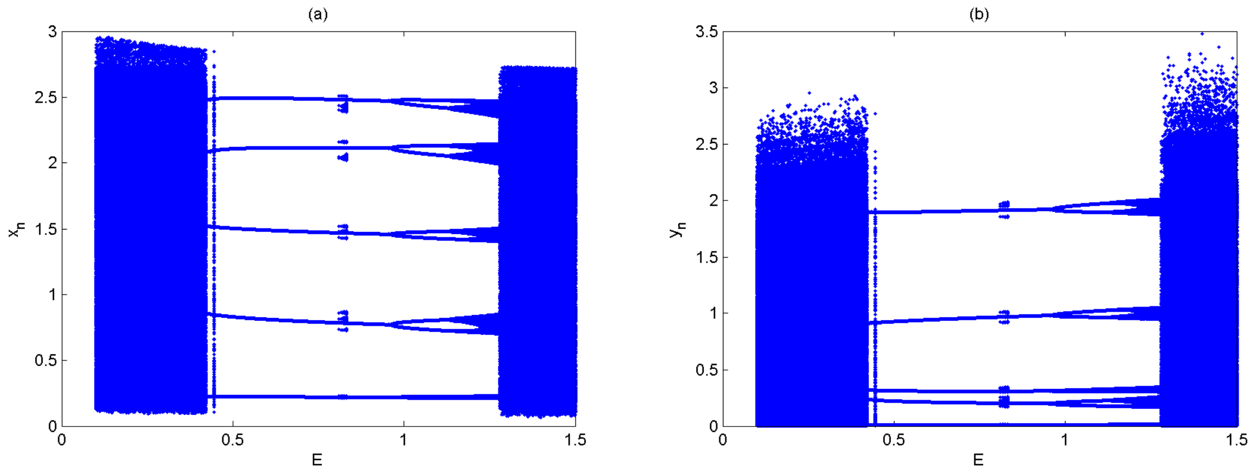

Figure 6.6: Bifurcation diagram for competing fish species with E of system (2.1) for fixed values $k_1 = 2.2, k_2 = 3.5, \alpha_1 = 1, \alpha_2 = 1.5, \beta_{12} = 1,$ $\beta_{21} = 1.1, \gamma_1 = 0.4, \gamma_2 = 0.62, d_1 = 1, d_2 = 1, q = 0.1$

7. Discussion

Kar and Chaudhuri [6] proposed system (1.1) and showed global stability and existence of bionomic equilibrium under certain conditions. The main novelty in our study is to introduce the non-linear Michaelis-Menten type harvesting in the first equation of system (1.1) and examine the dynamical behaviour of discrete version of the continuous system. Discrete-time harvesting models studied in [17, 18] did not consider the effect of toxicity of the interacting populations. Michaelis-Menten harvesting of the first species plays a significant role in determining the dynamics and bifurcations of the system. We discover interesting oscillations in the population size which are not observed in the continuous system. The parameters q and d_1 in the harvesting term influence the number and stability of the fixed points. The stability of boundary and interior fixed points is examined. As the trivial fixed point always exists and unstable when the intrinsic growth rate of the first fish species exceeds a certain threshold value, which in turn implies that the two species cannot go to extinction together. Neimark-Sacker and flip bifurcation, chaos control is investigated. Furthermore, the detailed mathematical proof of the global stability of the positive fixed point is given by using iteration scheme and the comparison principle of difference equations. Conditions of Theorem 5.4 indicate that if the intrinsic growth rates remain below one, then the global stability of the system may occur. But if we increase these rates, then the chaotic behaviour will appear (see Figure 6.5 a). The chaotic nature of the system is controlled by the hybrid control technique (see Figure 6.5b). In investigating bifurcation, we have identified that intra specific competition rate (α_1), toxicity rate (γ_2) and the intrinsic growth rate (k_2) have a major role in the system dynamics. It is observed that if the value of one the parameters α_1 or k_2 are increased we find a transition phase from stability to bifurcation within a limit cycle, to a periodic window and ultimately to chaos (see Figure 6.2 and Figure 6.4) whereas the opposite holds when we increase the value of the toxic inhibition rate (see Figure 6.3). Thus, the increase amount toxicity level can enhance the stability of the system. According to Figure 6.6, we can observe that the system is under control when harvesting effort E is low and chaotic when it is increased. Bifurcating behaviour and chaos have always been considered as an unwanted situation in biology. There will be a high risk of extinction of the species due to chaos. So, to prevent such extinction of the species, one can consider the application of a hybrid control method. It is to be noted that the competition terms used in (1.1) are instantaneous. In other words, two different species that compete for a given resource require a certain amount of time to get the resource. So the interference term of the model will be in the form of a Holling type II. Also, both the species can be harvested in the form mentioned above. Therefore, the model (2.1) can be reformulated as:

$$\begin{aligned} x_{n+1} &= x_n \exp(k_1 - \alpha_1 x_n - \frac{\beta_{12} y_n}{1 + a_1 x_n} - \gamma_1 x_n y_n - \frac{q_1 E}{d_1 E + d_2 x_n}), \\ y_{n+1} &= y_n \exp(k_2 - \alpha_2 y_n - \frac{\beta_{21} x_n}{1 + a_2 y_n} - \gamma_2 x_n y_n - \frac{q_2 E}{d_3 E + d_4 y_n}) \end{aligned} \tag{7.1}$$

Stability, bifurcation analysis and chaos control for model (7.1) is our future work for investigation.

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