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Research Article

Clairaut Semi-invariant Submersions From Locally Product Riemannian Manifolds

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Abstract

The purpose of this article is to analyze geometric features of Clairaut semi-invariant Riemannian submersions whose total manifolds are locally product Riemannian manifold and investigate fundamental results on such submersion. We also ensure an explicit example of Clairaut semi-invariant Riemannian submersion.

Keywords: Riemannian submersion, Clairaut submersion, Clairaut semi-invariant submersion, locally product Riemannian manifold.

Total Manifoldu Yerel Çarpım Riemann Manifoldu Olan Clairaut Yarı-değişmez Submersiyonlar

Öz

Bu makalenin amacı, total manifoldu yerel bir Riemann manifoldu olan Clairaut yarı-değişmez Riemann submersiyonlarının geometrik özelliklerini analiz etmek ve böyle bir submersiyon ile ilgili temel sonuçları araştırmaktır. Ayrıca Clairaut yarı-değişmez Riemann submersiyona açık bir örnek verilmektedir.

Anahtar Kelimeler: Riemann submersiyon, Clairaut submersiyon, Clairaut yarı-değişmez submersiyon, yerel çarpım Riemann manifold.

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1. Introduction

O'Neill [19] and Gray [9] defined Riemannian submersion between two Riemannian manifolds for the first time. In differential geometry, to equate geometric structures described on the above mentioned manifolds, Riemannian submersions are utilized broadly as differential maps. Riemannian submersions have important application areas in medical imaging, in robotics theory and Kaluza-Klein theory, and many more. We refer interested readers to [27] and reference therein for current progress and applications of Riemannian submersions. Afterwards, Watson [38] introduced almost Hermitian submersions and then Sahin [31] was presented the concept of Lagrangian submersion and anti-invariant submersions from almost Hermitian manifolds and this concept studied in [20, 32, 10, 5, 25, 14]. Şahin [28] generalized anti-invariant submersions, showing the advantages of studying properties of the total manifold of semi-invariant submersions. and the same idea investigated by Ozdemir et al. [20] and [12]. After, some researchers study some different types of Riemannian submersions such as pointwise slant submersion [23, 3, 7, 17], generic submersion [24, 1, 26, 22], Lorentzian Clairaut submersions [2], slant submersion [8, 29, 16, 13, 15], semi-slant submersion [21], hemi-slant submersion [33].

Clairaut's theorem [6], in the investigation of geodesic onto a surface of cycle (revolution), says that on the cycle surface $\overline{M}$ for any geodesic ς ($\varsigma: I_2 \subset \mathrm{R} \to \overline{M}$) the expression $r\sin\varphi$ is constant along ς, where the angle $\varphi(p)$ is the angle between the meridian curve and $\varsigma(p)$, $p \in I_2 \subset \mathrm{R}$. He also introduced and studied the theory of Riemannian submersions which content a generalization of Clairaut's theorem.

Clairaut anti-invariant submersions (CAIS) whose total manifold are paracosymplectic manifold are given in [11] with characterization theorems. CAIS's whose total manifolds are Kenmotsu and Sasakian were given by Tastan and Gerdan [34] and in [35], the authors also investigated CAIS from cosymplectic manifolds. Lee et al. [18] considered CAIS whose total manifold is Kahler manifolds. Gupta et al. [39] investigated Clairaut semi-invariant submersion from Kähler manifold.

In this paper, we investigate Clairaut semi-invariant Riemannian submersion (CSIRS) from a locally product Riemannian manifold onto a Riemannian manifold. In Section 2, we give some expressions that we will need in the next subsequent section. In Section 3, we describe CSIRS from locally product Riemannian manifold onto a Riemannian manifold and study the geometry of leaves of distributions and we present an example of the CSIRS whose total manifolds are locally product Riemannian.

2 Preliminaries

2.1 Locally Product Riemannian Manifolds

In this section, we give brief information for locally product Riemannian manifolds.

Let $\overline{M}$ be a smooth manifold of $(m+n)$-dimensional with a tensor field P of type $(1,1)$ such that

$$P^2 = I,\ (P \neq \pm I). \tag{2.1}$$

where I is the identity morphism of tangent bundle $T\overline{M}$. If $\overline{M}$ is equipped with the structure P, then $(\overline{M}, P)$ is an almost product manifold. If an almost product manifold $(\overline{M}, P)$ accepts a Riemannian metric $g_{\overline{M}}$ such that for all any $\overline{U_1}, \overline{U_2} \in T\overline{M}$,

$$g_{\overline{M}}\left(P\overline{U_1}, P\overline{U_2}\right) = g_{\overline{M}}\left(\overline{U_1}, \overline{U_2}\right) \text{ or } g_1(P\overline{U_1}, \overline{U_2}) = g_{\overline{M}}(\overline{U_1}, P\overline{U_2}). \tag{2.2}$$

Then, $\overline{M}$ is called an almost product Riemannian manifold. An almost product Riemannian manifold $\overline{M}$ is called a locally product Riemannian manifold if the equation hold

$$(\overline{\nabla}_{\overline{U_1}} P)\overline{U_2} = \overline{\nabla}_{\overline{U_1}} P\overline{U_2} - P\overline{\nabla}_{\overline{U_1}}\overline{U_2} = 0, \tag{2.3}$$

where $\overline{U_1}, \overline{U_2} \in T\overline{M}$ and $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$ [36].

2.2 Riemannian Submersions

In this section, we recall the fundamental definitions and notions of a Riemannian submersion.

Let $\left(\overline{M}, g_{\overline{M}}\right)$ and $\left(\overline{N}, g_{\overline{N}}\right)$ be Riemannian manifolds with $\dim\left(\overline{N}\right) < \dim(\overline{M})$. A surjective mapping $F: \left(\overline{M}, g_{\overline{M}}\right) \rightarrow \left(\overline{N}, g_{\overline{N}}\right)$ is called a Riemannian submersion if it satisfies the following conditions:

(i). The fibers $F^{-1}(a) \in \overline{N}, a \in \overline{N}$, are $k-dimensional$ Riemannian submanifolds of $\overline{N}$, where $k = \dim(\overline{M}) - \dim\left(\overline{N}\right)$.

In which case, for a vector field X_1 on $\overline{M}$, if it is always tangent to fibers then it is called vertical and if it is always orthogonal to fibers then it is called horizontal. If a vector field X_1 on $\overline{M}$ is horizontal and F–related to a vector field X_{1_*} on $\overline{N}$, then it is called basic, i.e., for all $, a \in \overline{N}$, $F_* X_{1a} = X_{1_*} F_{*(a)}$, where F_* is the derivative map of F.

(ii). F_{*q} preserves the length of the horizontal vectors, i.e., for all $q \in \overline{M}$ and for any horizontal vectors $X_1, Y_1 \in (\ker F_*)^{\perp}$ at q, $g_{\overline{M}}(X_1, Y_1) = g_{\overline{N}}(F_* X_1, F_* Y_1)$.

A Riemannian submersion $F: \left(\overline{M}, g_{\overline{M}}\right) \rightarrow \left(\overline{N}, g_{\overline{N}}\right)$ specifies two tensor fields $\mathcal{T}$ and $\mathcal{A}$ on $\overline{M}$ of types (1,2) for all $E, G \in \chi(\overline{M})$, by the formulas [19]:

$$\mathcal{T}(E, G) = \mathcal{T}_E G = h\overline{\nabla}_{vE} vG + v\overline{\nabla}_{vE} hG, \tag{2.4}$$

$$\mathcal{A}(E, G) = \mathcal{A}_E G = v\overline{\nabla}_{hE} hG + h\overline{\nabla}_{hE} vG, \tag{2.5}$$

where, v and h are the projections on the vertical distribution and horizontal distribution, repectively. $\overline{\nabla}$ is the Levi-Civita connection of $g_{\overline{M}}$

Let X_1, X_2 be horizontal and U_1, U_2 be vertical vector fields on $\overline{M}$, then we get

$$\mathcal{A}_{X_1}X_2 = -\mathcal{A}_{X_2}X_1 = \frac{1}{2}v[X_1, X_2], \tag{2.6}$$

$$\mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_2}U_1. \tag{2.7}$$

From (2.4) and (2.5), we get

$$\overline{\nabla}_{U_1}U_2 = \hat{\nabla}_{U_1}U_2 + \mathcal{T}_{U_1}U_2, \tag{2.8}$$

$$\overline{\nabla}_{U_1}X_1 = h\overline{\nabla}_{U_1}X_1 + \mathcal{T}_{U_1}X_1, \tag{2.9}$$

$$\overline{\nabla}_{X_1}U_1 = v\overline{\nabla}_{X_1}U_1 + \mathcal{A}_{X_1}U_1, \tag{2.10}$$

$$\overline{\nabla}_{X_1}X_2 = h\overline{\nabla}_{X_1}X_2 + \mathcal{A}_{X_1}X_2, \tag{2.11}$$

for any $X_1, X_2 \in \Gamma(\ker F_*)^{\perp}$and $U_1, U_2 \in \Gamma(\ker F_*)$. Also, if X_1 is basic then $h\overline{\nabla}_{U_1}X_1 = h\overline{\nabla}_{X_1}U_1 = \mathcal{A}_{X_1}U_1$. We say that $(\ker F_*)^{\perp}$ is totally geodesic if and only if $\mathcal{A} \equiv \{0\}$. From (2.8), we can also see that on the fibers, $\mathcal{T}$ take actions as the second fundamental form.

Let F be a surjective mapping between Riemannian manifolds $\overline{M}$ and $\overline{N}$. Then for $E, F \in \Gamma(T\overline{M})$ the second fundamental form of F is described as

$$(\overline{\nabla}F_*)(E, G) = \overline{\nabla}_E^F F_* G - F_*(\overline{\nabla}_E G), \tag{2.12}$$

where $\overline{\nabla}$ is the Riemannian connection and $\overline{\nabla}^F$ is the pull-back connection. From [4], the second fundamental form is well-known to be symmetric. Besides, F is called totally geodesic if $(\overline{\nabla}F_*)(E, G) = 0$ for all $E, F \in \Gamma(T\overline{M})$.

The fibers of F is called totally umbilical if

$$\mathcal{T}_{U_1}U_2 = g_{\overline{M}}(U_1, U_2)H, \tag{2.13}$$

for any $U_1, U_2 \in \Gamma(\ker F_*)$, where H stands for the mean curvature vector field of the fiber of F [20].

2.3 Semi-invariant Submersion

In this section, we present results on the geometry of semi-invariant submersions from locally product Riemannian (l.p.R) manifolds.

Definition 1 [20] Let $(\overline{M}, g_{\overline{M}}, P)$ be a l.p.R manifold and $(\overline{N}, g_{\overline{N}})$ be a Riemannian manifold. If there is a distribution $D_1 \subset \ker F_*$ such that

$$\ker F_* = D_1 \oplus D_2, PD_1 = D_1, PD_2 \subset (\ker F_*)^{\perp}, \tag{2.14}$$

then a Riemannian submersion $F: (\overline{M}, g_{\overline{M}}, P) \to (\overline{N}, g_{\overline{N}})$ is named as semi-invariant submersion, where the orthogonal complement of D_1 in $\ker F_*$ is D_2. In this case, the horizontal distribution $(\ker F_*)^{\perp}$ can be decomposed as

$$(\ker F_*)^{\perp} = PD_2 \oplus \eta, \tag{2.15}$$

where the orthogonal complementary distribution of PD_2 in $(\ker F_*)^{\perp}$ is η, and η is invariant with respect to P.

Let $F: (\overline{M}, g_{\overline{M}}, P) \to (\overline{N}, g_{\overline{N}})$ be a semi-invariant submersion from almost l.p.R manifold onto a Riemannian manifold. For any $U_1 \in \Gamma(\ker F_*)$, set

$$PU_1 = \phi U_1 + \omega U_1, \tag{2.16}$$

where $\omega U_1 \in \Gamma(\ker F_*)^{\perp}$ and $\phi U_1 \in \Gamma(\ker F_*)$. Also, for $X_1 \in \Gamma(\ker F_*)^{\perp}$ we set,

$$PX_1 = BX_1 + CX_1, \tag{2.17}$$

where $CX_1 \in \Gamma(\ker F_*)^{\perp}$ and $BX_1 \in \Gamma(\ker F_*)$.

3 Main Theorem and Proof

3.1 Clairaut Semi-invariant Submersion From Locally Product Riemannian Manifold

In this part, we give new Clairaut conditions for semi-invariant submersion from l.p.R manifold. To this end, firstly we recall some basic auxiliary results.

Definition 2 [6] Let $F: (\overline{M}, g_{\overline{M}}) \to (\overline{N}, g_{\overline{N}})$ be a Riemannian submersion and ς a geodesic on $\overline{M}$. If there exists a positive function r on $\overline{M}$, such that the function $(r \circ \varsigma)\sin\varphi$ is constant, then F is named as a Clairaut submersion, where the angle between the horizontal space at $\varsigma(p)$ and $\dot{\varsigma}(p)$ is $\varphi(p)$, for any $p \in I_2$.

In [6], Bishop also introduced Clairaut submersion, and he gave the conditions a Riemannian submersion to be a clairaut submersion.

Theorem 1 [6]Let $F: (\overline{M}, g_{\overline{M}}) \to (\overline{N}, g_{\overline{N}})$ be a Riemannian submersion between two Riemannian manifolds with connected fibers. If each fiber is totally umbilical and has the mean curvature vector field $H = -\mathrm{grad}\beta$, then F is a Clairaut submersion with $r = e^{\beta}$, where according to $g_{\overline{M}}$, the gradient of the function β is $\mathrm{grad}\beta$.

Herein, after giving several auxiliary results, we state new Clairaut conditions for semi-invariant Riemannian submersions from l.p.R manifolds.

Theorem 2 Let $F:(\overline{M}, g_{\overline{M}}, P) \to (\overline{N}, g_{\overline{N}})$ be a semi-invariant Riemannian submersion from a l.p.R manifold onto a Riemannian manifold. If $\varsigma: I_2 \subset R \to \overline{M}$ is a regular curve and the horizontal and vertical parts of the tangent vector field $\dot{\varsigma}(p)$ of $\varsigma(p)$ are $X_1(p)$ and $U_1(p)$ respectively, then ς is a geodesic if and only if the equations satisfy along ς

$$v\overline{\nabla}_{\dot{\varsigma}}BX_1 + v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1 = 0, \tag{3.1}$$

$$h\overline{\nabla}_{\dot{\varsigma}}BX_1 + h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1 = 0. \tag{3.2}$$

Proof. From (2.3), we get

$$\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = P(\overline{\nabla}_{\dot{\varsigma}}P\dot{\varsigma}). \tag{3.3}$$

Since $\dot{\varsigma} = U_1(p) + X_1(p)$, we can write

$$\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = P(\overline{\nabla}_{U_1(p)+X_1(p)}P(U_1(p) + X_1(p)))$$

$$= P(\overline{\nabla}_{U_1}PU_1 + \overline{\nabla}_{U_1}PX_1 + \overline{\nabla}_{X_1}PU_1 + \overline{\nabla}_{X_1}PX_1).$$

Using (2.16) and (2.17), we get

$$\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = P(\overline{\nabla}_{U_1}(\phi U_1 + \omega U_1) + \overline{\nabla}_{U_1}(BX_1 + CX_1) \tag{3.4}$$

$$+\overline{\nabla}_{X_1}(\phi U_1 + \omega U_1) + \overline{\nabla}_{X_1}(BX_1 + CX_1))$$

$$= P(\overline{\nabla}_{U_1}\phi U_1 + \overline{\nabla}_{U_1}\omega U_1 + \overline{\nabla}_{U_1}BX_1$$

$$+\overline{\nabla}_{U_1}CX_1 + \overline{\nabla}_{X_1}\phi U_1 + \overline{\nabla}_{X_1}\omega U_1 + \overline{\nabla}_{X_1}BX_1 + \overline{\nabla}_{X_1}CX_1).$$

Using (2.8)-(2.11), by direct computations, we have

$$\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = P(\mathcal{T}_{U_1}\phi U_1 + v\overline{\nabla}_{U_1}\phi U_1 + \mathcal{T}_{U_1}\omega U_1 + h\overline{\nabla}_{U_1}\omega U_1 + \mathcal{A}_{X_1}\phi U_1 + v\overline{\nabla}_{X_1}\phi U_1$$

$$+h\overline{\nabla}_{X_1}\omega U_1 + \mathcal{A}_{X_1}\omega U_1 + \mathcal{T}_{U1}BX_1 + v\overline{\nabla}_{U_1}BX_1 + \mathcal{T}_{U1}CX_1 + h\overline{\nabla}_{U_1}CX_1$$

$$+\mathcal{A}_{X_1}BX_1 + v\overline{\nabla}_{X_1}BX_1 + h\overline{\nabla}_{X_1}CX_1 + \mathcal{A}_{X_1}CX_1)$$

$$= P(v\overline{\nabla}_{\dot{\varsigma}}BX_1 + v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1$$

$$+h\overline{\nabla}_{\dot{\varsigma}}BX_1 + h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1).$$

By separating the vertical and horizontal parts of this equation, then we obtain

$$vP\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = v\overline{\nabla}_{\dot{\varsigma}}BX_1 + v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, \tag{3.5}$$

$$hP\overline{\nabla}_{\dot{\varsigma}}\dot{\varsigma} = h\overline{\nabla}_{\dot{\varsigma}}BX_1 + h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1. \tag{3.6}$$

Hence the theorem is proved.

Theorem 3 Let F be a semi-invariant Riemannian submersion from a l.p.R manifold $(\overline{M}, g_{\overline{M}}, P)$ onto a Riemannian manifold $(\overline{N}, g_{\overline{N}})$ and let $\varsigma: I_2 \subset R \to \overline{M}$ be a regular curve and the horizontal and vertical parts of the tangent vector field $\dot{\varsigma}(p)$ of $\varsigma(p)$ respectively are $X_1(p)$ and $U_1(p)$. Then F is a Clairaut submersion with $r = e^{\beta}$ if and only if along ς, the equation is obtained as

$$-\|U_1\|^2 g_{\overline{M}}(\mathrm{grad}\beta, X_1) = g_{\overline{M}}(v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1$$

$$+(\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, BX_1)$$

$$+g_{\overline{M}}(h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1$$

$$+(\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1, CX_1).$$

Proof. Let $\varsigma(p)$ be a geodesic on $\overline{M}$, $U_1(p) = v\dot{\varsigma}(p)$ and $X_1(p) = h\dot{\varsigma}(p)$. Let $\sqrt{k}$ be constant speed of ς on $\overline{M}$ that is, $k = g_{\overline{M}}(\dot{\varsigma}(p), \dot{\varsigma}(p)) = \|\dot{\varsigma}(p)\|^2$. Hence we conclude that,

$$g_{\overline{M}}(U_1(p), U_1(p)) = k\sin^2\varphi(p), \tag{3.7}$$

$$g_{\overline{M}}(X_1(p), X_1(p)) = k\cos^2\varphi(p), \tag{3.8}$$

where $\varphi(p)$ denotes the angle between $\dot{\varsigma}(p)$ and the horizontal space at $\varsigma(p)$. Differentiating (3.7), we have

$$\frac{d}{dp} g_{\overline{M}}(X_1(p), X_1(p)) = 2g_{\overline{M}}(\overline{\nabla}_{\dot{\varsigma}(p)}X_1(p), X_1(p)) = -2k\cos\varphi(p)\sin\varphi(p)\frac{d\varphi}{dp}. \tag{3.9}$$

Using (2.2), we get

$$\frac{d}{dp} g_{\overline{M}}(X_1(p), X_1(p)) = \frac{d}{dp} g_{\overline{M}}(PX_1(p), PX_1(p)).$$

From (2.3) and (2.17), we have

$$\frac{d}{dp} g_{\overline{M}}(X_1(p), X_1(p)) = 2g_{\overline{M}}(\overline{\nabla}_{\dot{\varsigma}}X_1, X_1)$$

$$= 2g_{\overline{M}}(P\overline{\nabla}_{\dot{\varsigma}}X_1, PX_1)$$

$$= 2g_{\overline{M}}(\overline{\nabla}_{\dot{\varsigma}}PX_1, PX_1)$$

$$= 2g_{\overline{M}}(\overline{\nabla}_{\dot{\varsigma}}BX_1 + \overline{\nabla}_{\dot{\varsigma}}CX_1, PX_1)$$

$$= 2g_1(\overline{\nabla}_{U_1+X_1}BX_1 + \overline{\nabla}_{U_1+X_1}CX_1, BX_1 + CX_1)$$

$$= 2g_{\overline{M}}(\overline{\nabla}_{U_1}BX_1 + \overline{\nabla}_{X_1}BX_1 + \overline{\nabla}_{U_1}CX_1 + \overline{\nabla}_{X_1}CX_1, BX_1 + CX_1).$$

Since F is a semi-invariant submersion and using equations (2.8)-(2.11), we obtain

$$\frac{d}{dp} g_{\overline{M}}(X_1(p), X_1(p) = 2g_{\overline{M}}(v\overline{\nabla}_{\dot{\varsigma}}BX_1, BX_1) + 2g_{\overline{M}}(h\overline{\nabla}_{\dot{\varsigma}}CX_1, CX_1).$$

From (3.5) and (3.6), we get

$$\frac{d}{dp} g_{\overline{M}}(X_1(p), X_1(p) = -2g_{\overline{M}}(v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, BX_1)$$

$$-2g_{\overline{M}}(h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1, CX_1).$$

From (3.9), we have

$$g_{\overline{M}}(v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, BX_1) \tag{3.10}$$

$$+g_{\overline{M}}(h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1, CX_1)$$

$$= k\cos\varphi(p)\sin\varphi(p)\frac{d\varphi}{dp}.$$

Now, F is a CSIRS with $r = e^{\beta}$ if and only if $\frac{d}{dp}((r \circ \varsigma)\sin\varphi(p)) = 0$. Therefore

$$\frac{d}{dp}((e^{\beta} \circ \varsigma)\sin\varphi(p)) = 0 \Leftrightarrow (e^{\beta} \circ \varsigma)(\frac{d\beta}{dp}\dot{\varsigma}(p)\sin\varphi(p) + \cos\varphi(p)\frac{d\varphi}{dp}) = 0.$$

Since r is a positive function, then

$$\frac{d\beta}{dp}\dot{\varsigma}(p)\sin\varphi + \cos\varphi\frac{d\varphi}{dp} = 0. \tag{3.11}$$

By multiplying (3.11) with non-zero factor $k\sin\varphi$, then we obtain

$$-\frac{d\beta}{dp}\dot{\varsigma}(p)k\sin^2\varphi = k\cos\varphi\sin\varphi\frac{d\varphi}{dp}. \tag{3.12}$$

Since the right-hand sides of equations (3.10) and (3.12) are equal,

$$g_{\overline{M}}(v\overline{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, BX_1)$$

$$+g_{\overline{M}}(h\overline{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1, CX_1)$$

$$= -\frac{d\beta}{dp}\dot{\varsigma}(p)k\sin^2\varphi.$$

From (3.7), we get

$$g_{\bar{M}}(v\bar{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, BX_1)$$

$$+g_{\bar{M}}(h\bar{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1, CX_1)$$

$$= -\frac{d\beta}{dp}\dot{\varsigma}(p) g_{\bar{M}}(U_1, U_1).$$

Thus, we have

$$g_{\bar{M}}(v\bar{\nabla}_{\dot{\varsigma}}\phi U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})CX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\omega U_1, BX_1)$$

$$+g_{\bar{M}}(h\bar{\nabla}_{\dot{\varsigma}}\omega U_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})BX_1 + (\mathcal{T}_{U1} + \mathcal{A}_{X_1})\phi U_1, CX_1)$$

$$= -\|U_1\|^2 g_{\bar{M}}(grad\beta, X_1).$$

Since $\frac{d\beta}{dp}\varsigma(p) = \dot{\varsigma}[\beta](p) = g_{\bar{M}}(grad\beta, \dot{\varsigma}(p)) = g_{\bar{M}}(grad\beta, U_1(p) + X_1(p)) = g_{\bar{M}}(grad\beta, X_1)$. Hence the theorem is proved.

Theorem 4 Let F be a semi-invariant Riemannian submersion from a l.p.R manifold $(\bar{M}, g_{\bar{M}}, P)$ onto a Riemannian manifold $(\bar{N}, g_{\bar{N}})$ with $r = e^{\beta}$. The following equation is provided as:

$$g_{\bar{M}}(\hat{\nabla}_{U_1}\phi U_2 + v\bar{\nabla}_{U_1}\omega U_2, BX_1) + g_{\bar{M}}(\mathcal{A}_{\omega U_2}U_1 + h\bar{\nabla}_{U_1}\phi U_2, CX_1)$$
$$= -g_{\bar{M}}(U_1, U_2) g_{\bar{M}}(grad\beta, X_1),$$

for $X_1 \in \Gamma(\eta)$ and $U_1, U_2 \in \Gamma(D_2)$ such that ωU_2 is basic.

Proof. From (2.13) and Theorem 1, we get

$$\mathcal{T}_{U_1}U_2 = -g_{\bar{M}}(U_1, U_2)grad\beta, \tag{3.13}$$

where $U_1, U_2 \in \Gamma(D_2)$. If inner product is applied to in (3.13) with $X_1 \in \Gamma(\eta)$, then we have

$$g_{\bar{M}}(\mathcal{T}_{U_1}U_2, X_1) = -g_{\bar{M}}(U_1, U_2) g_{\bar{M}}(grad\beta, X_1).$$

Let $X_1 \in \Gamma(\eta)$ and $U_1, U_2 \in \Gamma(D_2)$, then using (2.1), (2.2) and (2.3), we have

$$g_{\bar{M}}(\bar{\nabla}_{U_1}PU_2, PX_1) = g_{\bar{M}}(\bar{\nabla}_{U_1}U_2, X_1). \tag{3.14}$$

Using (2.8) in (3.14), we get

$$g_{\bar{M}}(\mathcal{T}_{U_1}U_2, X_1) = g_{\bar{M}}(\bar{\nabla}_{U_1}U_2, X_1).$$

Therefore, we have

$$g_{\bar{M}}(\bar{\nabla}_{U_1}PU_2, PX_1) = -g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, X_1).$$

Using (2.4),(2.5),(2.8)-(2.10),(2.16) and (2.17), we have

$$g_{\bar{M}}(\hat{\nabla}_{U_1}\phi U_2, BX_1) + g_{\bar{M}}(h\bar{\nabla}_{U_1}\phi U_2, CX_1)$$

$$+g_{\bar{M}}(v\bar{\nabla}_{U_1}\omega U_2, BX_1) + g_{\bar{M}}(h\bar{\nabla}_{U_1}\omega U_2, CX_1)$$

$$= -g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, X_1).$$

Since ωU_2 is basic, so $h\bar{\nabla}_{U_1}\omega U_2 = \mathcal{A}_{\omega U_2}U_1$, Thus we have

$$g_{\bar{M}}(\hat{\nabla}_{U_1}\phi U_2 + v\bar{\nabla}_{U_1}\omega U_2, BX_1)$$

$$+g_{\bar{M}}(\mathcal{A}_{\omega U_2}U_1 + h\bar{\nabla}_{U_1}\phi U_2, CX_1)$$

$$= -g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, X_1).$$

Theorem 5 Let F be a CSIRS from a l.p.R manifold $(\bar{M}, g_{\bar{M}}, P)$ onto a Riemannian manifold $(\bar{N}, g_{\bar{N}})$ with $r = e^{\beta}$. Then either β is constant on $P(D_2)$ or the fibres of F are 1-dimensional.

Proof. Let F be a CSIRS. For $U_1, U_2 \in \Gamma(D_2)$, taking inner product in (3.13) with PX_1, then we have

$$g_{\bar{M}}(\mathcal{T}_{U_1}U_2, PX_1) = -g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, PX_1), \tag{3.15}$$

for all $X_1 \in \Gamma(D_2)$. In (3.15), using (2.2), (2.3) and (2.8) we acquire

$$g_{\bar{M}}(\bar{\nabla}_{U_1}PU_2, X_1) = -g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, PX_1).$$

Utilizing (2.9) and (3.15) in this equation, we obtain

$$g_{\bar{M}}(U_1, PU_2)g_{\bar{M}}(grad\beta, X_1) = g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, PX_1).$$

Taking $PU_2 = X_1$ and $U_2 = PX_1$, we obtain

$$g_{\bar{M}}(U_1, X_1)g_{\bar{M}}(grad\beta, PU_2) = g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, PX_1). \tag{3.16}$$

Taking $X_1 = U_1$ and replacement the role of U_1 and U_2, then we have

$$g_{\bar{M}}(U_2, U_2)g_{\bar{M}}(grad\beta, PU_1) = g_{\bar{M}}(U_1, U_2)g_{\bar{M}}(grad\beta, PU_2). \tag{3.17}$$

Now, just taking $X_1 = U_1$ in (3.16), we get

$$g_{\bar{M}}(U_1, U_1) g_{\bar{M}}(grad\beta, PU_2) = g_{\bar{M}}(U_1, U_2) g_{\bar{M}}(grad\beta, PU_1). \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$g_{\bar{M}}(grad\beta, PU_1) = \frac{\left(g_{\bar{M}}(U_1, U_2)\right)^2}{\|U_1\|^2 . \|U_2\|^2} g_{\bar{M}}(grad\beta, PU_1).$$

From the condition of equality in the Schwarz inequality, if $grad\beta \in \Gamma(PD_2)$, then it means that the claim of Theorem (5) is satisfied.

Lastly, we give an example of a CSIRS from a l.p.R manifold.

3.2 Example

Let $\overline{\mathrm{M}}$ given by

$$\bar{M} = \{(u_1, u_2, u_3, u_4, u_5, u_6) \in \mathbb{R}^6 : (u_2, u_3, u_4, u_5, u_6) \neq 0, u_1 \neq 0\}.$$

We describe the Riemannian metric $g_{\bar{M}}$ on $\bar{M}$ as

$$g_{\bar{M}} = 3e^{-2u_1} du_1^2 + 3e^{-2u_1} du_2^2 + e^{-2u_1} du_3^2 + e^{-2u_1} du_4^2 + e^{-2u_1} du_5^2 + 3e^{-2u_1} du_6^2.$$

We take the product structure $(P, g_{\bar{M}})$ on $\bar{M}$ as $P(a, b, c, d, e, f) = (-b, -a, d, c, -f, -e)$. Let $\bar{N} = \{(v_1, v_2, v_3) \in \mathbb{R}^3\}$ be a Riemannian manifold with Riemannian metric $g_{\bar{N}}$ on $\bar{N}$ given by

$$g_{\bar{N}} = e^{-2u_1} dv_1^2 + e^{-2u_1} dv_2^2 + e^{-2u_1} dv_3^2.$$

A P $-$basis can be given by

$$\left\{e_1 = e^{u_1} \frac{\partial}{\partial u_1}, e_2 = e^{u_1} \frac{\partial}{\partial u_2}, e_3 = e^{u_1} \frac{\partial}{\partial u_3}, e_4 = e^{u_1} \frac{\partial}{\partial u_4}, e_5 = e^{u_1} \frac{\partial}{\partial u_5}, e_6 = e^{u_1} \frac{\partial}{\partial u_6}\right\},$$

on $T_q \bar{M}$ and

$$\left\{e_1^* = \frac{\partial}{\partial y_1}, e_2^* = \frac{\partial}{\partial y_2}, e_3^* = \frac{\partial}{\partial y_3}\right\},$$

on $T_{F(q)} \bar{N}$ for all $q \in \bar{M}$. Now, we assume a map $F: (\bar{M}, P, g_{\bar{M}}) \to (\bar{N}, g_{\bar{N}})$ by

$$F(u_1, u_2, u_3, u_4, u_5, u_6) = (u_1 - u_3, u_2 - u_4, u_5 + u_6).$$

Then, we have

$$\ker F_* = ppan\{U_1 = e_1 + e_3, U_2 = e_2 + e_4, U_3 = e_5 - e_6\},$$

$$(\ker F_*)^{\perp} = ppan\{X_1 = e_1 - e_3, X_2 = e_2 - e_4, X_3 = e_5 + e_6\}.$$

Hence it is easy to see that

$$g_{\overline{M}}(X_i, X_i) = g_{\overline{N}}(F_*(X_i), F_*(X_i)) = 4,$$

$$g_{\overline{M}}(PU_1, PU_1) = g_{\overline{N}}(F_*(PU_1), F_*(PU_1)) = 4,$$

$$g_{\overline{M}}(PU_2, PU_2) = g_{\overline{N}}(F_*(PU_2), F_*(PU_2)) = 4,$$

for $i = 1,2,3$.Thus F is a Riemannian submersion. On the other hand, we obtain

$$PU_1 = -X_2, PU_2 = -X_1, PU_3 = U_3,$$

where P is the product structure of $\mathbb{R}^6$. Therefore F is a semi-invariant Riemannian submersion with

$$D_1 = \langle U_3 = e_5 - e_6 \rangle,$$

$$D_2 = \langle U_1 = e_1 + e_3, U_2 = e_2 + e_4 \rangle,$$

such that $\ker F_* = D_1 \oplus D_2$. Next, we will look for a smooth function β on $\overline{M}$ satisfying $T_U U = -g_{\overline{M}}(U, U) grad\beta$, for all $U \in \Gamma(\ker F_*)$. We can simply calculate that

$$\overline{\nabla}_{e_1} e_1 = 0, \overline{\nabla}_{e_1} e_3 = 0, \overline{\nabla}_{e_3} e_1 = -e^{2u_1} \frac{\partial}{\partial u_3}, \overline{\nabla}_{e_3} e_3 = \frac{e^{2u_1}}{3} \frac{\partial}{\partial u_1} + e^{2u_1} \frac{\partial}{\partial u_3},$$

$$\overline{\nabla}_{e_2} e_2 = e^{2u_1} \frac{\partial}{\partial u_1}, \overline{\nabla}_{e_2} e_4 = 0, \overline{\nabla}_{e_4} e_2 = 0, \overline{\nabla}_{e_4} e_4 = \frac{e^{2u_1}}{3} \frac{\partial}{\partial u_1},$$

$$\overline{\nabla}_{e_5} e_5 = \frac{e^{2u_1}}{3} \frac{\partial}{\partial u_1}, \overline{\nabla}_{e_5} e_6 = 0, \overline{\nabla}_{e_6} e_5 = 0, \overline{\nabla}_{e_6} e_6 = e^{2u_1} \frac{\partial}{\partial u_1}$$

Hence, we have

$$\overline{\nabla}_{U_1} U_1 = e^{2u_1} \frac{1}{3} \frac{\partial}{\partial u_1}, \overline{\nabla}_{U_2} U_2 = \frac{4}{3} e^{2u_1} \frac{\partial}{\partial u_1}, \overline{\nabla}_{U_3} U_3 = \frac{4}{3} e^{2u_1} \frac{\partial}{\partial u_1},$$

$$\overline{\nabla}_{U_1} U_2 = \overline{\nabla}_{U_1} U_3 = \overline{\nabla}_{U_2} U_3 = 0.$$

Now, if we take $U = \lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3$, for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ then

$$T_U U = \lambda_1^2 T_{U_1} U_1 + \lambda_2^2 T_{U_2} U_2 + \lambda_3^2 T_{U_3} U_3$$

$$+2\lambda_1 \lambda_2 T_{U_1} U_2 + 2\lambda_1 \lambda_3 T_{U_1} U_3 + 2\lambda_2 \lambda_3 T_{U_2} U_3.$$

From (2.8)-(2.10), by direct calculations, we have

$$T_U U = (\lambda_1^2 + 4\lambda_2^2 + 4\lambda_3^2)\frac{1}{3}e^{2u_1}\frac{\partial}{\partial u_1}.$$

Since $U = \lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3$, then by direct calculations, we obtain

$$g_{\bar{M}}(U, U) = 4(\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

Moreover, for any smooth function β on $\mathbb{R}^6$, the gradient of β with respect to the metric $g_{\bar{M}}$ is given by

$$\nabla\beta = \sum_{i,j=1}^{6} g_{\bar{M}}^{ij}\frac{\partial\beta}{\partial u_i}\frac{\partial}{\partial u_j}$$

$$= \frac{e^{2u_1}}{3}\frac{\partial\beta}{\partial u_1}\frac{\partial}{\partial u_1} + \frac{e^{2u_1}}{3}\frac{\partial\beta}{\partial u_2}\frac{\partial}{\partial u_2} + e^{2u_1}\frac{\partial\beta}{\partial u_3}\frac{\partial}{\partial u_3}$$

$$+e^{2u_1}\frac{\partial\beta}{\partial u_4}\frac{\partial}{\partial u_4} + e^{2u_1}\frac{\partial\beta}{\partial u_5}\frac{\partial}{\partial u_5} + e^{2u_1}\frac{\partial\beta}{\partial u_6}\frac{\partial}{\partial u_6}.$$

Hence $\nabla\beta = -\frac{\lambda_1^2+4\lambda_2^2+4\lambda_3^2}{4(\lambda_1^2+\lambda_2^2+\lambda_3^2)}\frac{e^{2u_1}}{3}\frac{\partial}{\partial u_1}$ for the function $\beta = -\frac{\lambda_1^2+4\lambda_2^2+4\lambda_3^2}{4(\lambda_1^2+\lambda_2^2+\lambda_3^2)}u_1$. Therefore, it can be seen that $T_U U = -g_{\bar{M}}(U, U)grad\beta$. Hence F is a CSIRS.

4 Conclusion

In this paper, we tried to study Clairaut semi-invariant Riemannian submersions whose total manifolds are locally product Riemannian manifold and we investigated the various fundamental geometric properties on the fibers and distributions of these submersions. As future research, we plan to focus on studying Clairaut's semi-invariant Riemannian submersions between different kinds of the manifolds.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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