INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 1 PAGE 48–61 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1178802

Some Characterizations of Translation Surface Generated by Spherical Indicatrices of Timelike Curves in Minkowski 3-space

Akhilesh Yadav and Ajay Kumar Yadav*

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this paper, we study translation surfaces generated by spherical indicatrices of timelike curves in Minkowski 3-space and find necessary and sufficient conditions for the translation surfaces to be flat or minimal. Further, we obtain necessary and sufficient conditions for generating curves of the translation surfaces to be geodesic, asymptotic line and line of curvature. Finally for such translation surfaces we obtain the axis when they are constant angle surfaces.

Keywords: *Tangent indicatrices, principal normal indicatrices, binormal indicatrices, timelike curves.*

AMS Subject Classification (2020): *Primary: 53A04; 53A05; Secondary: 53B30.*

1. Introduction

A Darboux surface is a surface which is union of 'equivalent' curves, i.e., the curves are images of one another by isometries of space, called its generating curves. Kinematically, a Darboux surface is defined as the movement of a curve by rigid motions of the space. Hence, a parametrization of such surfaces can be given as $X(u, v) = A(v).\alpha(u) + \beta(v)$, where $\alpha,\ \beta$ are two space curves and A is an orthogonal matrix. A translation surface is special case of Darboux surfaces when the orthogonal matrix A is identity matrix and both curves intersect each other. Thus, parametrization of generalized type of a translation surface in 3-dimensional Euclidean space is given by

$$X(u, v) = \alpha(u) + \beta(v).$$

Translation surface which is known as double curve in differential geometry are base for roofing structures. The construction and design of free form glass roofing structures are generally created with the help of curved (formed) glass panes or planar triangular glass facets.

Many authors studied translation surfaces in Euclidean space as well as semi-Euclidean space. In [15], Liu obtained some characterizations about the translation surfaces with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space E^3 and 3-dimensional Minkowski space E_1^3. In [17], Muntenau and Nistor studied the second fundamental form of the translation surfaces in 3-dimensional Euclidean space E^3, and obtained some characterizations by using the second Gaussian curvature K_{II} of the translation surfaces. In [8], Çetin and Tunçer studied surfaces parallel to translation surfaces in Euclidean 3-space. In [2], Ali et al. gave some results on some special points of the translation surfaces in E^3. Since the translation surfaces are surfaces produced by two space curves, some basic calculations of the surface can be stated in terms of Frenet vectors and curvatures of the space curves. In [7], Çetin et al. investigated translation surfaces according to Frenet frames in Minkowski 3-space, and studied some properties of these surfaces. Furthermore, they calculated first fundamental form, second fundamental form, Gaussian curvature and mean curvature of the translation surface. Finally, they gave the conditions for the generator curves of the translation surface being a geodesic, an asymptotic line and a principal line.

Received : *23-08-2022,* ***Accepted :*** *28-03-2023*
*** Corresponding author***

Non-lightlike ruled surfaces of constant slope parallel to a tangent of a timelike general helix and normal of timelike slant helix were studied by Ali [1]. It was shown that all non-lightlike ruled surfaces of constant slope are developable but not minimal surfaces. In [3], Arfah discussed the causal characterization of spherical indicatrices of timelike curves in Minkowski 3-space, and provided the concept of spherical indicatrix of tangent, normal and binormal vectors of timelike curves with their casual properties. Acar and Aksoyak studied translation surfaces generated by tangent, normal and binormal indicatrices of space curves in E^3, respectively, and obtain some characterizations based on the fact that such surfaces are flat or minimal [4].
Motivated by these studies, in this paper, we study translation surfaces generated by tangent, principal normal and binormal indicatrices of timelike curves α and β in Minkowski 3-space E_1^3. We obtain some characterizations of such surfaces based when they are flat or minimal. Moreover we give the condition for the generator curves of the translation surface being geodesic, asymptotic and line of curvature. Finally for such translation surfaces we obtain the axis when they are constant angle surfaces and also find the condition for one of the curve α and β is a slant helix.

2. Preliminaries

The Minkowski 3-space denoted by E_1^3 is a three dimensional real vector space R^3 endowed with the metric tensor $\langle .,. \rangle = -dx^2 + dy^2 + dz^2$. The (Lorentzian) scalar and cross product are defined by

$$\begin{cases} \langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3, \\ x \times y = (x_2y_3 - x_3y_2, x_1y_3 - x_3y_1, x_2y_1 - x_1y_2), \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ belong to E_1^3. This space is also known as Lorentz-Minkowski space. A vector $x \in E_1^3$ is said to be spacelike when $\langle x, x \rangle > 0$ or $x = 0$, timelike when $\langle x, x \rangle < 0$ and lightlike (null) when $\langle x, x \rangle$ = 0. A curve in E_1^3 is called spacelike, timelike or lightlike when the velocity vector of the curve is spacelike, timelike or lightlike, respectively.
Let $\gamma = \gamma(s) : I \to E_1^3$ be an arbitrary timelike curve. The curve γ is said to be a unit speed (or parameterized by the arc-length parameter s) if $\langle \gamma'(s), \gamma'(s) \rangle = -1$ for any $s \in I$. Let $\{t(s), n(s), b(s)\}$ be the moving Frenet frame of γ which satisfies the following conditions

$$\begin{cases} \langle t, t \rangle = -\langle n, n \rangle = -\langle b, b \rangle = -1, \\ \langle t, n \rangle = \langle t, b \rangle = \langle b, n \rangle = 0, \\ t \times n = b, \; n \times b = -t, \; b \times t = n, \\ det(t, n, b) = 1. \end{cases} \tag{2.2}$$

For the timelike curve γ the Frenet-Serret equations are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} = \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.3}$$

where the $'$ denotes the derivative with respect to s. κ and τ are the curvature and torsion of the curve, respectively.

Definition 2.1. [20] Let v and w be two spacelike vectors in E_1^3. Then, we have the followings:
(a) If v and w span a spacelike plane, then there exists a unique non-negative real number $\theta \geq 0$ such that $\langle v, w \rangle = \|v\|\|w\| \cos\theta$.
(b) If v and w span a timelike plane such that both vectors lie in the same spacelike component of the plane, then there exists a unique non-negative real number $\theta \geq 0$ such that $\langle v, w \rangle = \|v\|\|w\| \cosh\theta$.

Definition 2.2. [20] Let v be a spacelike vector and w be a timelike vector in E_1^3. Then, there is a unique non-negative real number $\theta \geq 0$ such that $\langle v, w \rangle = \|v\|\|w\| \sinh\theta$.

Definition 2.3. [18] Let v be a timelike vector and w be a timelike vector in same time cone of E_1^3, i.e., $\langle v, w \rangle < 0$. Then, there is a unique non-negative real number $\theta \geq 0$, such that $\langle v, w \rangle = -\|v\|\|w\| \cosh\theta$.

Definition 2.4. [6] A unit speed curve $\gamma = \gamma(s) : I \to E_1^3$ is a general helix if there exists a fixed unit vector d, called axis of the helix, such that $\langle t, d \rangle$ is constant along the curve, where $t(s) = \gamma'(s)$ is the unit tangent vector of γ.

Theorem 2.1. [6] *A unit speed curve $\gamma = \gamma(s) : I \to E_1^3$ is a general helix if and only if $\frac{\tau}{\kappa}(s)$ is a non zero constant.*

Definition 2.5. [12] A unit speed curve $\gamma = \gamma(s) : I \to E_1^3$ is a slant helix if there exists a fixed unit vector d, called axis of the slant helix, such that $\langle n, d\rangle$ is constant along the curve, where $n(s)$ is the principal normal vector of γ.

Definition 2.6. Let $\gamma = \gamma(s) : I \to E_1^3$ be a unit speed curve with Frenet frame $\{t(s), n(s), b(s)\}$. When we move along the curve the locus of the tip of the unit vectors t, n and b determines new curves on the unit sphere which are known as spherical indicatrices of the curve. In particular, the spherical indicarix of t, n and b are known as tangent indicatrix t(s), principal normal indicatrix n(s) and binormal indicatrix b(s), respectively.

A surface in E_1^3 is said to be a spacelike, timelike or lightlike if the metric on the surface is positive definite, indefinite or degenerate, respectively. Type of the surface can also be expressed in terms of the causal character of the normal on the surface by the following lemma [10].

Lemma 2.1. [14] *A surface in Minkowski 3-space is spacelike, timelike or lightlike if and only if, at every point of the surface there exists a normal that is timelike, spacelike or lightlike, respectively.*

Let $M : X = X(u, v) \in E_1^3$ be a regular surface. Then, the unit normal vector field of the surface M is determined by

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \tag{2.4}$$

where X_u and X_v are derivatives of X with respect to u and v, respectively. The coefficients of the first fundamental form and second fundamental form are given by

$$E = \langle X_u, X_u\rangle,\ F = \langle X_u, X_v\rangle,\ G = \langle X_v, X_v\rangle$$

and

$$l = \langle X_{uu}, N\rangle,\ m = \langle X_{uv}, N\rangle,\ n = \langle X_{vv}, N\rangle.$$

Gaussian and mean curvatures of the surface M are expressed as follows [5]:

$$K = \langle N, N\rangle \frac{ln - m^2}{EG - F^2}, \tag{2.5}$$

and

$$H = \frac{1}{2}\frac{En + Gl - 2Fm}{|EG - F^2|}, \tag{2.6}$$

respectively.

Definition 2.7. A surface in E_1^3 is called flat when the Gaussian curvature vanishes, and it is called minimal when the mean curvature vanishes.

Definition 2.8. A constant angle surface or constant slope surface M in E_1^3 is a surface whose unit normal N makes a constant angle with a fixed vector U, i.e., $\langle N, U\rangle = constant$.

3. Translation surface generated by spherical indicatrices of timelike curves in Minkowski 3-space

Let $\alpha : I \to E_1^3$ and $\beta : J \to E_1^3$ be two timelike curves with arc-length parameters u and v, respectively in E_1^3. Let $\{t_\alpha, n_\alpha, b_\alpha, \kappa_\alpha, \tau_\alpha\}$ and $\{t_\beta, n_\beta, b_\beta, \kappa_\beta, \tau_\beta\}$ be the Frenet apparatus of the curves α and β, respectively. In this section, we examine the translation surfaces generated by the tangent indicatrices t_α, t_β, principal normal indicatrices n_α, n_β and binormal indicatrices b_α, b_β of the curves α and β and find out some characterizations of the surfaces as well as of the generating curves of the surfaces. Throughout the paper, we will assume that $\kappa_\alpha, \kappa_\beta, \tau_\alpha$ and τ_β are all non-zero so that the curves $t_\alpha, t_\beta, n_\alpha, n_\beta$ and b_α, b_β are regular curves.

3.1. Translation Surfaces generated by tangent indicatrices of two timelike curves in E_1^3

Let α, β be timelike curves in E_1^3. Translation surface generated by tangent indicatrices of the space curves α, β in E_1^3 is defined by

$$M_1 : X(u,v) = t_\alpha(u) + t_\beta(v), \tag{3.1}$$

where t_α and t_β are both timelike vectors. Thus, n_α, b_α and n_β, b_β are all spacelike vectors. Differentiating the above equation (3.1) with respect to u and v, we obtain $X_u = t'_\alpha$ and $X_v = t'_\beta$. Now, using Frenet equations (2.3) for the curves α and β, we get

$$X_u = \kappa_\alpha n_\alpha, \ X_v = \kappa_\beta n_\beta. \tag{3.2}$$

The unit normal vector N of the surface M_1 is given by

$$N(u,v) = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \tag{3.3}$$

where $X_u \times X_v = \kappa_\alpha \kappa_\beta (n_\alpha \times n_\beta)$ and $\|X_u \times X_v\| = \sqrt{-\epsilon(EG - F^2)}$, where $\epsilon = \langle N, N\rangle$.
Now, we have two cases according to causality of the plane spanned by n_α and n_β.
Case (i): We assume that n_α and n_β span timelike plane. Thus, by Definition 2.1(b), we have $\langle n_\alpha, n_\beta\rangle = \cosh\theta$ and the coefficients of first fundamental form are obtained as follows

$$E = \kappa_\alpha^2, \ F = \kappa_\alpha\kappa_\beta \cosh\theta, \ G = \kappa_\beta^2, \tag{3.4}$$

where θ is the smooth hyperbolic angle function between n_α and n_β.
In this case, $EG - F^2 = -\kappa_\alpha^2\kappa_\beta^2 \sinh^2\theta < 0$ shows that the surface M_1 is timelike, and hence, the unit normal N is spacelike, i.e., $\langle N, N\rangle = 1$. Thus, we obtain the spacelike unit normal as

$$N(u,v) = \frac{n_\alpha \times n_\beta}{\sinh\theta}, \tag{3.5}$$

so that $\langle N, n_\alpha\rangle = \langle N, n_\beta\rangle = 0$.
Now, suppose the hyperbolic angle between t_α and N is ϕ, and the hyperbolic angle between t_β and N is ψ, then N can be expressed as follows [13]:

$$N(u,v) = -\sinh\phi\, t_\alpha + \cosh\phi\, b_\alpha,$$
$$N(u,v) = -\sinh\psi\, t_\beta + \cosh\psi\, b_\beta.$$

The coefficients of second fundamental form of the timelike surface M_1 are given by

$$\begin{cases} l = \kappa_\alpha^2(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi), \\ m = 0, \\ n = \kappa_\beta^2(\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi). \end{cases} \tag{3.6}$$

Now using the equations (2.5), (2.6), and above calculations, we obtain the following results.

Theorem 3.1. *The Gaussian curvature K and the mean curvature H of the timelike translation surface M_1 are given as follows, respectively,*

$$K = -\frac{(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi)(\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi)}{\sinh^2\theta},$$
$$H = \frac{(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi) + (\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi)}{2\sinh^2\theta}.$$

Corollary 3.1. *The timelike translation surface M_1 is flat if and only if either $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$ or $\tanh\psi = -\frac{\tau_\beta}{\kappa_\beta}$.*

Proof. By putting $K = 0$ in the Theorem 3.1, we get the desired result. □

Corollary 3.2. *The timelike translation surface M_1 be minimal if and only if $\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi = -\sinh\psi - \frac{\tau_\beta}{\kappa_\beta}\cosh\psi$.*

Proof. By putting $H = 0$ in the Theorem 3.1, we get the required result. □

Case (ii): We suppose that n_α and n_β span spacelike plane. Thus, by Definition 2.1(a), we have $\langle n_\alpha, n_\beta\rangle = \cos\theta$ and the coefficients of first fundamental form are obtained as follows

$$E = \kappa_\alpha^2,\ F = \kappa_\alpha\kappa_\beta\cos\theta,\ G = \kappa_\beta^2, \tag{3.7}$$

where θ is the smooth angle function between n_α and n_β.
In this case, $EG - F^2 = \kappa_\alpha^2\kappa_\beta^2\sin^2\theta > 0$ shows that the surface M_1 is spacelike, hence, the unit normal N is timelike, i.e., $\langle N, N\rangle = -1$. Thus, we obtain the timelike unit normal as

$$N(u,v) = \frac{n_\alpha \times n_\beta}{\sin\theta}, \tag{3.8}$$

so that $\langle N, n_\alpha\rangle = \langle N, n_\beta\rangle = 0$.
Now, suppose the hyperbolic angle between t_α and N is ϕ and the hyperbolic angle between t_β and N is ψ then N can be expressed as follows [13]:

$$N(u,v) = \cosh\phi\ t_\alpha + \sinh\phi\ b_\alpha,$$
$$N(u,v) = \cosh\psi\ t_\beta + \sinh\psi\ b_\beta.$$

The coefficients of second fundamental form of the spacelike surface M_1 are given by

$$\begin{cases} l = \kappa_\alpha^2(\frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi - \cosh\phi), \\ m = 0, \\ n = \kappa_\beta^2(\frac{\tau_\beta}{\kappa_\beta}\sinh\psi - \cosh\psi). \end{cases} \tag{3.9}$$

Now, using the equations (2.5), (2.6), and above calculations, we obtain the following results.

Theorem 3.2. *The Gaussian curvature K and the mean curvature H of the spacelike translation surface M_1 are given as follows, respectively,*

$$K = -\frac{(\frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi - \cosh\phi)(\frac{\tau_\beta}{\kappa_\beta}\sinh\psi - \cosh\psi)}{\sin^2\theta},$$
$$H = \frac{(\frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi - \cosh\phi) + (\frac{\tau_\beta}{\kappa_\beta}\sinh\psi - \cosh\psi)}{2\sin^2\theta}.$$

Corollary 3.3. *The spacelike translation surface M_1 is flat if and only if either $\coth\phi = \frac{\tau_\alpha}{\kappa_\alpha}$, or $\coth\psi = \frac{\tau_\beta}{\kappa_\beta}$.*

Proof. By putting $K = 0$ in the Theorem 3.2, we get the desired result. □

Corollary 3.4. *The spacelike translation surface M_1 is minimal if and only if $\frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi - \cosh\phi = -\frac{\tau_\beta}{\kappa_\beta}\sinh\psi + \cosh\psi$.*

Proof. By putting $H = 0$ in the Theorem 3.2, we get the required result. □

Theorem 3.3. *If the timelike (spacelike) translation surface M_1 is flat then either the angle between N and t_α is a function of u only or the angle between N and t_β is a function that depends only on v.*

Proof. Let the spacelike translation surface M_1 be flat then by Corollary 3.3, if we assume $\coth\phi = \frac{\tau_\alpha}{\kappa_\alpha}$, then since right hand side is a function of u only, the angle ϕ between N and t_α is a function that depends only on u. Further, if we assume $\coth\psi = \frac{\tau_\beta}{\kappa_\beta}$, then since right hand side is a function of v only, the angle ψ between N and t_β is a function that depends only on v. In case of timelike surface, same arguments work. □

Theorem 3.4. *Let the timelike (spacelike) translation surface M_1 be flat. If the curves α and β are helices then either the angle ϕ between N and t_α is a non-zero constant or the angle ψ between N and t_β is a non-zero constant.*

Proof. Let the timelike surface M_1 be flat. If the curve α and β are helices, then $\frac{\tau_\alpha}{\kappa_\alpha}$ and $\frac{\tau_\beta}{\kappa_\beta}$ are non-zero constant functions. Then, by Corollary 3.1, $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$ implies that ϕ is a non-zero constant. Further, $\tanh\psi = -\frac{\tau_\beta}{\kappa_\beta}$ implies that ψ is a non-zero constant. In case of spacelike surfaces, same arguments work. □

Theorem 3.5. *Let the timelike (spacelike) translation surface M_1 be flat. If the curves α and β are helices, then the surface M_1 is a constant angle surface.*

Proof. We suppose that the surface M_1 is flat and the curves α and β are helices. Then, by Theorem 3.4, either ϕ or ψ is a constant function. Let $\phi = \phi_\circ$ be constant. Since α is a helix, there exists a unit fixed direction U_α which makes a constant angle with unit tangent vector t_α of the curve α. Suppose U_α is spacelike vector, then $\langle t_\alpha, U_\alpha\rangle = \sinh\delta_\circ$ = constant, which on differentiating with respect to u gives $\langle n_\alpha, U_\alpha\rangle = 0$, when $\kappa_\alpha \neq 0$. Thus we can express U_α as $U_\alpha = -\sinh\delta_\circ t_\alpha + \cosh\delta_\circ b_\alpha$. Then, $\langle N, U_\alpha\rangle = \langle -\sinh\phi_\circ\, t_\alpha + \cosh\phi_\circ\, b_\alpha, -\sinh\delta_\circ\, t_\alpha + \cosh\delta_\circ\, b_\alpha\rangle = -\sinh\phi_\circ\sinh\delta_\circ + \cosh\phi_\circ\cosh\delta_\circ$, which is a constant. Thus, the surface is a constant angle surface. In case U_α is a timelike vector, similar arguments show that $\langle N, U_\alpha\rangle$ = const. □

Theorem 3.6. *Let the translation surface M_1 be minimal. If the curves α and β are planar curves, then the angle between N and t_α, and the angle between N and t_β are same (upto sign).*

Proof. Let the surface M_1 be minimal. Let α and β be planar curves, then $\tau_\alpha = \tau_\beta = 0$. Then, by Corollary 3.2, we get $\sinh\phi = -\sinh\psi$, which implies $\phi = -\psi$. □

Theorem 3.7. *The generating curve t_α is an asymptotic curve on*
(i) the timelike translation surface M_1 if and only if $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$,
(ii) the spacelike translation surface M_1 if and only if $\coth\phi = \frac{\tau_\alpha}{\kappa_\alpha}$.

Proof. Let M_1 be a timelike translation surface. Then, the normal curvature of the tangent indicatrix t_α is given by [21],
$\kappa_n = \frac{1}{\kappa_\alpha^2}\langle t''_\alpha, N\rangle = \frac{1}{\kappa_\alpha^2}\langle \kappa_\alpha^2 t_\alpha + \kappa'_\alpha n_\alpha + \kappa_\alpha\tau_\alpha b_\alpha, -\sinh\phi\, t_\alpha + \cosh\phi\, b_\alpha\rangle = \sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi$.
Now, we know that t_α is asymptotic if and only if the normal curvature of t_α is zero. Thus, t_α is asymptotic if and only if $\langle t''_\alpha, N\rangle = 0$, which implies $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$. Similarly, we can prove for the spacelike surface. □

Corollary 3.5. *The Gaussian and the mean curvatures of the translation surface M_1 are given as*

$$K = -\frac{\kappa_{n_1}\kappa_{n_2}}{\sinh^2\theta},\ H = \frac{\kappa_{n_1} + \kappa_{n_2}}{2\sinh^2\theta},$$

respectively, where κ_{n_1} and κ_{n_2} are normal curvatures of the generating curves t_α and t_β. Moreover, M_1 is flat if and only if either the generating curve t_α or t_β is an asymptotic curve on M_1.

Proof. By using Theorems 3.1 and 3.7, we obtain the Gaussian curvature as $K = -\frac{\kappa_{n_1}\kappa_{n_2}}{\sinh^2\theta}$ and the mean curvature as $H = \frac{\kappa_{n_1}+\kappa_{n_2}}{2\sinh^2\theta}$. Thus, $K = 0$ if and only if either $\kappa_{n_1} = 0$ or $\kappa_{n_2} = 0$. □

Theorem 3.8. *The generating curve t_α is a geodesic curve on*
(i) the timelike translation surface M_1 if and only if $\tanh\phi = -\frac{\kappa_\alpha}{\tau_\alpha}$,
(ii) the spacelike translation surface M_1 if and only if $\coth\phi = \frac{\kappa_\alpha}{\tau_\alpha}$.

Proof. Let M_1 be a timelike translation surface. Then, the geodesic curvature of the tangent indicatrix t_α is given by [21],
$\kappa_g = \frac{1}{\kappa_\alpha^2}\langle t''_\alpha, N \times \frac{t'_\alpha}{|t'_\alpha|}\rangle = \frac{1}{\kappa_\alpha^2}\langle t''_\alpha, N\times n_\alpha\rangle = \frac{1}{\kappa_\alpha^2}\langle \kappa_\alpha^2 t_\alpha + \kappa'_\alpha n_\alpha + \kappa_\alpha\tau_\alpha b_\alpha, -\sinh\phi\, b_\alpha + \cosh\phi\, t_\alpha\rangle = \frac{1}{\kappa_\alpha^2}(-\kappa_\alpha^2\cosh\phi - \kappa_\alpha\tau_\alpha\sinh\phi) = -\cosh\phi - \frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi$.
Now, we know that t_α is geodesics if and only if the geodesic curvature of t_α is zero. Thus, t_α is a geodesic curve on M_1 iff $\langle t''_\alpha, N\times n_\alpha\rangle = 0$, which implies $\tanh\phi = -\frac{\kappa_\alpha}{\tau_\alpha}$. Similarly, we can prove for the spacelike translation surface. □

Theorem 3.9. *The generating curve t_α is a line of curvature on M_1 if and only if the angle between N and t_α is constant along t_α.*

Proof. The geodesic torsion of the curve t_α is given by [21], $\tau_g = -\frac{1}{\kappa_\alpha}\langle N_u, N\times n_\alpha\rangle$
$= -\frac{1}{\kappa_\alpha}\langle -\phi_u\cosh\phi\, t_\alpha - \kappa_\alpha\sinh\phi\, n_\alpha + \phi_u\sinh\phi\, b_\alpha - \tau_\alpha\cosh\phi\, n_\alpha, -\sinh\phi\, b_\alpha + \cosh\phi\, t_\alpha\rangle$
$= -\frac{1}{\kappa_\alpha}(\cosh^2\phi - \sinh^2\phi)\phi_u = -\frac{\phi_u}{\kappa_\alpha}$.
We know that the curve t_α is a line of curvature on M_1 iff $\tau_g = 0$, so the curve t_α is a line of curvature on M_1 iff $\phi_u = 0$, i.e., the angle between N and t_α is constant along t_α. □

Proposition 3.1. [16] *A constant angle surface in Minkowski space is a flat surface.*

Theorem 3.10. *Let M_1 be a constant angle timelike translation surface generated by tangent indicatrices t_α and t_β of the space curves α and β, and N be the unit normal of M_1 such that the angle between N and t_α is not constant. If $\langle N, U_\alpha \rangle = a$ $(const.)$, where U_α is a fixed unit vector called the axis of the surface, then the axis U_α is given by*

$$U_\alpha = -a \sinh\phi \, t_\alpha + a \cosh\phi \, b_\alpha - a\tau_g \cosh\phi \, n_\alpha,$$

which is a spacelike vector.

Proof. Let M_1 be a constant angle timelike surface. The unit normal to the surface M_1 is given by $N(u, v) = -\sinh\phi \, t_\alpha + \cosh\phi \, b_\alpha$. Then, $\langle N, U_\alpha \rangle = a$, which implies

$$-\sinh\phi \, \langle t_\alpha, U_\alpha \rangle + \cosh\phi \, \langle b_\alpha, U_\alpha \rangle = a. \tag{3.10}$$

Now, by Proposition 3.1, M_1 is a flat surface, so without loss of generality, we can assume by Corollary 3.1 that $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$, which implies

$$\frac{a\cosh\phi}{\kappa_\alpha} = -\frac{a\sinh\phi}{\tau_\alpha}. \tag{3.11}$$

Differentiating the equation $\langle N, U_\alpha \rangle$ = a with respect to u, we get $\langle N_u, U_\alpha \rangle$ = 0 and differentiating N with respect to u, we obtain

$$N_u = -\phi_u \cosh\phi \, t_\alpha - (\kappa_\alpha \sinh\phi + \tau_\alpha \cosh\phi) \, n_\alpha + \phi_u \sinh\phi \, b_\alpha.$$

By using (3.11) in above the equation, we get $N_u = -\phi_u \cosh\phi \, t_\alpha + \phi_u \sinh\phi \, b_\alpha$. Thus, $\langle N_u, U_\alpha \rangle = 0$ implies $\phi_u(-\cosh\phi \, \langle t_\alpha, U_\alpha \rangle + \sinh\phi \, \langle b_\alpha, U_\alpha \rangle) = 0$, and since $\phi_u \neq 0$, we get

$$-\cosh\phi \, \langle t_\alpha, U_\alpha \rangle + \sinh\phi \, \langle b_\alpha, U_\alpha \rangle = 0. \tag{3.12}$$

By using (3.10) and (3.12), we obtain $\langle t_\alpha, U_\alpha \rangle = a\sinh\phi$ and $\langle b_\alpha, U_\alpha \rangle = a\cosh\phi$.
Thus, U_α can be written as $U_\alpha = -a\sinh\phi \, t_\alpha + a\cosh\phi \, b_\alpha + \langle n_\alpha, U_\alpha \rangle \, n_\alpha$ and differentiating U_α with respect to u, we get

$$U'_\alpha = (-a\phi_u \cosh\phi + \kappa_\alpha \, \langle n_\alpha, U_\alpha \rangle) \, t_\alpha + (a\phi_u \sinh\phi + \tau_\alpha \, \langle n_\alpha, U_\alpha \rangle) \, b_\alpha.$$

Now, since U_α is a constant unit vector $U'_\alpha = 0$. So, we get $-a\phi_u \cosh\phi + \kappa_\alpha \, \langle n_\alpha, U_\alpha \rangle = 0$ and $a\phi_u \sinh\phi + \tau_\alpha \, \langle n_\alpha, U_\alpha \rangle = 0$, which together imply $\langle n_\alpha, U_\alpha \rangle = \frac{a\phi_u \cosh\phi}{\kappa_\alpha} = -\frac{a\phi_u \sinh\phi}{\tau_\alpha}$.
Hence, we obtain the axis U_α as follows

$$U_\alpha = -a\sinh\phi \, t_\alpha + a\cosh\phi \, b_\alpha + \frac{a\phi_u \cosh\phi}{\kappa_\alpha} \, n_\alpha = -a\sinh\phi \, t_\alpha + a\cosh\phi \, b_\alpha - \frac{a\phi_u \sinh\phi}{\tau_\alpha} \, n_\alpha,$$

Now, from Theorem 3.9, we have $\tau_g = -\frac{\phi_u}{\kappa_\alpha} \implies \phi_u = -\tau_g \kappa_\alpha$. Thus,

$$U_\alpha = -a\sinh\phi \, t_\alpha + a\cosh\phi \, b_\alpha - a\tau_g \cosh\phi \, n_\alpha. \tag{3.13}$$

Now, $\langle U_\alpha, U_\alpha \rangle = a^2 + a^2\tau_g^2 \cosh^2\phi > 0$ implies that the axis of timelike constant angle translation surface M_1 is a spacelike vector. Similarly, we can obtain the result for the spacelike translation surface. □

Corollary 3.6. *Let M_1 be a constant angle timelike translation surface generated by tangent indicatrices of the space curves α and β with the spacelike unit vector U_α as the axis of the surface, and N be the unit normal of the surface such that the angle between N and t_α is not constant. Then, either the curve α or the curve β is a slant helix with the same axis.*

Proof. Let M_1 be the constant angle timelike translation surface generated by tangent indicatrices of the space curves α and β. Then, since the surface is flat by Proposition 3.1, and by using Corollary 3.1 and Theorem 3.10, we get $a^2 + a^2\tau_g^2 \cosh^2\phi = 1$. Thus, $\tau_g \cosh\phi = \pm\sqrt{\frac{1-a^2}{a^2}}$= constant, which implies $\langle n_\alpha, U_\alpha \rangle = -a\tau_g \cosh\phi$ = constant. Hence, the curve α is a slant helix. Similarly, when we assume that $\tanh\psi = -\frac{\tau_\beta}{\kappa_\beta}$, then we find that β is a slant helix. □

Example 3.1. Let α and β be two timelike space curves in Minkowski 3-space E_1^3 given by $\alpha(s) = \frac{1}{6}(s^3 + 6s, 3s^2, s^3)$ and $\beta(t) = (\sqrt{2}\sinh t, 1 - \cosh t, \sinh t)$, where α and β are curves given by the arc-length parameters s and t, respectively. The tangent indicatrices of the curve α and β are given by $t_\alpha(s) = (\frac{s^2}{2} + 1, s, \frac{s^2}{2})$ and $t_\beta(t) = (\sqrt{2}\cosh t, -\sinh t, \cosh t)$.
The translation surface generated by the tangent indicatrices is given by
$M_1 : X(s,t) = t_\alpha(s) + t_\beta(t) = (\frac{s^2}{2} + \sqrt{2}\cosh t + 1, s - \sinh t, \frac{s^2}{2} + \cosh t)$.

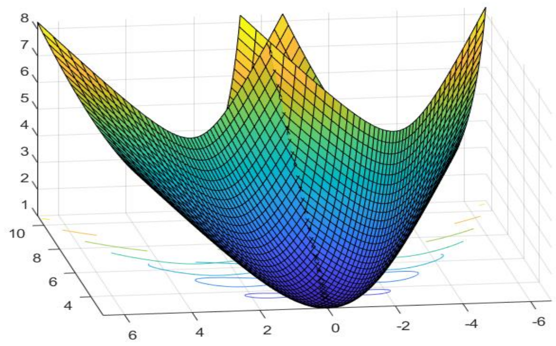

Figure 1. Translation surface generated by tangent indicatrices of timelike curves.

3.2. Translation Surfaces generated by principal normal indicatrices of two timelike space curves in E_1^3

Let α, β be timelike curves in E_1^3. Translation surface generated by principal normal indicatrices of the space curves α, β in E_1^3 is defined by

$$M_2 : X(u,v) = n_\alpha(u) + n_\beta(v), \tag{3.14}$$

where n_α and n_β are both spacelike vectors.
Differentiating the above equation (3.14) with respect to u and v, we obtain $X_u = n'_\alpha,\ X_v = n'_\beta$. By using Frenet equations (2.3) for the curves α and β, we get

$$X_u = \kappa_\alpha t_\alpha + \tau_\alpha b_\alpha,\ X_v = \kappa_\beta t_\beta + \tau_\beta b_\beta, \tag{3.15}$$

where $\kappa_\alpha,\ \tau_\alpha$ and $\kappa_\beta,\ \tau_\beta$ are curvatures and torsions of the curves α and β, respectively.
Now, the coefficients of first fundamental form are obtained as follows

$$E = \tau_\alpha^2 - \kappa_\alpha^2,\ F = \kappa_\alpha\kappa_\beta(\lambda_1 + \frac{\tau_\beta}{\kappa_\alpha}\lambda_3) + \tau_\alpha\kappa_\beta(\lambda_7 + \frac{\tau_\beta}{\kappa\beta}\lambda_9),\ G = \tau_\beta^2 - \kappa_\beta^2, \tag{3.16}$$

where

$$\begin{cases} \lambda_1 = \langle t_\alpha, t_\beta\rangle, \lambda_2 = \langle t_\alpha, n_\beta\rangle, \lambda_3 = \langle t_\alpha, b_\beta\rangle, \\ \lambda_4 = \langle n_\alpha, t_\beta\rangle, \lambda_5 = \langle n_\alpha, n_\beta\rangle, \lambda_6 = \langle n_\alpha, b_\beta\rangle, \\ \lambda_7 = \langle b_\alpha, t_\beta\rangle, \lambda_8 = \langle b_\alpha, n_\beta\rangle, \lambda_9 = \langle b_\alpha, b_\beta\rangle. \end{cases} \tag{3.17}$$

Thus, the Frenet vector fields of the curve α can be expressed as linear combination of $\{t_\beta, n_\beta, b_\beta\}$ as follows

$$\begin{cases} t_\alpha = -\lambda_1 t_\beta + \lambda_2 n_\beta + \lambda_3 b_\beta, \\ n_\alpha = -\lambda_4 t_\beta + \lambda_5 n_\beta + \lambda_6 b_\beta, \\ b_\alpha = -\lambda_7 t_\beta + \lambda_8 n_\beta + \lambda_9 b_\beta. \end{cases} \tag{3.18}$$

Similarly, the Frenet vector fields of the curve β can be expressed as linear combination of $\{t_\alpha, n_\alpha, b_\alpha\}$ as follows

$$\begin{cases} t_\beta = -\lambda_1 t_\alpha + \lambda_4 n_\alpha + \lambda_7 b_\alpha, \\ n_\beta = -\lambda_2 t_\alpha + \lambda_5 n_\alpha + \lambda_8 b_\alpha, \\ b_\beta = -\lambda_3 t_\alpha + \lambda_6 n_\alpha + \lambda_9 b_\alpha. \end{cases} \tag{3.19}$$

Now, the unit normal vector N of the translation surface M_2 is given as

$$N(u,v)=\frac{\kappa_\alpha\kappa_\beta(t_\alpha\times t_\beta)+\kappa_\alpha\tau_\beta(t_\alpha\times b_\beta)+\tau_\alpha\kappa_\beta(b_\alpha\times t_\beta)+\tau_\alpha\tau_\beta(b_\alpha\times b_\beta)}{\sqrt{|EG-F^2|}}, \tag{3.20}$$

where

$$EG-F^2=(\tau_\alpha^2-\kappa_\alpha^2)(\tau_\beta^2-\kappa_\beta^2)-\big[\kappa_\alpha\kappa_\beta(\lambda_1+\frac{\tau_\beta}{\kappa_\alpha}\lambda_3)+\tau_\alpha\kappa_\beta\big(\lambda_7+\frac{\tau_\beta}{\kappa_\beta}\lambda_9)\big]^2.$$

Thus, by using (2.2), (3.19) and (3.20), the unit normal N can be written as

$$N_1(u,v)=\frac{\kappa_\alpha\kappa_\beta\big[\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_4+\frac{\tau_\beta}{\kappa_\beta}\lambda_6)t_\alpha+\big\{(-\lambda_7-\frac{\tau_\beta}{\kappa_\beta}\lambda_9)+\frac{\tau_\alpha}{\kappa_\alpha}(-\lambda_1-\frac{\tau_\beta}{\kappa_\beta}\lambda_3)\big\}n_\alpha+(\lambda_4+\frac{\tau_\beta}{\kappa_\beta}\lambda_6)b_\alpha\big]}{\sqrt{|EG-F^2|}}.$$

Also by using (2.2), (3.18) and (3.20), the unit normal N can be written as

$$N_2(u,v)=\frac{\kappa_\alpha\kappa_\beta\big[-\frac{\tau_\beta}{\kappa_\beta}(\lambda_2+\frac{\tau_\alpha}{\kappa_\alpha}\lambda_8)t_\beta+\big\{(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7)\big\}n_\beta-(\lambda_2+\frac{\tau_\alpha}{\kappa_\alpha}\lambda_8)b_\beta\big]}{\sqrt{|EG-F^2|}}.$$

The coefficients of second fundamental form of the translation surface M_2 are obtained as follows

$$\begin{cases} l=-\frac{\kappa_\alpha\kappa_\beta}{\sqrt{|EG-F^2|}}\big[\kappa_\alpha(\frac{\tau_\alpha}{\kappa_\alpha})'(\lambda_4+\lambda_6\frac{\tau_\beta}{\kappa_\beta})-(\kappa_\alpha^2-\tau_\alpha^2)\{(\lambda_7+\frac{\tau_\beta}{\kappa_\beta}\lambda_9)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_1+\frac{\tau_\beta}{\kappa_\beta}\lambda_3)\}\big],\\ m=0,\\ n=\frac{\kappa_\alpha\kappa_\beta}{\sqrt{|EG-F^2|}}\big[-\kappa_\beta(\frac{\tau_\beta}{\kappa_\beta})'(\lambda_2+\lambda_8\frac{\tau_\alpha}{\kappa_\alpha})+(\kappa_\beta^2-\tau_\beta^2)\{(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7)\}\big]. \end{cases} \tag{3.21}$$

Now, using the equations (2.5), (2.6), and above calculations, we obtain the following results.

Theorem 3.11. *The Gaussian curvature K and the mean curvature H of the translation surface M_2 are obtained as follows, respectively,*

$$\begin{aligned} K=\frac{\kappa_\alpha^2\kappa_\beta^2}{(EG-F^2)^2}\Bigg[&\Big(\kappa_\alpha(\frac{\tau_\alpha}{\kappa_\alpha})'(\lambda_4+\lambda_6\frac{\tau_\beta}{\kappa_\beta})-(\kappa_\alpha^2-\tau_\alpha^2)\{(\lambda_7+\frac{\tau_\beta}{\kappa_\beta}\lambda_9)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_1+\frac{\tau_\beta}{\kappa_\beta}\lambda_3)\}\Big)\\ &\times\Big(-\kappa_\beta(\frac{\tau_\beta}{\kappa_\beta})'(\lambda_2+\lambda_8\frac{\tau_\alpha}{\kappa_\alpha})+(\kappa_\beta^2-\tau_\beta^2)\{(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7)\}\Big)\Bigg], \end{aligned}$$

$$\begin{aligned} H=&\frac{\kappa_\alpha\kappa_\beta}{2(\sqrt{|EG-F^2|})^3}\Bigg[(\tau_\beta^2-\kappa_\beta^2)\Big(\kappa_\alpha(\frac{\tau_\alpha}{\kappa_\alpha})'(\lambda_4+\lambda_6\frac{\tau_\beta}{\kappa_\beta})-(\kappa_\alpha^2-\tau_\alpha^2)\{(\lambda_7+\frac{\tau_\beta}{\kappa_\beta}\lambda_9)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_1+\frac{\tau_\beta}{\kappa_\beta}\lambda_3)\}\Big)\\ &+(\tau_\alpha^2-\kappa_\alpha^2)\Big(-\kappa_\beta(\frac{\tau_\beta}{\kappa_\beta})'(\lambda_2+\lambda_8\frac{\tau_\alpha}{\kappa_\alpha})+(\kappa_\beta^2-\tau_\beta^2)\{(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7)\}\Big)\Bigg]. \end{aligned}$$

Theorem 3.12. *Let the translation surface M_2 be flat. If the curves α and β are planar curves, then either t_α and b_β are orthogonal or t_β and b_α are orthogonal.*

Proof. Let the surface M_2 be flat, then $K=0$. Also the curves α and β are planar curves then $\tau_\alpha=\tau_\beta=0$, which implies $\lambda_3=0$ or $\lambda_7=0$. Thus, either $\langle t_\alpha,b_\beta\rangle=0$ or $\langle t_\beta,b_\alpha\rangle=0$. Hence, either t_α and b_β are orthogonal or t_β and b_α are orthogonal. □

Theorem 3.13. *Let the translation surface M_2 be minimal. Then the following occurs,*

$$\begin{aligned} (\tau_\beta^2-\kappa_\beta^2)&\Big(\kappa_\alpha(\frac{\tau_\alpha}{\kappa_\alpha})'(\lambda_4+\lambda_6\frac{\tau_\beta}{\kappa_\beta})-(\kappa_\alpha^2-\tau_\alpha^2)\{(\lambda_7+\frac{\tau_\beta}{\kappa_\beta}\lambda_9)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_1+\frac{\tau_\beta}{\kappa_\beta}\lambda_3)\}\Big)\\ &=(\tau_\alpha^2-\kappa_\alpha^2)\Big(\kappa_\beta(\frac{\tau_\beta}{\kappa_\beta})'(\lambda_2+\lambda_8\frac{\tau_\alpha}{\kappa_\alpha})-(\kappa_\beta^2-\tau_\beta^2)\{(\lambda_3+\frac{\tau_\beta}{\kappa_\beta}\lambda_1)+\frac{\tau_\alpha}{\kappa_\alpha}(\lambda_9+\frac{\tau_\beta}{\kappa_\beta}\lambda_7)\}\Big). \end{aligned}$$

Theorem 3.14. *If M_2 is minimal and the curves α and β are planar, then the hyperbolic angle between t_α and b_β is same as the hyperbolic angle between t_β and b_α.*

Proof. Suppose the curves α and β are planar, then $\tau_\alpha = 0 = \tau_\beta$. Now, by Theorem 3.11, we have, $\lambda_7 = \lambda_3,\ i.e.,\ \langle t_\alpha, b_\beta\rangle = \langle t_\beta, b_\alpha\rangle$. □

Example 3.2. Let α and β be two timelike curves in Minkowski 3-space E_1^3 given by
$\alpha(s) = (\frac{2s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}})$ and $\beta(t) = (\sqrt{2}\sinh t, 1-\cosh t, \sinh t)$,
where s and t are arc-length parameters of the curves α and β, respectively. The principal normal indicatrices of the curve α and β are given by
$n_\alpha(s) = (0, -\cos\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}})$ and $n_\beta(t) = (\sqrt{2}\sinh t, -\cosh t, \sinh t)$.
The translation surface generated by the principal normal indicatrices is given by
$M_2 : X(s,t) = n_\alpha(s) + n_\beta(t) = (\sqrt{2}\sinh t, -\cos\frac{s}{\sqrt{3}} - \cosh t, -\sin\frac{s}{\sqrt{3}} + \sinh t)$.

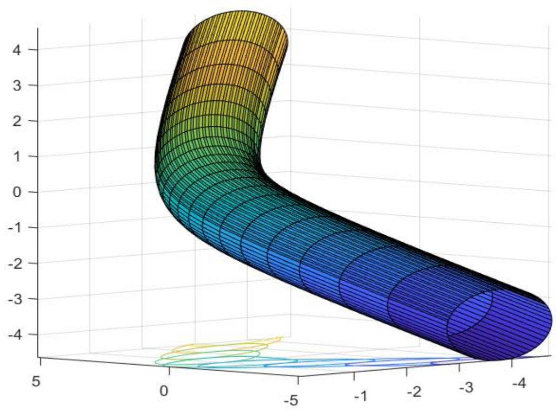

Figure 2. Translation surface generated by Principal normal indicatrices of timelike curves.

3.3. *Translation Surfaces generated by binormal indicatrices of two timelike space curves in E_1^3*

Let α, β be timelike space curves in E_1^3. Translation surface generated by binormal indicatrices of the space curves α, β in E_1^3 is defined by

$$M_3 : X(u,v) = b_\alpha(u) + b_\beta(v), \tag{3.22}$$

since α and β are both timelike curves, thus b_α and b_β are spacelike vectors.
Differentiating the above equation (3.22) with respect to u and v, we obtain $X_u = b'_\alpha,\ X_v = b'_\beta$. Now, using Frenet equations (2.3) for the curves α and β, we get

$$X_u = -\tau_\alpha n_\alpha,\ X_v = -\tau_\beta n_\beta, \tag{3.23}$$

where τ_α and τ_β are the torsions of the curves α and β, respectively. The unit normal vector N of the surface M_3 is given by

$$N(u,v) = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \tag{3.24}$$

where $X_u \times X_v = \tau_\alpha\tau_\beta(n_\alpha \times n_\beta)$ and $\|X_u \times X_v\| = \sqrt{-\epsilon(EG - F^2)},\ \epsilon = \langle N, N\rangle$.
Now, we have two cases according to causality of the plane spanned by n_α and n_β.
Case (i) We suppose that, n_α and n_β span timelike plane. Thus, by Definition 2.1(b), we have $\langle n_\alpha, n_\beta\rangle = \cosh\theta$ and the coefficients of first fundamental form are obtained as follows

$$E = \tau_\alpha^2,\ F = \tau_\alpha\tau_\beta\cosh\theta,\ G = \tau_\beta^2, \tag{3.25}$$

where θ is the smooth hyperbolic angle function between n_α and n_β.
In this case, $EG - F^2 = -\tau_\alpha^2\tau_\beta^2\sinh^2\theta < 0$ shows that the surface M_3 is timelike, and hence, the unit normal N is spacelike, i.e., $\langle N, N\rangle = 1$. Thus, we obtain the spacelike unit normal as

$$N(u,v) = \frac{n_\alpha \times n_\beta}{\sinh\theta}, \tag{3.26}$$

so that $\langle N, n_\alpha\rangle = \langle N, n_\beta\rangle = 0$.
Now, suppose the hyperbolic angle between t_α and N is ϕ and between t_β and N is ψ, then N can be expressed as follows [13]:

$$N(u,v) = -\sinh\phi\, t_\alpha + \cosh\phi\, b_\alpha,$$
$$N(u,v) = -\sinh\psi\, t_\beta + \cosh\psi\, b_\beta.$$

The coefficients of second fundamental form of the timelike surface M_3 are given by

$$\begin{cases} l = -\tau_\alpha\kappa_\alpha(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi), \\ m = 0, \\ n = -\tau_\beta\kappa_\beta(\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi). \end{cases} \tag{3.27}$$

Now, using (2.5), (2.6), and above calculations, we obtain the following results.

Theorem 3.15. *The Gaussian curvature K and the mean curvature H of the timelike translation surface M_3 are found as follows, respectively,*

$$K = -\frac{\kappa_\alpha\kappa_\beta(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi)(\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi)}{\tau_\alpha\tau_\beta\sinh^2\theta},$$

$$H = -\frac{\kappa_\alpha\kappa_\beta\left[\frac{\tau_\beta}{\kappa_\beta}(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi) + \frac{\tau_\alpha}{\kappa_\alpha}(\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi)\right]}{2\tau_\alpha\tau_\beta\sinh^2\theta}.$$

Corollary 3.7. *The timelike translation surface M_3 is flat if and only if*

$$\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha} \;\; or \;\; \tanh\psi = -\frac{\tau_\beta}{\kappa_\beta}.$$

Proof. By putting $K = 0$ in Theorem 3.15, we get the desired result. □

Corollary 3.8. *Let the timelike translation surface M_3 be minimal. Then,*

$$\frac{\tau_\beta}{\kappa_\beta}(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi) = -\frac{\tau_\alpha}{\kappa_\alpha}(\sinh\psi + \frac{\tau_\beta}{\kappa_\beta}\cosh\psi).$$

Proof. By putting $H = 0$ in Theorem 3.15, we get the stated result. □

Case (ii) We assume that, n_α and n_β span spacelike plane. Thus, by Definition 2.1(a), $\langle n_\alpha, n_\beta\rangle = \cos\theta$ and the coefficients of first fundamental form are obtained as follows

$$E = \tau_\alpha^2,\; F = \tau_\alpha\tau_\beta\cos\theta,\; G = \tau_\beta^2, \tag{3.28}$$

where θ is the smooth angle function between n_α and n_β.
In this case, $EG - F^2 = \tau_\alpha^2\tau_\beta^2\sin^2\theta > 0$ shows that the surface M_3 is spacelike, and hence, the unit normal N is timelike, i.e., $\langle N, N\rangle = -1$. Thus, we obtain the timelike unit normal as

$$N(u,v) = \frac{n_\alpha \times n_\beta}{\sin\theta}, \tag{3.29}$$

so that $\langle N, n_\alpha\rangle = \langle N, n_\beta\rangle = 0$.
Now, suppose the hyperbolic angle between t_α and N is ϕ and between t_β and N is ψ, then N can be expressed as follows [13] :

$$N(u,v) = \cosh\phi\, t_\alpha + \sinh\phi\, b_\alpha,$$
$$N(u,v) = \cosh\psi\, t_\beta + \sinh\psi\, b_\beta.$$

The coefficients of second fundamental form of the spacelike surface M_3 are given by

$$\begin{cases} l = \tau_\alpha\kappa_\alpha(\cosh\phi - \frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi), \\ m = 0, \\ n = \tau_\beta\kappa_\beta(\cosh\psi - \frac{\tau_\beta}{\kappa_\beta}\sinh\psi). \end{cases} \tag{3.30}$$

Now, using (2.5), (2.6), and above calculations, we obtain the following results.

Theorem 3.16. *The Gaussian curvature K and the mean curvature H of the spacelike translation surface M_3 are found as follows, respectively,*

$$K = -\frac{\kappa_\alpha \kappa_\beta (\cosh\phi - \frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi)(\cosh\psi - \frac{\tau_\beta}{\kappa_\beta}\sinh\psi)}{\tau_\alpha \tau_\beta \sin^2\theta},$$

$$H = \frac{\kappa_\alpha \kappa_\beta \left[\frac{\tau_\beta}{\kappa_\beta}(\cosh\phi - \frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi) + \frac{\tau_\alpha}{\kappa_\alpha}(\cosh\psi - \frac{\tau_\beta}{\kappa_\beta}\sinh\psi)\right]}{2\tau_\alpha \tau_\beta \sin^2\theta}.$$

Corollary 3.9. *The spacelike translation surface M_3 is flat if and only if*

$$\coth\phi = \frac{\tau_\alpha}{\kappa_\alpha} \ \ or \ \ \coth\psi = \frac{\tau_\beta}{\kappa_\beta}.$$

Proof. By putting $K = 0$ in Theorem 3.16, we get the desired result. □

Corollary 3.10. *Let the spacelike translation surface M_3 be minimal. Then,*

$$\frac{\tau_\beta}{\kappa_\beta}(\cosh\phi - \frac{\tau_\alpha}{\kappa_\alpha}\sinh\phi) = -\frac{\tau_\alpha}{\kappa_\alpha}(\cosh\psi - \frac{\tau_\beta}{\kappa_\beta}\sinh\psi).$$

Proof. By putting $H = 0$ in Theorem 3.16, we get the stated result. □

Theorem 3.17. *The generating curve b_α is an asymptotic curve on*
(i) the timelike translation surface M_3 iff $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$,
(ii) the spacelike translation surface M_3 iff $\coth\phi = \frac{\tau_\alpha}{\kappa_\alpha}, \phi \neq 0$.

Proof. Following similar steps as the Theorem 3.7, we find that the normal curvature of the curve b_α on timelike translation surface M_3 is given by $\kappa_n = -\frac{\kappa_\alpha}{\tau_\alpha}(\sinh\phi + \frac{\tau_\alpha}{\kappa_\alpha}\cosh\phi)$. Hence, b_α is asymptotic on M_3 iff $\tanh\phi = -\frac{\tau_\alpha}{\kappa_\alpha}$. Similarly, we can prove for the spacelike surface. □

Corollary 3.11. *The Gaussian and mean curvatures of the translation surface M_3 are given as*

$$K = -\frac{\kappa_{n_1}\kappa_{n_2}}{\sinh^2\theta}, \ H = \frac{\kappa_{n_1} + \kappa_{n_2}}{2\sinh^2\theta},$$

respectively, where κ_{n_1} and κ_{n_2} are normal curvatures of the generating curves b_α and b_β. Moreover, M_3 is flat if and only if either the curve b_α or b_β is an asymptotic curve on M_3.

Proof. Proof is same as in the proof of Corollary 3.5. □

Theorem 3.18. *The generating curve b_α is a geodesic curve on*
(i) the timelike translation surface M_3 iff $\tanh\phi = -\frac{\kappa_\alpha}{\tau_\alpha}$,
(ii) the spacelike translation surface M_3 iff $\coth\phi = \frac{\kappa_\alpha}{\tau_\alpha}, \phi \neq 0$.

Proof. By similar steps as the Theorem 3.8, we obtain the geodesic curvature of the curve b_α on timelike translation surface M_3 as $\kappa_g = -(\sinh\phi + \frac{\kappa_\alpha}{\tau_\alpha}\cosh\phi)$. Hence, b_α is a geodesic curve on M_3 iff $\kappa_g = 0$ iff $\tanh\phi = -\frac{\kappa_\alpha}{\tau_\alpha}$. Similarly, we can prove for the spacelike surface. □

Theorem 3.19. *The generating curve b_α is a line of curvature on timelike (spacelike) translation surface M_3 if and only if the angle between N and t_α is constant along b_α.*

Proof. Following the steps of the Theorem 3.9, we get that the geodesic curvature of b_α as $\tau_g = \frac{\phi_u}{\tau_\alpha}$. We know that the curve b_α is a line of curvature on M_3 iff $\tau_g = 0$, so the curve b_α is a line of curvature on M_3 iff $\phi_u = 0$, i.e., the angle between N and t_α is constant along b_α. □

Theorem 3.20. *Let M_3 be a constant angle timelike translation surface generated by binormal indicatrices b_α and b_β of the space curves α and β, and N be the unit normal of M_3 such that the angle between N and t_α is not constant. If $\langle N, U_\alpha\rangle = a$ $(const.)$, where U_α is a fixed unit vector called the axis of the surface, then the axis U_α is given by*

$$U_\alpha = -a\sinh\phi\, t_\alpha + a\cosh\phi\, b_\alpha - a\tau_g\sinh\phi\, n_\alpha,$$

which is a spacelike vector.

Proof. After following similar steps to that of Theorem 3.10, we obtain the axis U_α as follows

$$U_\alpha = -a\sinh\phi\, t_\alpha + a\cosh\phi\, b_\alpha + \frac{a\phi_u\cosh\phi}{\kappa_\alpha}\, n_\alpha = -a\sinh\phi\, t_\alpha + a\cosh\phi\, b_\alpha - \frac{a\phi_u\sinh\phi}{\tau_\alpha}\, n_\alpha.$$

Now, by Theorem 3.19, we have $\tau_g = \frac{\phi_u}{\tau_\alpha} \implies \phi_u = \tau_\alpha\tau_g$. So,

$$U_\alpha = -a\sinh\phi\, t_\alpha + a\cosh\phi\, b_\alpha - a\tau_g\sinh\phi\, n_\alpha.$$

Now, $\langle U_\alpha, U_\alpha\rangle = a^2 + a^2\tau_g^2\sinh^2\phi > 0$ implies that the axis of the translation surface M_3 is a spacelike vector. Similarly, we can obtain the result for the spacelike translation surface. □

Corollary 3.12. *Let M_3 be a constant angle timelike translation surface generated by binormal indicatrices of the space curves α and β with the spacelike unit vector U_α as the axis of the surface and N be the unit normal of the surface such that the angle between N and t_α is not constant. Then, either the curve α or the curve β is a slant helix with the same axis.*

Proof. The proof is same as in the proof of Corollary 3.6. □

Example 3.3. Let α and β be two timelike curves in Minkowski 3-space E_1^3 given by
$\alpha(s) = (\frac{2s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}})$ and $\beta(t) = (\sqrt{2}\sinh t, \sqrt{2}\cosh t, t)$,
where α and β are curves given by the arc-length parameters s and t, respectively. The binormal indicatrices of the curve α and β are given by
$b_\alpha(s) = (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin\frac{s}{\sqrt{3}}, \frac{2}{\sqrt{3}}\cos\frac{s}{\sqrt{3}})$, and $b_\beta(t) = (-\cosh t, -\sinh t, -\sqrt{2})$.
The translation surface generated by the binormal indicatrices is given by
$M_3 : X(s,t) = b_\alpha(s) + b_\beta(t) = (\frac{1}{\sqrt{3}} - \cosh t, -\frac{2}{\sqrt{3}}\sin\frac{s}{\sqrt{3}} - \sinh t, \frac{2}{\sqrt{3}}\cos\frac{s}{\sqrt{3}} - \sqrt{2})$.

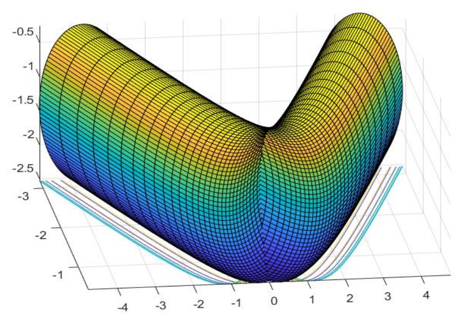

Figure 3. Translation surface generated by binormal indicatrices of timelike curves.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

This work is supported by Council of Scientific & Industrial Research, Government of India [09/013(0953)/2020-EMR-I].

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Ali, A. T.: *Non-lightlike constant angle ruled surfaces in Minkowski 3-space*. Journal of Geometry and Physics. **157** (1-2), 103833 (2020). https://doi.org/10.1016/j.geomphys.2020.103833

[2] Ali, A. T., Abdel-Aziz, H. S., Sorour, A. H.: *On curvatures and points of the translation surfaces in Euclidean 3-space*. J. Egyptian Math. Soc. **23** (1), 167-172 (2015). https://doi.org/10.1016/j.joems.2014.02.007

[3] Arfah, A.: *On Causal Characterization of Spherical Indicatrices of Timelike Curves in Minkowski 3-Space*. Hagia Safia J. of Geometry. **2** (2), 26-37 (2020).

[4] Acar, N., Aksoyak, F. K.: *Some Characterizations of translation surfaces generated by spherical indicatrices of space curves*. Palestine Journal of Mathematics. **11** (1), 456-468 (2022).

[5] Baba-Hamed, Ch., Mohammed, B., Zoubir, H.: *Translation Surfaces in the Three-Dimensional Lorentz-Minkowski Space Satisfying* $\Delta r_i = \lambda_i r_i$. Int. J. Math. Anal. **4** (17), 797-808 (2010).

[6] Barros, M.: *General helices and a theorem of Lancret*. Proc. American Mathematical Soc. **125** (5), 1503-1509 (1997).

[7] Çetin, M., Kocayiğit, H., Önder, M.: *Translation surfaces according to Frenet frame in Minkowski 3-space*. International Journal of Physical Sciences. **7** (47), 6135-6143 (2012).

[8] Çetin, M., Tunçer, Y.: *Parallel surfaces to translation surfaces in Euclidean 3- space*. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics. **64** (2), 47-54 (2015).

[9] Couto, I. T., Lymberopoulas, A.: Introduction to Lorentz Geometry, curves and surfaces. CRC Press. Florida (2021). https://doi.org/10.1201/9781003031574-1

[10] do Carmo, M. P.: Differential geometry of curves and surfaces. Prentice-Hall Inc. New Jersey (1976).

[11] Fu, Y., Yang, D.: *On constant slope Spacelike surfaces in 3-dimensional Minkowski space*. Journal of Mathematical Analysis and Application. **385** (1), 208-220 (2012). https://doi.org/10.1016/j.jmaa.2011.06.040

[12] Izumiya, S., Takeuchi, N.: *New special curves and developable surfaces*. Turk. J. Math. **28** (2), 531-537 (2004).

[13] Kılıç, O., Çalışkan, A.: *The Frenet and Darboux Instantaneous Rotation Vectors for curves and spacelike Surfaces*. Mathematical and Computational Applications. **1** (2), 77-86 (1996).

[14] Kühnel, W.: Differential Geometry: Curves - Surfaces - Manifolds. American Math. Soc. Rhode Island (2015).

[15] Liu, H.: *Translation surfaces with constant mean curvature in 3-dimensional spaces*. Journal of Geometry. **64** (1-2), 141-149 (1999). https://doi.org/10.1007/BF01229219

[16] López, R., Munteanu, M. I.: *Constant angle surfaces in Minkowski space*. Bulletin of the Belgian Math. Soc. **18** (2), 271-286 (2011). https://doi.org/10.36045/bbms/1307452077

[17] Munteanu, M. I., Nistor, A. I.: *On the geometry of the second fundamental form of translation surfaces in* E^3. Houston J. Math. **37** (4), 1087-1102 (2011).

[18] O'Neill, B.: Semi-Riemannian geometry with application to relativity. Academic Press. London (1983).

[19] Pressley, A.: Elementary differential geometry. Springer. London (2001).

[20] Ratcliffe, J. G.: Foundations of Hyperbolic Manifolds. Springer. New York (2006).

[21] Ünlütürk, Y., Savcı, Ü. Z.: *On non-unit speed curves in Mikowski 3-space*. Scientia Magna. **8** (4), 66-74 (2012).

[22] Yılmaz, S., Özyılmaz, E., Turgut, M.: *New Spherical Indicatrices and their Characterizations*. Analele stiintifice ale Universitatii Ovidius Constanta. **18** (2), 337-354 (2010).

Affiliations

AKHILESH YADAV
ADDRESS: Banaras Hindu University, Dept. of Mathematics, 221005, Varanasi, India.
E-MAIL: akhilesha68@gmail.com, akhileshyadav@bhu.ac.in
ORCID ID: 0000-0003-3990-857X

AJAY KUMAR YADAV
ADDRESS: Banaras Hindu University, Dept. of Mathematics, 221005, Varanasi, India.
E-MAIL: ajaykumar74088@gmail.com, yajay@bhu.ac.in
ORCID ID: 0000-0002-9627-4596