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One–generator quasi–abelian codes revisited*

Research Article

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Abstract: The class of 1-generator quasi-abelian codes over finite fields is revisited. Alternative and explicit characterization and enumeration of such codes are given. An algorithm to find all 1-generator quasi-abelian codes is provided. Two 1-generator quasi-abelian codes whose minimum distances are improved from Grassl's online table are presented.

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1. Introduction

As a family of codes with good parameters, rich algebraic structures, and wide ranges of applications (see [8], [9], [11], [10], [13], [14] , and references therein), quasi-cyclic codes have been studied for a half-century. Quasi-abelian codes, a generalization of quasi-cyclic codes, have been introduced in [15] and extensively studied in [7].

Given finite abelian groups $H \leq G$ and a finite field $\mathbb{F}_q$, an H-*quasi-abelian code* is defined to be an $\mathbb{F}_q[H]$-submodule of $\mathbb{F}_q[G]$. Note that H-quasi-abelian codes are not only a generalization of quasi-cyclic codes (see [7], [8], [9], and [15]) if H is cyclic but also of abelian codes (see [1] and [2]) if $G = H$, and of cyclic codes (see [12]) if $G = H$ is cyclic. The characterization and enumeration of quasi-abelian codes have been established in [7]. An H-quasi-abelian code C is said to be of 1-*generator* if C is a cyclic $\mathbb{F}_q[H]$-module. Such a code can be viewed as a generalization of 1-generator quasi-cyclic codes which are more frequently studied and applied (see [11], [13], and [14]). Analogous to the case of 1-generator quasi-cyclic codes, the number of 1-generator quasi-abelian codes has been determined in [7]. However, an explicit construction and an algorithm to determine all 1-generator quasi-abelian codes have not been well studied.

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In this paper, we give an alternative discussion on the algebraic structure of 1-generator quasi-abelian codes and an algorithm to find all 1-generator quasi-abelian codes. Examples of new codes derived from 1-generator quasi-abelian codes are presented.

The paper is organized as follows. In Section 2, we recall some notations and basic results. An alternative discussion on the algebraic structure of 1-generator quasi-abelian codes is given in Section 3 together with an algorithm to find all 1-generator quasi-abelian codes and the number of such codes. Examples of new codes derived from 1-generator quasi-abelian codes are presented in Section 4.

2. Preliminaries

Let $\mathbb{F}_q$ denote a finite field of order q and let G be a finite abelian group of order n, written additively. Denote by $\mathbb{F}_q[G]$ the *group ring* of G over $\mathbb{F}_q$. The elements in $\mathbb{F}_q[G]$ will be written as $\sum_{g\in G}\alpha_g Y^g$, where $\alpha_g \in \mathbb{F}_q$. The addition and the multiplication in $\mathbb{F}_q[G]$ are given as in the usual polynomial rings over $\mathbb{F}_q$ with the indeterminate Y, where the indices are computed additively in G. We note that $Y^0 = 1$ is the identity of $\mathbb{F}_q[G]$, where 1 is the identity in $\mathbb{F}_q$ and 0 is the identity of G.

Given a ring $\mathcal{R}$, a linear code of length n over $\mathcal{R}$ refers to a submodule of the $\mathcal{R}$-module $\mathcal{R}^n$. A *linear code* in $\mathbb{F}_q[G]$ refers to an $\mathbb{F}_q$-subspace C of $\mathbb{F}_q[G]$. This can be viewed as a linear code of length n over $\mathbb{F}_q$ by indexing the n-tuples by the elements in G. The *Hamming weight* $\mathrm{wt}(\boldsymbol{u})$ of $\boldsymbol{u} = \sum_{g\in G} u_g Y^g \in \mathbb{F}_q[G]$ is defined to be the number of nonzero term u_g's in $\boldsymbol{u}$. The *minimum Hamming distance* a code C is defined by $\mathrm{d}(C) := \min\{\mathrm{wt}(\boldsymbol{u}) \mid \boldsymbol{u} \in C, \boldsymbol{u} \neq 0\}$. A linear code C in $\mathbb{F}_q[G]$ is referred to as an $[n, k, d]_q$ code if C has $\mathbb{F}_q$-dimension k and minimum Hamming distance d.

Given a subgroup H of G, a code C in $\mathbb{F}_q[G]$ is called an *H-quasi-abelian code* if C is an $\mathbb{F}_q[H]$-module, i.e., C is closed under the multiplication by the elements in $\mathbb{F}_q[H]$. Such a code will be called a *quasi-abelian code* if H is not specified or where it is clear in the context. An H-quasi-abelian code C is said to be of 1-*generator* if C is a cyclic $\mathbb{F}_q[H]$-module. Since every H-quasi-abelian code C in $\mathbb{F}_q[G]$ is an $\mathbb{F}_q[H]$-module, it is also an $\mathbb{F}_q[A]$-module for all cyclic subgroups of H. It follows that C is quasi-cyclic of index $|G|/|A|$. However, being 1-generator H-quasi-abelian does not imply that C is 1-generator quasi-cyclic. Therefore, it makes sense to study 1-generator H-quasi-abelian codes.

Assume that $H \le G$ such that $|H| = m$ and the index $[G : H] = \frac{n}{m} = l$. Let $\{\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_l\}$ be a fixed set of representatives of the cosets of H in G. Let $R := \mathbb{F}_q[H]$. Define $\Phi : \mathbb{F}_q[G] \to R^l$ by

$$\Phi\left(\sum_{h\in H}\sum_{i=1}^{l} \alpha_{h+\mathfrak{g}_i} Y^{h+\mathfrak{g}_i}\right) = (\boldsymbol{\alpha}_1(Y), \boldsymbol{\alpha}_2(Y), \dots, \boldsymbol{\alpha}_l(Y)), \tag{1}$$

where $\boldsymbol{\alpha}_i(Y) = \sum_{h\in H} \alpha_{h+\mathfrak{g}_i} Y^h \in R$, for all $i \in \{1, 2, \dots, l\}$. It is not difficult to see that Φ is an R-module isomorphism, and hence, the next lemma follows.

Lemma 2.1. *The map Φ induces a one-to-one correspondence between H-quasi-abelian codes in $\mathbb{F}_q[G]$ and linear codes of length l over R.*

Throughout, assume that $\gcd(q, |H|) = 1$, or equivalently, $\mathbb{F}_q[H]$ is semisimple. Following [7, Section 3], the group ring $R = \mathbb{F}_q[H]$ is decomposed as follows.

For each $h \in H$, denote by $\mathrm{ord}(h)$ the order of h in H. The *q-cyclotomic class* of H containing $h \in H$, denoted by $S_q(h)$, is defined to be the set

$$S_q(h) := \{q^i \cdot h \mid i = 0, 1, \dots\} = \{q^i \cdot h \mid 0 \le i \le \nu_h\},$$

where $q^i \cdot h := \sum_{j=1}^{q^i} h$ in H and ν_h is the multiplicative order of q in $\mathbb{Z}_{\mathrm{ord}(h)}$.

An *idempotent* in a ring R is a non-zero element e such that $e^2 = e$. An idempotent e is said to be *primitive* if for every other idempotent f, either $ef = e$ or $ef = 0$. The primitive idempotents in R

are induced by the q-cyclotomic classes of H (see [4, Proposition II.4]). Every idempotent e in R can be viewed as a unique sum of primitive idempotents in R. The $\mathbb{F}_q$-*dimension* of an idempotent $e \in R$ is defined to be the $\mathbb{F}_q$-dimension of Re.

From [7, Subsection 3.2], $R := \mathbb{F}_q[H]$ can be decomposed as

$$R = Re_1 + Re_2 + \cdots + Re_s,$$

where $e_1, e_2, \ldots, e_s$ are the primitive idempotents in R. Moreover, every ideal in R is of the form Re, where e is an idempotent in R.

3. 1-generator quasi-abelian codes

In [7], characterization and enumeration of 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ have been given. In this section, we give alternative characterization and enumeration of such codes. The characterization in Subsection 3.1 allows us to express an algorithm to find all 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ in Subsection 3.2.

Using the R-module isomorphism Φ defined in (1), to study 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$, it suffices to consider cyclic R-submodules $R\boldsymbol{a}$, where $\boldsymbol{a} = (a_1, a_2, \ldots, a_l) \in R^l$.

For each $\boldsymbol{a} = (a_1, a_2, \ldots, a_l) \in R^l$, there exists a unique idempotent $e \in R$ such that $Re = Ra_1 + Ra_2 + \cdots + Ra_l$. The element e is called the *idempotent generator element* for $R\boldsymbol{a}$. An idempotent $f \in R$ of largest $\mathbb{F}_q$-dimension such that

$$f\boldsymbol{a} = 0$$

is called the *idempotent check element* for $R\boldsymbol{a}$.

Let $S = \mathbb{F}_{q^l}[H]$, where $\mathbb{F}_{q^l}$ is an extension field of $\mathbb{F}_q$ of degree l. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a fixed basis of $\mathbb{F}_{q^l}$ over $\mathbb{F}_q$. Let $\varphi : R^l \to S$ be an R-module isomorphism defined by

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_l) \mapsto A = \sum_{i=1}^{l} \alpha_i a_i. \tag{2}$$

Using the map φ, the code $R\boldsymbol{a}$ can be regarded as an R-module RA in S.

Lemma 3.1 ([7, Lemma 6.1])**.** *Let $\boldsymbol{a} \in R^l$ and let e and f be the idempotent generator and idempotent check elements of $R\boldsymbol{a}$, respectively, Then*

$$e + f = 1$$

and

$$\dim_{\mathbb{F}_q}(R\boldsymbol{a}) = \dim_{\mathbb{F}_q}(Re) = m - \dim_{\mathbb{F}_q}(Rf).$$

For a ring $\mathcal{R}$, denote by $\mathcal{R}^*$ and $\mathcal{R}^\times$ the set of non-zero elements and the group of units of $\mathcal{R}$, respectively.

In order to enumerate and determine all 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$, we need the following result.

Lemma 3.2. *Let $\boldsymbol{a}, \boldsymbol{b} \in R^l$ and let e be the idempotent generator of $R\boldsymbol{a}$. Let $A = \varphi(\boldsymbol{a})$ and $B = \varphi(\boldsymbol{b})$, where φ is defined in (2). Then $R\boldsymbol{a} = R\boldsymbol{b}$ if and only if there exists $u \in (Re)^\times$ such that $\boldsymbol{b} = u\boldsymbol{a}$.*

Equivalently, $RA = RB$ if and only if there exists $u \in (Re)^\times$ such that $B = uA$.

Proof. Write $\boldsymbol{a} = (a_1, a_2, \ldots, a_l)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_l)$, where $a_i, b_i \in R$ for all $i \in \{1, 2, \ldots, l\}$.

Assume that $R\boldsymbol{a} = R\boldsymbol{b}$. Then $\boldsymbol{b} = v\boldsymbol{a}$ for some $v \in R$. Let $u = ve \in Re$. Note that, for each $i \in \{1, 2, \ldots, l\}$, we have $a_i = r_i e$ for some $r_i \in R$. Then $ua_i = (ve)(r_i e) = vr_i e^2 = v(r_i e) = va_i = b_i$ for all $i \in \{1, 2, \ldots, l\}$. Hence, $\boldsymbol{b} = u\boldsymbol{a}$ and

$$Re = R\boldsymbol{a} = R\boldsymbol{b} = R(u\boldsymbol{a}) = uR\boldsymbol{a} = uRe.$$

Since $u \in Re$ and $Re = uRe$, we have $u \in (Re)^\times$.

Conversely, assume that there exists $u \in (Re)^\times$ such that $\boldsymbol{b} = u\boldsymbol{a}$. Then $R\boldsymbol{b} = Ru\boldsymbol{a} \subseteq R\boldsymbol{a}$. We need to show that $\dim_{\mathbb{F}_q}(R\boldsymbol{a}) = \dim_{\mathbb{F}_q}(R\boldsymbol{b})$. Let e' be an idempotent generator of $R\boldsymbol{b}$. We have

$$Re' = R\boldsymbol{b} = R(u\boldsymbol{b}) = u(R\boldsymbol{b}) = u(Re) = Re$$

since $u \in (Re)^\times$. Hence, by Lemma 3.1, we have

$$\dim_{\mathbb{F}_q}(R\boldsymbol{a}) = \dim_{\mathbb{F}_q}(Re) = \dim_{\mathbb{F}_q}(Re') = \dim_{\mathbb{F}_q}(R\boldsymbol{b}).$$

Therefore, $R\boldsymbol{b} = R\boldsymbol{a}$ as desired. □

3.1. The enumeration of 1-generator quasi-abelian codes

First, we focus on the number of 1-generator H-quasi-abelian codes of a given idempotent generator in $\mathbb{F}_q[H]$. Using the fact that the idempotents in $\mathbb{F}_q[H]$ are known, the number of 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ can be concluded.

Proposition 3.3. *Let $\{e_1, e_2, \ldots, e_r\}$ be a set of primitive idempotents of R and $e = e_1 + e_2 + \cdots + e_r$. Then the following statements hold.*

i) *$e_1, e_2, \ldots, e_r$ are pairwise orthogonal (non-zero) idempotents of Se.*

ii) *e_j is the identity of Se_j for all $j \in \{1, 2, \ldots, r\}$.*

iii) *e is the identity of Se.*

iv) *$Se = Se_1 \oplus Se_2 \oplus \cdots \oplus Se_r$.*

Proof. For *i*), it is clear that $e_1, e_2, \ldots, e_r$ are pairwise orthogonal (non-zero) idempotents in S. They are in Se since $e_j = e_j e \in Se$ for all $j \in \{1, 2, \ldots, r\}$. The statements *ii*) and *iii*) follow since $se_j = se_j^2 = (se_j)e_j$ for all $se_j \in Se_j$ and $se = se^2 = (se)e$ for all $se \in Se$. The last statement can be verified using *i*). □

Corollary 3.4. *Let $\{e_1, e_2, \ldots, e_r\}$ be a set of primitive idempotents of R and $e = e_1 + e_2 + \cdots + e_r$. Then the following statements hold.*

i) *$e_1, e_2, \ldots, e_r$ are pairwise orthogonal (non-zero) idempotents of Re.*

ii) *e_j is the identity of Re_j for all $j \in \{1, 2, \ldots, r\}$.*

iii) *e is the identity of Re.*

iv) *$Re = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$, where Re_j is isomorphic to an extension field of $\mathbb{F}_q$ for all $j \in \{1, 2, \ldots, r\}$.*

Let $\Omega = \left\{ \sum_{j=1}^{r} A_j \,\middle|\, A_j \in (Se_j)^* \right\} \subset Se$. Then we have the following results.

Lemma 3.5. *Let $A = \sum_{i=1}^{l} \alpha_i a_i \in S$, where $a_i \in R$, and let $b \in R$. Then $RA \subseteq Sb$ if and only if $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Rb$.*

Proof. Assume that $RA \subseteq Sb$. Then $A = Bb$ for some $B \in S$. Write $B = \sum_{i=1}^{l} \alpha_i b_i$, where $b_i \in R$. Then $a_i = bb_i$ for all $i \in \{1, 2, \ldots, l\}$. Hence, we have

$$\sum_{i=1}^{l} r_i a_i = \sum_{i=1}^{l} r_i b b_i = \left(\sum_{i=1}^{l} r_i b_i \right) b \in Rb$$

for all $\sum_{i=1}^{l} r_i a_i \in Ra_1 + Ra_2 + \cdots + Ra_l$.

Conversely, it suffices to show that $A \in Sb$. Since $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Rb$, we have $a_i \in Rb$ for all $i \in \{1, 2, \ldots, l\}$. Then, for each $i \in \{1, 2, \ldots, l\}$, there exists $r_i \in R$ such that $a_i = r_i b$. Hence,

$$A = \sum_{i=1}^{l} \alpha_i a_i = \sum_{i=1}^{l} \alpha_i r_i b = \left(\sum_{i=1}^{l} \alpha_i r_i \right) b \in Sb$$

as desired. □

Lemma 3.6. *Let $A = \sum_{i=1}^{l} \alpha_i a_i \in Se$, where $a_i \in R$. Then $A \in \Omega$ if and only if*

$$Re = Ra_1 + Ra_2 + \cdots + Ra_l.$$

Proof. First, we note that $RA \subseteq Se$ since $A \in Se$. Then $Ra_1 + Ra_2 + \cdots + Ra_l \subseteq Re$ by Lemma 3.5.

Assume that $A \in \Omega$. Then $A = A_1 + A_2 + \cdots + A_r$, where $A_j \in (Se_j)^*$. We have $Ae_j = A_j \neq 0$ for all $j \in \{1, 2, \ldots, r\}$. Suppose that $Ra_1 + Ra_2 + \cdots + Ra_l \subsetneq Re$. By Corollary 3.4, we have $Re = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$. Then

$$Ra_1 + Ra_2 + \cdots + Ra_l \subseteq \widehat{Re_j} = R(e - e_j)$$

for some $j \in \{1, 2, \ldots, r\}$, where $\widehat{Re_j} := Re_1 \oplus \cdots \oplus Re_{j-1} \oplus Re_{j+1} \oplus \cdots \oplus Re_r$. By Lemma 3.5, we have

$$0 \neq A_j = Ae_j \in RA \subseteq S(e - e_j),$$

a contradiction. Therefore, $Ra_1 + Ra_2 + \cdots + Ra_l = Re$.

Conversely, assume that $Re = Ra_1 + Ra_2 + \cdots + Ra_l$. Then $RA \subseteq Se$ by Lemma 3.5. Since $A \in Se$, by Theorem 3.3, we have $A = A_1 + A_2 + \cdots + A_r$, where $A_j \in Se_j$ for all $j \in \{1, 2, \ldots, r\}$. Suppose that $A_j = 0$ for some $j \in \{1, 2, \ldots, r\}$. Then $RA = \widehat{RA_j} \subseteq \widehat{Se_j} = S(e - e_j)$. By Lemma 3.5, we have

$$Re = Ra_1 + Ra_2 + \cdots + Ra_l \subseteq R(e - e_j)$$

which is a contradiction. Hence, $A_j \in (Se_j)^*$ for all $j \in \{1, 2, \ldots, r\}$. □

Corollary 3.7. *Let $A = \sum_{i=1}^{l} \alpha_i a_i \in Se_j$, where $a_i \in R$. Then $A \in (Se_j)^*$ if and only if $Re_j = Ra_1 + Ra_2 + \cdots + Ra_l$.*

Let $j \in \{1, 2, \ldots, r\}$ and let k_j denote the $\mathbb{F}_q$-dimension of e_j. Then Re_j is isomorphic to a finite field of q^{k_j} elements.

Define an equivalence relation on $(Se_j)^*$ by

$$A \sim B \iff \exists u \in (Re_j)^{\times} \text{ such that } A = uB.$$

For $A \in (Se_j)^*$, denote by $[A]$ the equivalence class of A and let $[(Se_j)^*] = \{[A] \mid A \in (Se_j)^*\}$.

Lemma 3.8. *Let $j \in \{1, 2, \ldots, r\}$. Then $|[A]| = q^{k_j} - 1$ for all $A \in (Se_j)^*$.*

Proof. Let $A \in (Se_j)^*$ and define $\rho : (Re_j)^\times \to [A]$,

$$u \mapsto uA.$$

From the definition of $\sim$, ρ is a well-defined surjective map. For each $u_1, u_2 \in (Re_j)^\times$, if $u_1 A = u_2 A$, then $(u_1 - u_2)A = 0$. Write $A = \sum_{i=1}^{l} \alpha_i a_i$, where $a_i \in R$. Then $a_i(u_1 - u_2) = 0$ for all $i \in \{1, 2, \ldots, l\}$. Since $A \in (Se_j)^*$, by Corollary 3.7, we can write $e_j = \sum_{i=1}^{i} r_i a_i$, where $r_i \in R$. Hence,

$$e_j(u_1 - u_2) = \left(\sum_{i=1}^{i} r_i a_i\right)(u_1 - u_2) = \sum_{i=1}^{i} r_i a_i(u_1 - u_2) = 0 \in Re_j.$$

Since e_j is the identity of Re_j, it follows that $u_1 = u_2 \in (Re_j)^\times$. Hence, ρ is a bijection. Therefore, $|[A]| = |(Re_j)^\times| = |\mathbb{F}_{q^{k_j}}^*| = q^{k_j} - 1$. □

Corollary 3.9. *For each $i \in \{1, 2, \ldots, r\}$, we have*

$$|[(Se_j)^*]| = \frac{|(Se_j)^*|}{|[A]|} = \frac{q^{lk_j} - 1}{q^{k_j} - 1}.$$

Let $[\Omega] = \prod_{j=1}^{r} [(Se_j)^*]$. *Then* $|[\Omega]| = \prod_{j=1}^{r} \frac{q^{lk_j} - 1}{q^{k_j} - 1}$.

The number of 1-generator quasi-abelian codes sharing a idempotent has been determined in [7, Corollary 6.1]. Here, an alternative proof using a different technique is provided.

Theorem 3.10. *Let $\mathfrak{C}$ denote the set of all 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ with idempotent generator e. Then there exists a one-to-one correspondence between $[\Omega]$ and $\mathfrak{C}$. Hence, the number of 1-generator quasi-abelian codes having e as their idempotent generator is*

$$\prod_{j=1}^{r} \frac{q^{lk_j} - 1}{q^{k_j} - 1}.$$

Proof. Define $\sigma : [\Omega] \to \mathfrak{C}$,

$$([A_1], [A_2], \ldots, [A_r]) \mapsto R\boldsymbol{a},$$

where $A := A_1 + A_2 + \cdots + A_r \in Se$ is viewed as $A = \sum_{i=1}^{l} \alpha_i a_i$ and $\boldsymbol{a} := (a_1, a_2, \ldots, a_l)$.

Since $A_j \in (Se_j)^*$ for all $j \in \{1, 2, \ldots, r\}$, we have $A \in \Omega$. Then $Re = Ra_1 + Ra_2 + \cdots + Ra_l$ by Lemma 3.6, and hence, $R\boldsymbol{a}$ is a 1-generator quasi-abelian code with idempotent generator e, i.e., $R\boldsymbol{a} \in \mathfrak{C}$.

For $([A_1], [A_2], \ldots, [A_r]) = ([B_1], [B_2], \ldots, [B_r]) \in [\Omega]$, there exists $u_j \in (Re_j)^\times$ such that $A_j = u_j B_j$ for all $j \in \{1, 2, \ldots, r\}$. Let $u := u_1 + u_2 + \cdots + u_r$. Then

$$u\left(u_1^{-1} + u_2^{-1} + \cdots + u_r^{-1}\right) = e_1 + e_2 + \cdots + e_r = e$$

is the identity of Re (see Corollary 3.4), where u_j^{-1} refers to the inverse of u_j in Re_j. Hence, u is a unit in $(Re)^\times$. Let $B := \sum_{j=1}^{r} B_j$. Then

$$A = \sum_{j=1}^{r} A_j = \sum_{j=1}^{r} u_j B_j = uB.$$

Hence, $R\boldsymbol{a} = R\boldsymbol{b}$ by Lemma 3.2. Therefore, σ is a well-defined map.

For $([A_1],[A_2],\ldots,[A_r]),([B_1],[B_2],\ldots,[B_r]) \in [\Omega]$, if $R\boldsymbol{a} = R\boldsymbol{b}$, then, by Lemma 3.2, there exists $u \in (Re)^\times$ such that $A = uB$. Then $A_j = uB_j = ue_jB_j$ since e_j is the identity of Se_j by Proposition 3.3. Since $A_j \in (Se_j)^*$, ue_j is a non-zero in Re_j which is a finite field. Thus ue_j is a unit in $(Re_j)^\times$. Hence,

$$([A_1],[A_2],\ldots,[A_r]) = ([B_1],[B_2],\ldots,[B_r])$$

which implies that σ is an injective map.

To verify that σ is surjective, let $R\boldsymbol{a} \in \mathfrak{C}$, where $\boldsymbol{a} = (a_1,a_2,\ldots,a_l) \in R^l$. Then $Re = Ra_1 + Ra_2 + \cdots + Ra_l$. Hence, by Lemma 3.6, we conclude that

$$A := \sum_{i=1}^{l} \alpha_i a_i \in \Omega.$$

Write $A = \sum_{j=1}^{r} A_j$, where $A_j \in (Se_j)^*$. Then $([A_1],[A_2],\ldots,[A_r]) \in [\Omega]$, and hence,

$$\sigma(([A_1],[A_2],\ldots,[A_r])) = R\boldsymbol{a}.$$

□

3.2. The generators for 1-generator quasi-abelian codes

In this subsection, we establish an algorithm to find all 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$. Note that every idempotent in $R := \mathbb{F}_q[H]$ can be written as a unique sum of primitive idempotents in R. Hence, it is sufficient to study H-quasi-abelian codes of a given idempotent generator.

Let $e = e_1 + e_2 + \cdots + e_r$ be an idempotent in R, where, for each $j \in \{1,2,\ldots,r\}$, e_j is the primitive idempotent in R induced by a q-cyclotomic class $S_q(h_j)$ for some $h_j \in H$.

For each $j \in \{1,2,\ldots,r\}$, assume that e_j is decomposed as

$$e_j = e_{j1} + e_{j2} + \cdots + e_{js_j},$$

where, for each $i \in \{1,2,\ldots,s_j\}$, e_{ji} is the primitive idempotent in S defined corresponding to a q^l-cyclotomic class $S_{q^l}(h_{ji})$ for some $h_{ji} \in S_q(h_j)$.

Note that all the elements in $S_q(h_j)$ have the same order. Hence, the q^l-cyclotomic classes $S_{q^l}(h_{ji})$ have the same size for all $1 \le i \le s_j$. Without loss of generality, we assume that e_{j1} is defined corresponding to $S_{q^l}(h_j)$. For each $j \in \{1,2,\ldots,r\}$, let k_j and d_j denote the $\mathbb{F}_q$-dimension of e_j and the $\mathbb{F}_{q^l}$-dimension of e_{j1}, respectively. Then k_j and d_j are the smallest positive integers such that

$$q^{k_j} \cdot h_j = h_j \text{ and } q^{ld_j} \cdot h_j = h_j.$$

Then $k_j | ld_j$ which implies that $\frac{k_j}{\gcd(l,k_j)} | d_j$. Since $q^{l\frac{k_j}{\gcd(l,k_j)}} \cdot h_j = q^{k_j \frac{l}{\gcd(l,k_j)}} \cdot h_j = h_j$, we have $d_j | \frac{k_j}{\gcd(l,k_j)}$. It follows that $d_j = \frac{k_j}{\gcd(l,k_j)}$. Hence, e_{ji}'s have the same q^l-size $d_j = \frac{k_j}{\gcd(l,k_j)}$ and $s_j = \gcd(l,k_j)$.

Using arguments similar to those in the proof of Proposition 3.3, we conclude the following result.

Proposition 3.11. *Let $\{e_1,e_2,\ldots,e_r\}$ be a set of primitive idempotents of R. Assume that $e_j = e_{j1} + e_{j2} + \cdots + e_{js_j}$, where e_{ji} is a primitive idempotent in S for all $i \in \{1,2,\ldots,s_j\}$. Then the following statements hold.*

i) For $j \in \{1,2,\ldots,r\}$, the elements $e_{j1},e_{j2},\ldots,e_{js_j}$ are pairwise orthogonal (non-zero) idempotents of Se_j.

ii) e_{ji} is the identity of Se_{ji} for all $j \in \{1,2,\ldots,r\}$ and $i \in \{1,2,\ldots,s_j\}$.

iii) $e_j = e_{j1} + e_{j2} + \cdots + e_{js_j}$ *is the identity of* Se_j *for all* $j \in \{1, 2, \ldots, r\}$.

iv) *For* $j \in \{1, 2, \ldots, r\}$, *we have* $Se_j = Se_{j1} \oplus Se_{j2} \oplus \cdots \oplus Se_{js_j}$, *where* Se_{ji} *is an extension field of* $\mathbb{F}_q$ *of order* q^{ld_j} *for all* $i \in \{1, 2, \ldots, s_j\}$.

Theorem 3.12. *Let* $j \in \{1, 2, \ldots, r\}$ *be fixed. For* $i \in \{1, 2, \ldots, s_j\}$, *let* π_i *be a primitive element of* Se_{ji}, *a finite field of* q^{ld_j} *elements. Let* $L_j = \frac{q^{ld_j}-1}{q^{k_j}-1}$ *and* $T_j = \{\infty, 0, 1, 2, \ldots, q^{ld_j} - 2\}$. *Then the elements*

$$\pi_t^{\nu_t} + \pi_{t+1}^{\nu_{t+1}} + \cdots + \pi_{s_j}^{\nu_{s_j}}, \tag{3}$$

for all $1 \le t \le s_j$, $0 \le \nu_t \le L_j - 1$, *and* $\nu_{t+1}, \nu_{t+2}, \ldots, \nu_{s_j} \in T_j$, *are a complete set of representatives of* $[(Se_j)^*]$. *(By convention,* $\pi_i^\infty = 0$*.)*

Proof. Note that the number of elements in (3) is

$$L_j q^{ld_j(s_j-1)} + L_j q^{ld_j(s_j-2)} + \cdots + L_j = \frac{q^{lk_j}-1}{q^{k_j}-1} = |[(Se_j)^*]|.$$

Hence, it suffices to show that the elements in (3) are in different equivalence classes. Let

$$A = \pi_t^{\nu_t} + \pi_{t+1}^{\nu_{t+1}} + \cdots + \pi_{s_j}^{\nu_{s_j}} \text{ and } B = \pi_x^{\mu_x} + \pi_{x+1}^{\mu_{x+1}} + \cdots + \pi_{s_j}^{\mu_{s_j}},$$

where $0 \le \nu_t, \mu_x \le L_j - 1$, $\nu_{t+1}, \nu_{t+2}, \ldots, \nu_{s_j} \in T_j$, and $\mu_{x+1}, \mu_{x+2}, \ldots, \mu_{s_j} \in T_j$. Assume that $[A] = [B]$. Then there exists $u \in (Re_j)^\times$ such that

$$\pi_t^{\nu_t} + \pi_{t+1}^{\nu_{t+1}} + \cdots + \pi_{s_j}^{\nu_{s_j}} = A = uB = u\pi_x^{\mu_x} + u\pi_{x+1}^{\mu_{x+1}} + \cdots + u\pi_{s_j}^{\mu_{s_j}}.$$

Since $\pi_t^{\nu_t} \in (Se_{jt})^\times$ and $u\pi_x^{\mu_x} \in (Se_{jx})^\times$, by the decomposition in Proposition 3.11, $t = x$ and $\pi_t^{\nu_t} = u\pi_t^{\mu_t} \in Se_{jt}$. Then $ue_{jt} = \pi_t^{\nu_t - \mu_t}$. Since $u \in (Re_j)^\times$, we have $u^{q^{k_j}-1} = e_j$, and hence, $e_{jt} = e_{jt}e_j = \pi_t^{(\nu_t-\mu_t)(q^{k_j}-1)}$. Since $0 \le \nu_t, \mu_t \le L_j - 1$ and π_t has order $q^{ld_j} - 1$, we conclude that $\nu_t = \mu_t$. Hence, $ue_{jt} = e_{jt} = e_j e_{jt}$ which implies $(u - e_j)e_{jt} = 0$ in Se_{jt}. It follows that

$$S(u - e_j) \subseteq S(e_{j1} + \cdots + e_{j,t-1} + e_{j,t+1} + \cdots + e_{js_j}) \subsetneq Se_j.$$

Since $u, e_j \in Re_j$, we have $u - e_j \in Re_j$ and $R(u - e_j) \subsetneq Re_j$. Hence, $R(u - e_j)$ is the zero ideal, i.e., $u = e_j$. Therefore, $A = uB = e_jB = B$ since e_j is the identity of Se_j. □

The following corollary now follows from Theorem 3.10 and Theorem 3.12.

Corollary 3.13. *Let* $\{e_1, e_2, \ldots, e_r\}$ *be a set of primitive idempotents of* R *and* $e = e_1 + e_2 + \cdots + e_r$. *Then all* 1*-generator quasi-abelian codes having* e *as their idempotent generator are of the form*

$$A_1 + A_2 + \cdots + A_r,$$

where $A_j \in (Se_j)^*$ *is as defined in (3).*

Combining the results above, we summarize the steps of finding all 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ as in Algorithm 1. We note that the 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ are possible to determined using [7, Theorem 6.1] which depend on linear codes of dimension 1 over various extension fields of $\mathbb{F}_q$. Using this concept, the algorithm might look more tedious and complicated.

An illustrative example for Algorithm 1 is given as follows.

Example 3.14. *Let* $q = 2$, $G = \mathbb{Z}_3 \times \mathbb{Z}_6$ *and* $H = \mathbb{Z}_3 \times 2\mathbb{Z}_6$. *Denote by* $a_0 := (0, 0)$, $a_1 := (1, 0)$, $a_2 := (2, 0)$, $a_3 := (0, 2)$, $a_4 := (1, 2)$, $a_5 := (2, 2)$, $a_6 := (0, 4)$, $a_7 := (1, 4)$, *and* $a_8 := (2, 4)$, *the elements in* H. *Then* $l = [G : H] = 2$ *and the elements in* H *can be partitioned into the following* 2*-cyclotomic*

For abelian groups $H \le G$ and a finite field $\mathbb{F}_q$ with $\gcd(q, |H|) = 1$ and $[G : H] = l$, do the following steps.

1. Compute the q-cyclotomic classes of H in G.
2. Compute the set $\{e_1, e_2, \ldots, e_r\}$ of primitive idempotents of $R = \mathbb{F}_q[H]$ (see [4, Proposition II.4]).
3. For each $1 \le j \le r$, compute a set B_j of a complete set of representatives of $[(Se_j)^*]$ (see Theorem 3.12).
4. Compute the idempotents of R, *i.e.*, the set

$$T = \left\{ \sum_{j=1}^{t} e_{i_j} \,\middle|\, 1 \le t \le r \text{ and } 1 \le i_1 < i_2 < \cdots < i_t \le r \right\}.$$

5. For each $e = \sum_{j=1}^{t} e_{i_j} \in T$, compute the 1-generator quasi-abelian codes having e as their idempotent generator of the form

$$A_1 + A_2 + \cdots + A_t,$$

where $A_j \in B_{i_j}$ (see Corollary 3.13).

6. Run e over all elements of T, the 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ are obtained.

Algorithm 1. Steps for determining all 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$

classes $S_2(a_0) = \{a_0\}$, $S_2(a_1) = \{a_1, a_2\}$, $S_2(a_3) = \{a_3, a_6\}$, $S_2(a_4) = \{a_4, a_8\}$, and $S_2(a_5) = \{a_7, a_5\}$. From [4, Proposition II.4], we note that

$$\begin{aligned}
e_1 =& Y^{a_0} + Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8},\\
e_2 =& Y^{a_1} + Y^{a_2} + Y^{a_4} + Y^{a_5} + Y^{a_7} + Y^{a_8},\\
e_3 =& Y^{a_3} + Y^{a_4} + Y^{a_5} + Y^{a_6} + Y^{a_7} + Y^{a_8},\\
e_4 =& Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_4} + Y^{a_6} + Y^{a_8},\\
e_5 =& Y^{a_1} + Y^{a_2} + Y^{a_3} + Y^{a_5} + Y^{a_6} + Y^{a_7}
\end{aligned}$$

are primitive idempotents of $R := \mathbb{F}_2[H]$ induced by $S_2(a_0)$, $S_2(a_1)$, $S_2(a_3)$, $S_2(a_4)$, and $S_2(a_5)$, respectively.

Let $e := e_1 + e_2 + e_3$. From Theorem 3.10, it follows that the number of 1-generator H-quasi abelian codes in $\mathbb{F}_2[G]$ with idempotent generator e is $3 \cdot 5 \cdot 5 = 75$.

Let $S := \mathbb{F}_4[H]$, where $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = 1 + \alpha\}$. Then $e_2 = e_{21} + e_{22}$ and $e_3 = e_{31} + e_{32}$, where

$$\begin{aligned}
e_{21} =& Y^{a_0} + \alpha^2 Y^{a_1} + \alpha Y^{a_2} + Y^{a_3} + \alpha^2 Y^{a_4} + \alpha Y^{a_5} + Y^{a_6} + \alpha^2 Y^{a_7} + \alpha Y^{a_8},\\
e_{22} =& Y^{a_0} + \alpha Y^{a_1} + \alpha^2 Y^{a_2} + Y^{a_3} + \alpha Y^{a_4} + \alpha^2 Y^{a_5} + 1Y^{a_6} + \alpha Y^{a_7} + \alpha^2 Y^{a_8},\\
e_{31} =& Y^{a_0} + Y^{a_1} + Y^{a_2} + \alpha^2 Y^{a_3} + \alpha^2 Y^{a_4} + \alpha^2 Y^{a_5} + \alpha Y^{a_6} + \alpha Y^{a_7} + \alpha Y^{a_8},\\
e_{32} =& Y^{a_0} + Y^{a_1} + Y^{a_2} + \alpha Y^{a_3} + \alpha Y^{a_4} + \alpha Y^{a_5} + \alpha^2 Y^{a_6} + \alpha^2 Y^{a_7} + \alpha^2 Y^{a_8}
\end{aligned}$$

are primitive idempotents in S induced by 4-cyclotomic classes $\{a_1\}$, $\{a_2\}$, $\{a_3\}$ and $\{a_6\}$, respectively.

Now, we have $k_1 = 1$, $k_2 = k_3 = 2$, $d_1 = d_2 = d_3 = 1$, $s_1 = 1$, and $s_2 = s_3 = 2$. It follows that $L_1 = \frac{2^2-1}{2-1} = 3$, $L_2 = L_3 = \frac{2^2-1}{2^2-1} = 1$, and $T_1 = T_2 = T_3 = \{\infty, 0, 1, 2\}$.

Then αe_1, αe_{21}, αe_{22}, αe_{31}, and αe_{32} are primitive elements of Se_1, Se_{21}, Se_{22}, Se_{31}, and Se_{32}, *respectively. Therefore, we have that*

$$\begin{aligned} B_1 &= \{e_1, \alpha e_1, \alpha^2 e_1\}, \\ B_2 &= \{e_{21}, e_{21}+e_{22}, e_{21}+\alpha e_{22}, e_{21}+\alpha^2 e_{22},\ e_{22}\},\ \textit{and} \\ B_2 &= \{e_{31}, e_{31}+e_{32}, e_{31}+\alpha e_{32}, e_{31}+\alpha^2 e_{32},\ e_{32}\} \end{aligned}$$

are complete sets of representatives of $[(Se_1)^*]$, $[(Se_2)^*]$, *and* $[(Se_3)^*]$, *respectively. Hence, all the generators of the* 75 1*-generator* H*-quasi abelian codes in* $\mathbb{F}_2[G]$ *with idempotent generator* e *are of the form*

$$A_1 + A_2 + A_3,$$

where $A_i \in B_i$ *for all* $i \in \{1, 2, 3\}$.

In order to find permutation inequivalent 1-generator H-quasi abelian codes, the following theorem is useful.

Theorem 3.15. *Let* $H \le G$ *be finite abelian groups of index* $[G : H] = l$ *and let* $\{\alpha^{q^i} \mid 1 \le i \le l\}$ *be a fixed basis of* $\mathbb{F}_{q^l}$ *over* $\mathbb{F}_q$. *If* $A = \sum_{i=1}^{l} a_i \alpha^{q^i} \in Se$, *then* A *and* $A^q = \sum_{i=1}^{l} a_i^q \alpha^{q^{i+1}}$ *generate permutation equivalent* H*-quasi abelian codes (viewed in* $\mathbb{F}_q[G]$*) with the same idempotent generator.*

Proof. Let e be the idempotent generator of a quasi-abelian code RA. Then

$$Ra_1^q + Ra_2^q + \cdots + Ra_l^q \subseteq Ra_1 + Ra_2 + \cdots + Ra_l = Re$$

Assume that $e = \sum_{i=1}^{l} r_i a_i$, where $r_i \in R$. It follows that

$$e = e^q = \sum_{i=1}^{l} r_i^q a_i^q \in Ra_1^q + Ra_2^q + \cdots + Ra_l^q.$$

Hence, we have $Re = Ra_1^q + Ra_2^q + \cdots + Ra_l^q$. Therefore, A and A^q generate 1-generator H-quasi-abelian codes with the same idempotent generator e.

Let $\psi : R \to R$ be a ring homomorphism defined by

$$\gamma \mapsto \gamma^q.$$

Let $\gamma = \sum_{h \in H} \gamma_h Y^h$ and $\beta = \sum_{h \in H} \beta_h Y^h$ be elements in R, where γ_h and β_h are elements in $\mathbb{F}_q$. If $\psi(\gamma) = \psi(\beta)$, then

$$0 = \gamma^q - \beta^q = (\gamma - \beta)^q = \sum_{h \in H} (\gamma_h - \beta_h) Y^{q \cdot h}.$$

By comparing the coefficients, we have $\gamma_h = \beta_h$ for all $h \in H$, *i.e.*, $\gamma = \beta$. Hence, ψ is a ring automorphism and

$$R(a_l^q, a_1^q, \ldots, a_{l-1}^q) = R(\psi(a_l), \psi(a_1), \ldots, \psi(a_{l-1})) = \Psi(R(a_l, a_1, \ldots, a_{l-1})), \tag{4}$$

where Ψ is a natural extension of ψ to R^l.

Since $\psi(\gamma) = \sum_{h \in H} \gamma_h Y^{q \cdot h}$, $\psi(\gamma)$ is just a permutation on the coefficients of γ. Hence, by (4), $\Psi \circ \Phi$ is a permutation on $\mathbb{F}_q[G]$ such that $\Phi^{-1}\left(R(a_l^q, a_1^q, \ldots, a_{l-1}^q)\right)$ is permutation equivalent to $\Phi^{-1}\left(R(a_l, a_1, \ldots, a_{l-1})\right)$ in $\mathbb{F}[G]$, where Φ is the R-module isomorphism defined in (1). Therefore, the result follows since $R(a_l, a_1, \ldots, a_{l-1})$ is permutation equivalent to $R(a_1, a_2, \ldots, a_l)$. □

4. Computational results

It has been shown in [6] and [7] that a family of quasi-abelian codes contains various new and optimal codes. Here, we present other 2 new codes from 1-generator quasi-abelian codes together with 1 new code obtained by shortening of one of these codes.

Given an abelian group $H = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ of order $n = n_1 n_2$, denote by $u = (u_0, u_1, u_2, \dots, u_{n-1}) \in \mathbb{F}_q^n$ the vector representation of

$$u = \sum_{j=0}^{n_2-1} \sum_{i=0}^{n_1-1} u_{jn_1+i} Y^{(i,j)} \text{ in } \mathbb{F}_q[H].$$

Let

$$C_{(a,b)} := \{(fa, fb) \mid f \in \mathbb{F}_q[H]\}, \tag{5}$$

where a and b are elements in $\mathbb{F}_q[H]$. Using (5), 2 quasi-abelian codes whose minimum distance improves on Grassl's online table [5] can be found. The codes C_1 and C_2 are presented in Table 1 and the generator matrices of C_1 and C_2 are

$$G_1 = \left[\begin{array}{c|c} I_{14} & \begin{matrix}
1&3&0&3&4&1&3&2&0&4&1&2&1&4&0&4&1&0&4&3&0&4\\
1&3&4&4&3&1&4&0&2&4&1&3&0&2&2&4&3&1&1&3&4&0\\
1&4&4&3&4&0&4&0&0&1&0&3&1&2&0&1&0&3&2&4&4&4\\
4&4&3&3&4&2&3&3&1&3&4&0&3&3&2&1&1&1&1&0&3&0\\
4&3&3&4&3&2&4&2&3&2&3&2&2&3&0&3&2&1&0&1&4&3\\
4&4&2&4&4&1&4&1&2&4&2&1&4&0&0&1&1&2&0&4&0&4\\
0&2&1&1&3&1&4&1&1&2&1&0&1&1&4&2&0&0&1&3&2&3\\
0&1&2&1&4&3&1&2&1&1&1&1&0&2&1&4&1&0&0&3&3&2\\
0&1&1&2&1&4&3&1&2&1&0&1&1&4&2&1&0&1&0&2&3&3\\
1&2&2&2&3&4&4&4&4&1&3&1&4&4&3&3&1&0&1&2&2&4\\
1&2&3&1&4&0&2&2&4&3&4&0&4&1&2&2&0&1&1&3&3&2\\
1&1&3&2&2&1&3&4&2&3&4&1&3&0&4&1&0&0&2&1&4&3\\
4&0&4&1&0&3&2&4&0&1&0&3&2&2&2&1&1&0&4&1&4&0\\
4&1&4&0&2&3&0&0&4&1&2&3&0&3&4&3&0&1&4&1&0&4
\end{matrix}\end{array}\right]$$

and

$$G_2 = \left[\begin{array}{c|c} I_{11} & \begin{matrix}
0&1&0&4&4&0&0&1&4&4&0&4&1&3&2&3&3&1&1&3&3&2&0&1&4\\
4&4&1&1&2&1&2&4&1&4&3&2&1&4&4&3&2&4&2&0&1&1&0&1&2\\
1&0&4&0&0&0&4&4&4&1&4&1&0&2&3&3&1&1&3&3&2&3&1&4&0\\
0&1&0&0&4&0&4&1&0&3&1&3&0&3&1&4&1&3&4&1&4&3&3&4&1\\
4&4&0&0&0&0&1&1&4&3&3&4&1&4&3&1&4&1&3&0&3&1&3&0&1\\
1&0&0&0&0&4&4&0&3&1&3&0&1&1&4&3&3&4&1&4&3&1&4&1&3\\
1&1&4&0&4&0&4&3&2&1&0&0&4&1&3&1&2&3&3&2&3&4&2&4&2\\
4&0&0&4&0&0&1&4&1&0&2&3&3&1&1&3&3&2&3&1&4&0&4&4&1\\
0&4&1&1&2&1&1&2&1&3&2&1&2&4&2&2&4&4&3&1&2&0&0&3&3\\
1&1&0&0&4&4&4&2&2&2&2&2&2&0&0&0&0&0&0&3&3&3&3&3&3\\
0&0&1&1&1&1&1&2&2&2&4&4&4&1&1&1&1&1&1&1&1&1&4&4&4
\end{matrix}\end{array}\right],$$

respectively.

By puncturing C_2 at the first coordinate, a $[35, 11, 17]_5$ code can be obtained with minimum distance improved by 1 from Grassl's online table [5]. All the computations are done using MAGMA [3].

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Table 1. New codes from quasi-abelian codes

name	$C_{(a,b)}$	H	a and b
$C1$	$[36,14,15]_5$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$a = (3,3,3,0,0,1,4,3,4,0,4,4,4,4,3,0,1,0)$ $b = (2,4,1,1,3,3,0,0,4,4,1,0,0,1,4,2,2,4)$
$C2$	$[36,11,18]_5$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$a = (2,4,4,3,4,4,3,2,4,3,4,4,3,4,2,3,4,4)$ $b = (3,0,0,0,3,3,3,0,3,0,3,0,1,1,1,1,1,1)$

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