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The nonnegative Q−matrix completion problem

Research Article

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Abstract: In this paper, the nonnegative Q-matrix completion problem is studied. A real $n \times n$ matrix is a Q-matrix if for $k \in \{1, \ldots, n\}$, the sum of all $k \times k$ principal minors is positive. A digraph D is said to have nonnegative Q-completion if every partial nonnegative Q-matrix specifying D can be completed to a nonnegative Q-matrix. For nonnegative Q-completion problem, necessary conditions and sufficient conditions for a digraph to have nonnegative Q-completion are obtained. Further, the digraphs of order at most four that have nonnegative Q-completion have been studied.

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1. Introduction

A *partial matrix* is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A partial matrix M is *fully specified* if all entries of M are specified, i.e., if M is a matrix. Let $\langle n \rangle = \{1, \ldots, n\}$ and M be an $n \times n$ partial matrix, i.e., one with n rows and n columns. For a subset α of $\langle n \rangle$, the *principal partial submatrix* $M(\alpha)$ is the partial matrix obtained from M by deleting all rows and columns not indexed by α. A *principal minor* of M is the determinant of a fully specified principal submatrix of M. A *partial nonnegative* (*positive*) *matrix* is a partial matrix whose specified entries are nonnegative (positive).

A real $n \times n$ matrix B is a *P-matrix* (*P_0-matrix*) if every principal minor of B is positive (nonnegative). The matrix B is a *Q-matrix*, if for each $k \in \{1, \ldots, n\}$ the sum of all $k \times k$ principal minors of B is positive. A *nonnegative Q-matrix* is a Q-matrix whose all entries are nonnegative. The property of being P, P_0 or Q-matrix is invariant under permutation similarity.

For a given class Π of matrices (e.g., P, P_0 or Q-matrices) a *partial Π-matrix* is a partial matrix for which the specified entries fulfill the requirements of a Π-matrix. Thus, a *partial P_0-matrix* (*partial P-matrix*) is one in which all fully specified principal minors are nonnegative (positive). Similarly, a

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partial Q-matrix is a partial matrix M in which $S_k(M) > 0$ for every $k \in \{1, 2, \dots, n\}$ for which all $k \times k$ principal submatrices are fully specified.

A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A Π-*completion* of a partial Π-matrix M is a completion of M which is a Π-matrix. Matrix completion problems for several classes of matrices including the classes of P and P_0-matrices have been studied by a number of authors (e.g., [2, 3, 5, 7, 8, 10, 11]).

In 2009, DeAlba *et al.* [4] considered the Q-matrix completion problem. Since the property of being a Q-matrix is not inherited by principal submatrices, the authors observed that the Q-matrix completion problem is substantially different from the completion problems studied earlier and attracts special attention. In their paper, it was shown that for Q-matrix completion of a digraph, *stratification* (see Section 2) of its complement is necessary and *positive signing* of the stratified complement of the digraph is sufficient. (Here, positive signing of a digraph means a signing of each of its arcs with the property that for each even (resp. odd) cycle the product of the signs of arcs on the cycle is negative (resp. positive).) Further, the authors classified all digraphs of order up to order 4 as to Q-matrix completion.

Theorem 1.1. [4] *Let D be a digraph of order n that omits at least one loop.*

(i) If D has Q-completion, then $\overline{D}$ is stratified.

(ii) If $n \leq 4$ and $\overline{D}$ is stratified, then D has Q-completion.

Theorem 1.2. [4] *Let $D \neq K_n$ be an order n digraph that includes all loops and has Q-completion. Then for each $k = 2, 3, \dots, n$, either,*

(i) $\overline{D}$ has a permutation digraph of order k, or

(ii) for each $v \in V(D)$, $\overline{D} - v$ has a permutation digraph of order $k - 1$.

Theorem 1.3. [4] *Let D be a digraph such that $\overline{D}$ is stratified. If it is possible to sign the arcs of $\overline{D}$ so that the sign of every cycle is $+$, then D has Q-completion.*

For an extensive survey of matrix completion problems, we refer the relevant sections in *Handbook of Linear Algebra* [9] published by Chapman and Hall/CRC Press. In this paper, we make a combinatorial study of the completion problem of partial nonnegative Q-matrices in which digraphs will play an important role.

2. Preliminaries

Most of the definitions of the graph-theoretic terms used in this paper can be found in any standard reference, for example, in [1] and [6].

For our purpose, a *directed graph* or *digraph* $D = (V(D), A(D))$ of order $n > 0$ is a finite nonempty set $V(D)$, with $|V(D)| = n$ of objects called *vertices* together with a (possibly empty) set $A(D)$ of ordered pairs of vertices (not necessarily distinct), called *arcs* or *directed edges*. Sometimes, we simply write $v \in D$ (resp. $(u, v) \in D$) to mean $v \in V(D)$ (resp. $(u, v) \in A(D)$). If $x = (u, v)$ is an arc in D, we say that x is *incident* with u and v. If $x = (u, u)$, then x is called a *loop* at the vertex u. By K_n^* we denote the digraph with vertex set $\langle n \rangle = \{1, \dots, n\}$ and arc set $\langle n \rangle \times \langle n \rangle$, i.e., one with all possible arcs including loops on the vertex set $\langle n \rangle$.

A digraph H is a *subdigraph* of the digraph D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$. If $V(W) \subseteq V(D)$, the *subdigraph induced by* $V(W)$, i.e. W, is the digraph $W = (V(W), A(W))$ with $A(W)$ the set of all arcs of D between the vertices in W. The digraph W is a *spanning subdigraph* if $V(W) = V(D)$. The *complement of a digraph* D is the digraph $\overline{D}$, where $V(\overline{D}) = V(D)$ and $(v, w) \in A(\overline{D})$ if and only if $(v, w) \notin A(D)$. Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are *isomorphic*, if there is a bijection $\psi : V_1 \to V_2$ such that $A_2 = \{(\psi(u), \psi(v)) : (u, v) \in A_1\}$.

A (*directed*) *u*-*v* *path* P of length $k \geq 0$ in D is an alternating sequence $(u = v_0, x_1, v_1, \ldots, x_k, v_k = v)$ of vertices and arcs, where $v_i,\ 1 \leq i \leq k$, are distinct vertices and $x_i = (v_{i-1}, v_i)$. Then, the vertices v_i and the arcs x_i are said to be on P. Further, if $k \geq 2$ and $u = v$, then a u-v path is a *cycle* of length k. We then write $C_k = \langle v_1, v_2, \ldots, v_k \rangle$ and call C_k a k-cycle in D. Paths and cycles in a digraph D are considered to be subdigraphs of D in a natural way. A cycle C is *even* (resp. *odd*) if its length is even (resp. odd). A digraph D is said to be connected (resp. strongly connected) if for every pair u, v of vertices, D contains a u-v path (resp. both a u-v path and a v-u path). The maximal connected (resp. strongly connected) subdigraphs of D are called *components* (resp. *strong components*) of D.

Let π be a permutation of a nonempty finite set V. The digraph $D_\pi = (V, A_\pi)$, where $A_\pi = \{(v, \pi(v)) : v \in V\}$, is called a *permutation digraph*. Clearly, each component of a permutation digraph is a loop or a cycle. The digraph D_π is said to be *positive* (resp. *negative*) if π is an even permutation (resp. an odd permutation). It is clear that D_π is negative if and only if it has odd number of even cycles.

A *permutation subdigraph* H (of order k) of a digraph D is a permutation digraph that is a subdigraph of D (of order k). Further, H is *positive* (*negative*) if the corresponding permutation is even (odd). A digraph D is (*positively*) *stratified* if D has a (positive) permutation subdigraph of order k for every $k = 2, 3, \ldots, |D|$.

By $\mathcal{P}_k$ we denote the collection of all permutation subdigraphs of order k of K_n^*. Further, we denote by $\mathcal{P}_k^+$ (resp. $\mathcal{P}_k^-$) the collection of all positive (resp. negative) permutation subdigraphs of order k of K_n^*.

Let $B = [b_{ij}]$ be an $n \times n$ matrix. We have

$$\det B = \sum (sgn\pi) b_{1\pi(1)} \cdots b_{n\pi(n)} \tag{1}$$

where the sum is taken over all permutations π of $\langle n \rangle$. For a permutation digraph P of K_n^* we denote the product $\prod\{b_{ij} : (i,j) \in A(P)\}$ by $w(P, B)$. For $k \in \{1, \ldots, n\}$ we denote the sum of all $k \times k$ principal minors of B by $S_k(B)$. In view of (1), we have

$$S_k(B) = \sum_{P \in \mathcal{P}_k^+} w(P, B) - \sum_{P \in \mathcal{P}_k^-} w(P, B). \tag{2}$$

3. Partial nonnegative Q-matrices and their completions

Recall that a partial Q-matrix M is a partial matrix such that $S_k(M) > 0$ for every $k \in \{1, \ldots, n\}$ for which all $k \times k$ principal submatrices are fully specified. Let M be a partial nonnegative Q-matrix. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then their sum $S_1(M)$ (the trace of M) must be positive. If all $k \times k$ principal submatrices are fully specified for some $k \geq 2$, then M is fully specified and, therefore, is a nonnegative Q-matrix. Thus, a partial nonnegative Q-matrix is characterized as follows.

Proposition 3.1. *Let M be a partial nonnegative matrix. Then, M is a partial nonnegative Q-matrix if and only if exactly one of the following holds:*

(i) At least one diagonal entry of M is not specified.

(ii) All diagonal entries are specified, at least one diagonal entry is positive and M has an off-diagonal unspecified entry.

(iii) All entries of M are specified and M is a Q-matrix.

A completion B of a partial nonnegative Q-matrix M is called a *nonnegative Q-completion* of M, if B is a nonnegative Q-matrix. Since any matrix which is permutation similar to a Q-matrix is a Q-matrix, it is evident that if a partial nonnegative Q-matrix has a nonnegative Q-completion, so does any partial matrix which is permutation similar to M.

Any partial nonnegative matrix M with all diagonal entries unspecified has nonnegative Q-completion. A completion of M can be obtained by choosing sufficiently large values for the unspecified diagonal entries.

Let $M = [m_{ij}]$ be a partial nonnegative Q-matrix which contains both specified and unspecified diagonal entries. Consider the principal partial submatrix $M(\alpha)$ of M induced by $\alpha = \{i : m_{ii} \text{ is specified}\} \subseteq \langle n \rangle$. In case $M(\alpha)$ is fully specified, M may not have a nonnegative Q-completion. For example, the partial matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}$$

where $*$ denotes an unspecified entry, $M(1, 2)$ is fully specified. That M does not have a nonnegative Q-completion is evident; because for any completion B of M, $S_3(B) = \det B = 0$. For the case when $M(\alpha)$ is not fully specified or itself a Q-matrix, we have the following.

Theorem 3.2. *Let $M = [m_{ij}]$ be a partial nonnegative Q-matrix and $\alpha = \{i : m_{ii} \text{ is specified}\}$. If the principal partial submatrix $M(\alpha)$ of M has nonnegative Q-completion, then M has nonnegative Q-completion.*

Proof. If $\alpha = \{1, \ldots, n\}$, then $M = M(\alpha)$ has a nonnegative Q-completion, by the hypothesis. Otherwise, without any loss of generality, we assume $\alpha = \{1, \ldots, r\}$ for some $1 \leq r \leq n - 1$ and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where $M_{11} = M(1, \ldots, r)$ and $M_{22} = M(r + 1, \ldots, n)$. Let B_{11} be a nonnegative Q-completion of $M(1, \ldots, r)$. Then,

$$M' = \begin{bmatrix} B_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

is a partial nonnegative Q-matrix, since M_{22} has unspecified diagonal entries. For $t > 0$, consider the completion

$$B(t) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

of M obtained by taking $b_{ii} = t,\ i = r + 1, \ldots, n$, and $b_{ij} = 0$ for all other unspecified entries in M. Since B_{11} is a nonnegative Q-matrix we have $S_i(B_{11}) > 0$ for $1 \leq i \leq r$. Now, for $1 \leq j \leq n$,

$$S_j(B(t)) = \begin{cases} \binom{n-r}{j} t^j + p_j(t), & \text{if } j \leq n - r, \\ S_{j-n+r}(B_{11}) t^{n-r} + p_j(t), & \text{if } j > n - r, \end{cases}$$

where $p_j(t)$ is a polynomial in t of degree at most $j - 1$, if $j \leq n - r$, and of degree at most $n - r - 1$, if $j > n - r$. As a consequence, for sufficiently large values of t, $S_j(B(t)) > 0$ for $1 \leq j \leq n$ and $B(t)$ is a nonnegative Q-completion of M. □

The converse of Theorem 3.2 is not true. The following example shows that a partial nonnegative Q-matrix M may have Q-completion, even when $M(\alpha)$ does not have.

Example 3.3. *Consider the partial matrix,*

$$M = \begin{bmatrix} * & b_{12} & b_{13} & b_{14} \\ b_{21} & d_2 & b_{23} & * \\ * & * & d_3 & b_{34} \\ * & b_{42} & * & * \end{bmatrix}, \tag{3}$$

where $*$ denotes the unspecified entries. Here, $\alpha = \{2,3\}$ and $M(\alpha)$ does not have a Q-completion for $d_2 = b_{23} = 0$ and $d_3 = 1$. However, we show that for any choice of nonnegative values of the specified entries b_{ij}, M has nonnegative Q-completions. For $t > 0$ consider the completion

$$B(t) = \begin{bmatrix} t & b_{12} & b_{13} & b_{14} \\ b_{21} & d_2 & b_{23} & t \\ t & t & d_3 & b_{34} \\ t & b_{42} & t & t \end{bmatrix}$$

of M. Then,

$$\begin{aligned} S_1(B(t)) &= 2t + d_2 + d_3, \\ S_2(B(t)) &= t^2 + f_1(t), \\ S_3(B(t)) &= t^3 + f_2(t), \\ S_4(B(t)) &= t^4 + f_3(t), \end{aligned}$$

where $f_i(t)$ is a polynomial in t of degree at most i, $i = 1,2,3$. Consequently, $B(t)$ is a nonnegative Q-matrix for sufficiently large t, and therefore, M has nonnegative Q-completions.

4. Digraphs and nonnegative Q-completions

It is useful to associate a partial matrix with a digraph that describes the positions of the specified entries in the partial matrix. We say that an $n \times n$ partial matrix M *specifies* a digraph $D = (\langle n \rangle, A(D))$, if for $1 \le i,j \le n$, $(i,j) \in A(D)$ if and only if the (i,j)-th entry of M is specified. For example, the partial nonnegative Q-matrix M in Example 3.3 specifies the digraph D_1 in Figure 1.

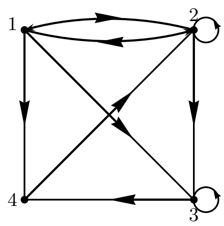


Figure 1. The digraph D_1

Theorem 4.1. *Suppose M is a partial nonnegative Q-matrix specifying the digraph D. If the partial submatrix of M induced by every strongly connected induced subdigraph of D has nonnegative Q-completion, then M has nonnegative Q-completion.*

Proof. We prove the result for the case when D has two strong components H_1 and H_2. The general result will then follow by induction. By a relabelling of the vertices of D, if required, we have

$$M = \begin{bmatrix} M_{11} & M_{12} \\ X & M_{22} \end{bmatrix},$$

where M_{ii} is a partial nonnegative Q-matrix specifying H_i, $i = 1,2$, and all entries in X are unspecified. By the hypothesis, M_{ii} has a nonnegative Q-completion B_{ii}. Consider the completion

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

by choosing all unspecified entries in M_{12} and X as 0. Then, for $2 \leq k \leq |D|$ we have

$$S_k(B) = S_k(B_{11}) + S_k(B_{22}) + \sum_{r=1}^{k-1} S_r(B_{11}) S_{k-r}(B_{22}) > 0,$$

Here, we mean $S_k(B_{ii}) = 0$ whenever k exceeds the size of B_{ii}. Thus M can be completed to a nonnegative Q-matrix. □

The proof of the following result is similar.

Theorem 4.2. *Suppose M is a partial nonnegative Q-matrix specifying the digraph D. If the partial submatrix of M induced by each component of D has a nonnegative Q-completion, then M has a nonnegative Q-completion.*

That the converse of Theorem 4.1 is not true can be seen from the following example.

Example 4.3. *Consider the digraph D_2 in Figure 2. We show that every partial nonnegative Q-matrix*

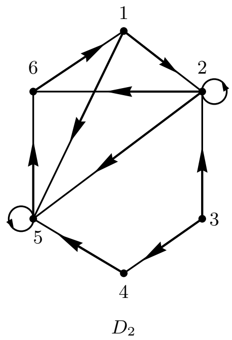


Figure 2. A digraph having nonnegative Q-completion

specifying the digraph D_2 has nonnegative Q-completion. To see this consider any partial nonnegative Q-matrix $M = [a_{ij}]$ specifying D_2. For $t > 1$ consider the completion

$$B(t) = \begin{bmatrix} t & a_{12} & 0 & 0 & a_{15} & 0 \\ 0 & a_{22} & t & 0 & a_{25} & a_{26} \\ 0 & a_{32} & t & a_{34} & t & 0 \\ 0 & 0 & 0 & t & a_{45} & 0 \\ 0 & t & 0 & 0 & a_{55} & a_{56} \\ a_{61} & 0 & 0 & 0 & 0 & t \end{bmatrix}$$

of M. Then, we have

$$\begin{aligned} S_1(B(t)) &= 4t + a_{22} + a_{55}, \\ S_2(B(t)) &= 6t^2 + f_1(t), \\ S_3(B(t)) &= 5t^3 + f_2(t), \\ S_4(B(t)) &= 4t^4 + f_3(t), \\ S_5(B(t)) &= 3t^5 + f_4(t), \\ S_6(B(t)) &= t^6 + f_5(t), \end{aligned}$$

where f_i are polynomials of degree at most i in t. It is clear that $S_k(B(t)) > 0$, $1 \le k \le 6$, for sufficiently large values of t. Hence M has nonnegative Q-completions.

On the other hand, the vertices $1, 2, 5$ and 6 induce a strong component H of D_2. Consider the partial nonnegative Q-matrix M_1 specifying H and with all specified entries as zero and

$$B_1 = \begin{bmatrix} x_{11} & 0 & 0 & x_{16} \\ x_{21} & 0 & 0 & 0 \\ x_{51} & x_{52} & 0 & 0 \\ 0 & x_{62} & x_{65} & x_{66} \end{bmatrix},$$

any nonnegative completion of M_1. Then

$$S_4(B_1) = -x_{16}x_{21}x_{52}x_{65} \le 0,$$

and consequently B_1 is not a nonnegative Q-matrix.

5. The nonnegative Q-completion problem

For a class Π of matrices (e.g., P, P_0 or Q-matrices) a digraph D is said to have Π-*completion*, if every partial Π-matrix specifying D can be completed to a Π-matrix. The Π-*matrix completion problem* refers to the study of digraphs which have Π-completion.

We say that a digraph D has *nonnegative* (*positive*) Q-*completion*, if every partial nonnegative (positive) Q-matrix specifying D can be completed to a nonnegative (positive) Q-matrix. The *nonnegative* (*positive*) Q-*matrix completion problem* aims at studying and classifying all digraphs D which have nonnegative (positive) Q-completion.

Example 5.1. *It follows from Example 4.3 that the digraph D_2 in Figure 2 has nonnegative Q-completion. However, its strong component H induced by the vertices $1, 2, 5$ and 6 does not have nonnegative Q-completion. In particular, this exhibits that the property of having nonnegative Q-completion is not preserved to induced subdigraphs.*

It is clear that if a digraph D has (nonnegative, positive) Q-completion, then any digraph which is isomorphic to D has (nonnegative, positive) Q-completion.

5.1. Sufficient conditions for nonnegative Q-completion

Proposition 5.2. *If a digraph $D \neq K_n^*$ of order n has nonnegative Q-completion, then any spanning subdigraph of D has nonnegative Q-completion.*

Proof. Let H be a spanning subdigraph of D. Let M_H be a partial nonnegative Q-matrix specifying the digraph H. Consider the partial matrix M_D obtained from M_H by specifying the entries corresponding to $(i, j) \in A(D) \setminus A(H)$ as 0. Since $D \neq K_n^*$, by Proposition 3.1, M_D is a partial nonnegative Q-matrix specifying D. Let B be a nonnegative Q-completion of M_D. Clearly, B is a nonnegative Q-completion of M_H. □

For a digraph $D = (V, A)$, a *weight function* on D is a real valued function ϕ defined on A. The triplet (V, A, ϕ) is then called a *weighted digraph*. For $e \in A$, $\phi(e)$ is called the *weight* of e. Further, for any permutation subdigraph P of D, we denote the sum of the weights of the edges on P by $\phi^*(P)$.

Theorem 5.3. *Let D be a digraph of order n. Suppose there is a weight function ϕ on $\overline{D}$ satisfying the following: for each $k \in \{1, \dots, n\}$, there is a positive permutation subdigraph P_k of order k in $\overline{D}$ such that,*

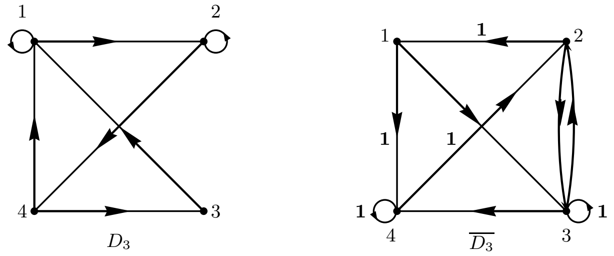


Figure 3. The digraph D_3 and its complement $\overline{D_3}$

(i) $\phi^*(P_k) > \phi^*(P)$ *for every negative permutation subdigraph P of $\overline{D}$ of size k, and*

(ii) $\phi^*(P_k) > \sum_{e \in S} \phi(e)$, *for any subset $S \subseteq A(\overline{D})$ with $|S| \leq k-1$.*

Then, D has nonnegative Q-completion.

Proof. Let $M = [m_{ij}]$ be a partial nonnegative Q-matrix specifying D. For $x > 1$ consider the completion $B(x) = [b_{ij}]$ of M defined by,

$$b_{ij} = \begin{cases} m_{ij}, & \text{if } (i,j) \in D \\ x^{\phi(e)}, & \text{if } e = (i,j) \in \overline{D}. \end{cases}$$

For $k \in \{1, \ldots, n\}$, we have

$$S_k(B(x)) = \sum_{P \in \mathcal{P}_k^+} w(P, B(x)) - \sum_{P \in \mathcal{P}_k^-} w(P, B(x)).$$

Since $P_k \in \mathcal{P}_k^+$, $w(P_k, B(x)) = x^{\phi^*(P_k)}$ is a term with a positive coefficient in $S_k(B(x))$. On the other hand, any $P \in \mathcal{P}_k^-$ is one of the following:

(i) a negative permutation subdigraph of $\overline{D}$ of order k,

(ii) a permutation subdigraph of K_n^* with at most $k-1$ arcs from $\overline{D}$.

In view of the properties of ϕ, we have,

$$w(P, B(x)) = \alpha x^t, \text{ where } \alpha \geq 0 \text{ and } t < \phi^*(P_k).$$

Consequently, for large values of x, we have $S_k(B(x)) > 0$ for $k = 1, \ldots, n$ and hence D has nonnegative Q-completion. □

Example 5.4. *Consider the digraph D_3 in Figure 3. Consider the weight function ϕ on the arcs of $\overline{D_3}$ obtained by assigning unit weights to the arcs $(1,4), (4,2), (2,1)$ and to the loops at 3 and 4, and assigning zero weights to all other arcs. The nonzero weights have been marked in bold-faces in $\overline{D_3}$ in Figure 3. We choose the following positive permutation digraphs with their respective weights in $\overline{D_3}$.*

P_1 - *the loop at 3,*	$\phi^*(P_1) = 1$
P_2 - *the union of the loops at 3 and 4,*	$\phi^*(P_2) = 2$
P_3 - *the 3-cycle $[1,4,2]$,*	$\phi^*(P_3) = 3$
P_4 - *the union of the loop at 3 and the 3-cycle $[1,4,2]$,*	$\phi^*(P_4) = 4.$

Further, the only even cycles in $\overline{D_3}$ are the cycles $[2,3]$ and $[1,3,4,2]$ of weights 0 and 2, respectively. It is clear that ϕ satisfies the conditions (i) and (ii) in Theorem 5.3, and therefore, D_3 has nonnegative Q-completion.

Corollary 5.5. *Let D be a digraph of order n such that*

(i) $\overline{D}$ is stratified, and

(ii) $\overline{D}$ does not have an even cycle.

Then D has nonnegative Q-completion.

Proof. Let $2 \leq k \leq n$. Since $\overline{D}$ is stratified, $\overline{D}$ has a permutation subdigraph P_k of order k. Since $\overline{D}$ does not have an even cycle, $\overline{D}$ does not have any negative permutation subdigraph. Thus, P_k is positive. Further, P_2 being even, it must be composed of two loops, and therefore, $\overline{D}$ has a (positive) permutation subdigraph of order 1 as well. We define $\phi(e) = 1$ for each $e \in A(\overline{D})$. Then the weight function ϕ in $\overline{D}$ satisfies conditions (i) and (ii) of Theorem 5.3. □

Remark 5.6. *One does not expect the converse of Theorem 5.3 to be true. However, the converse holds for all digraphs we have examined, including all digraphs of order at most four. That is, for each of these digraphs which have nonnegative Q-completion, there is a weight function on its complement satisfying the conditions (i) and (ii) in Theorem 5.3.*

5.2. Necessary conditions for nonnegative Q-completion

Proposition 5.7. *Let D be a digraph with at least two vertices. If D has nonnegative Q-completion, then D omits at least two loops.*

Proof. Suppose D omits at most one loop. Let M be a partial nonnegative Q-matrix specifying the digraph D with all specified entries as 0. Then for any nonnegative completion B of M, $S_2(B) \leq 0$. □

The converse of the Proposition 5.7 is not true. The digraph D_4 in Figure 4 omits 2 loops but does not have nonnegative Q-completion. For example,

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_{22} & x_{23} \\ x_{31} & 0 & x_{33} \end{bmatrix}$$

is a partial nonnegative Q-matrix specifying D_4 which has no nonnegative Q-completion. In fact, a

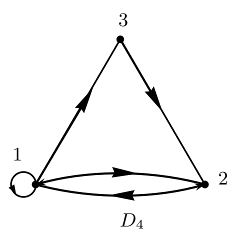


Figure 4. A digraph which does not have nonnegative Q-completion

digraph needs to satisfy stronger conditions to have nonnegative Q-completion, as the following result shows.

Theorem 5.8. *If a digraph D of order n $(n \geq 2)$ has nonnegative Q-completion, then $\overline{D}$ is positively stratified.*

Proof. Suppose D has nonnegative Q-completion. Assume $\overline{D}$ has no positive permutation digraph of order k for some $k \geq 2$. If M is the partial matrix that specifies D with all specified entries zero, and B is a nonnegative completion of M, then all $k \times k$ principal minors of B are nonpositive implying that B is not a nonnegative Q-matrix. □

Corollary 5.9. *Let D be a digraph of order n such that $|A(D)| > n(n-1)$. Then D does not have nonnegative Q-completion.*

Proof. If D has more than $n(n-1)$ arcs (including loops), then $\overline{D}$ has fewer than $n^2 - n(n-1) = n$ arcs. Thus $\overline{D}$ does not contain permutation subdigraph of order n. Therefore, by Theorem 5.8, D does not have nonnegative Q-completion. □

The converse of the Theorem 5.8 is not true, which can be seen from the following example.

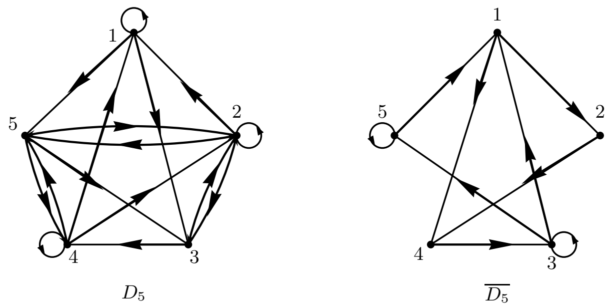


Figure 5. A digraph whose complement is positively stratified

Example 5.10. *The complement $\overline{D_5}$ of the digraph D_5 in Figure 5 is positively stratified. However, we show that D_5 does not have nonnegative Q-completion. Let M be the partial nonnegative Q-matrix specifying D_5 with all specified entries as zero. Consider a nonnegative completion*

$$B = \begin{bmatrix} 0 & x_{12} & 0 & x_{14} & 0 \\ 0 & 0 & 0 & x_{24} & 0 \\ x_{31} & 0 & d_3 & 0 & x_{35} \\ 0 & 0 & x_{43} & 0 & 0 \\ x_{51} & 0 & 0 & 0 & d_5 \end{bmatrix}$$

of M. For B to be a nonnegative Q-matrix, we have

$$S_1(B) = d_3 + d_5 > 0, \tag{4}$$
$$S_2(B) = d_3 d_5 > 0, \tag{5}$$
$$S_3(B) = x_{14}x_{43}x_{31} > 0, \tag{6}$$
$$S_4(B) = d_5 x_{14}x_{43}x_{31} - x_{12}x_{24}x_{43}x_{31} - x_{14}x_{43}x_{35}x_{51} > 0, \tag{7}$$
$$S_5(B) = x_{12}x_{24}x_{43}x_{35}x_{51} - x_{12}x_{24}x_{43}x_{31}d_5 > 0. \tag{8}$$

Clearly, the entries in B against all unspecified entries in M must be positive. Now, from (7) we have $d_5x_{14}x_{31} > x_{12}x_{24}x_{31} + x_{14}x_{35}x_{51}$ which yields $d_5x_{31} > x_{35}x_{51}$. On the other hand, (8) implies that $x_{35}x_{51} > d_5x_{31}$. Hence B cannot be a nonnegative Q-matrix.

Theorem 5.11. *Let D be a digraph with at least four vertices. Suppose $\overline{D}$ has more than one 2-cycle and does not have a 3-cycle. If D has nonnegative Q-completion, then D must omit more than three loops.*

Proof. For D to have nonnegative Q-completion, D must omit at least two loops, by Proposition 5.7. We prove that if D omits exactly three loops, then D does not have nonnegative Q-completion. Then, the result for the case when D omits exactly two loops will follow from Proposition 5.2.

Suppose D omits loops at the vertices $1, 2$ and 3. Now, we label the 2-cycles as $E_1, \dots, E_k$ $(k > 1)$ such that the number of vertices among $1, 2$ and 3 that E_j is incident with is in ascending order in j. Let M be the partial nonnegative Q-matrix specifying the digraph D with all specified entries as zero. Suppose that M has a nonnegative Q-completion $B = [b_{ij}]$. We put $d_1 = b_{11},\ d_2 = b_{22},\ d_3 = b_{33}$. For a 2-cycle $E = \langle r, s\rangle$ in $\overline{D}$, we write $w(E) = w(E, B) = b_{rs}b_{sr}$. Then, by (2) we have

$$S_3(A) = d_1d_2d_3 - (d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_3), \tag{9}$$

where $\sigma_t = \sum\{w(E_i) : E_i \text{ is not incident with } t\},\ t = 1, 2, 3$. Since $S_3(B) > 0$, (9) implies

$$d_1d_2 > \sigma_3,\ d_2d_3 > \sigma_1,\ d_3d_1 > \sigma_2, \tag{10}$$

and therefore

$$d_1d_2 + d_2d_3 + d_3d_1 > \sigma_1 + \sigma_2 + \sigma_3. \tag{11}$$

For $1 \le j \le k$, let

$$\begin{aligned}\beta_j &= \sum\Big\{w(E_i) : E_i \cap E_j = \emptyset \text{ and } i > j\Big\}\\ \gamma_j &= \sum\Big\{d_sd_t : E_j \text{ is incident with none of } s \text{ and } t\Big\}.\end{aligned}$$

Then, by (2) we have

$$S_4(A) = \sum_{j=1}^{k}\beta_j\, w(E_j) - \sum_{j=1}^{k}\gamma_j\, w(E_j) = \sum_{j=1}^{k}(\beta_j - \gamma_j)\, w(E_j). \tag{12}$$

However, we show that $\beta_j - \gamma_j \le 0$ for $1 \le j \le k$. Then, it will follow from (12) that $S_4(B) \le 0$, a contradiction to the fact that B is a Q-matrix. Fix $j \in \{1, 2, \dots, k\}$. We have

$$\gamma_j = \begin{cases} d_1d_2 + d_2d_3 + d_3d_1, & \text{if } E_j \text{ is not incident with any of the vertices } 1, 2, 3,\\ (d_1d_2d_3)/d_t, & \text{if } E_j \text{ is incident with only } t,\ t = 1, 2 \text{ or } 3,\\ 0, & \text{if } E_j \text{ is incident with two of the vertices } 1, 2 \text{ and } 3.\end{cases}$$

If E_j is not incident with any of the vertices $1, 2$ and 3, then from (11) we get

$$\beta_j \le \sum_{i=1}^{k} w(E_i) \le \sigma_1 + \sigma_2 + \sigma_3 < d_1d_2 + d_2d_3 + d_3d_1 = \gamma_j.$$

Next, suppose E_j is incident with exactly one vertex t in $\{1, 2, 3\}$. Consider the case $t = 1$. Since

$$\{E_i : E_i \cap E_j = \emptyset, i > j\} \subseteq \{E_i : 1 \text{ is not incident with } E_i\},$$

we have $\beta_j \le \sigma_1$. Therefore, from (10) we get $\beta_j \le \sigma_1 < d_2d_3 = \gamma_j$. The cases when $t = 2$ and 3 are similar. Finally, assume that E_j is incident to two among the vertices $1, 2$ and 3 so that $\gamma_j = 0$. Now, for any $i > j$ the 2-cycle E_i is incident with two vertices among $1, 2$ and 3. Consequently, by our choice of the ordering of the 2-cycles, E_i and E_j have a vertex in common yielding $\beta_j = 0$. Therefore, our assertion that $\beta_j - \gamma_j \le 0$ for $1 \le j \le k$ holds. $\square$

6. Nonnegative Q-completion of digraphs of small order

We have examined the digraphs of order at most four to nonnegative Q-completion. Clearly, any digraph of order 1 (with or without a loop) has nonnegative Q-completion. Any digraph of order 2 without a loop has nonnegative Q-completion.

There are only four non-isomorphic digraphs of order 3 without loops for which the digraphs obtained by attaching a loop at any of the vertices have nonnegative Q-completion. These digraphs are precisely the spanning subdigraphs of a 3-cycle.

Some types of digraphs of order four with respect to nonnegative Q-completion are presented below.

(a) Let $\widehat{D_6}$ be a digraph obtained from the digraph D_6 in Figure 6 by adding at most two loops at any of its vertices. Then, $\widehat{D_6}$ has nonnegative Q-completion. (In fact, $\widehat{D_6}$ satisfies the conditions of Theorem 5.3.) There are 41 non-isomorphic digraphs of order 4 without loops having similar property as D_6, i.e., all digraphs obtained from them by attaching at most two loops at any of the vertices have nonnegative Q-completion.

Figure 6. The digraph D_6 and its complement $\overline{D_6}$

(b) Let $\widehat{D_7}$ be any digraph obtained from the digraph D_7 in Figure 7 by attaching two loops. If $\widehat{D_7}$ has nonnegative Q-completion, then it must omit a loop at the vertex 2. In fact, in that case $\widehat{D_7}$ satisfies the conditions of Theorem 5.3. There are 66 non-isomorphic digraphs D of order 4 without loops having similar property as D_7, i.e., a digraph obtained from D by attaching at most two loops at its vertices has nonnegative Q-completion only when D omits a loop at a particular vertex.

Figure 7. The digraph D_7 and its complement $\overline{D_7}$

(c) Let $\widehat{D_8}$ be any digraph obtained from the digraph D_8 in Figure 8 by adding a loop at any of the vertices. Then, $\widehat{D_8}$ does not have nonnegative Q-completion (by Theorem 5.11). There are 22 non-isomorphic digraphs of order 4 without loops having similar property as D_8, i.e., any digraph obtained from them by attaching a loop at a vertex does not have nonnegative Q-completion.

Figure 8. The digraph D_8 and its complement $\overline{D_8}$

7. Comparison between Q-completion, and nonnegative and positive Q-completion problems

In this section, we compare the nonnegative Q-completion problem with the Q-completion and the positive Q-completion problems.

Proposition 7.1. *If a digraph D has nonnegative Q-completion, then D has positive Q-completion.*

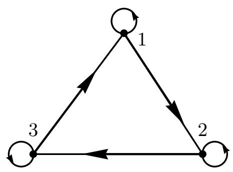
Proof. Suppose D has nonnegative Q-completion. Let $M = [a_{ij}]$ be a partial positive Q-matrix specifying the digraph D. Then M is a partial nonnegative Q-matrix specifying D. Let B be a nonnegative Q-completion of M. Then, perturbing the zero entries in B by small positive quantities, a positive Q-completion of M can be obtained. □

However, the converse is not true which can be seen from the following example.

Example 7.2. *Consider the digraph D_9 in Figure 9. Let $M = [m_{ij}]$ be a partial positive Q-matrix specifying D_9. We write*

$$M = \begin{bmatrix} m_{11} & m_{12} & x_{13} \\ x_{21} & m_{22} & m_{23} \\ m_{31} & x_{32} & m_{33} \end{bmatrix},$$

where x_{ij} are unspecified. It is easy to see that putting sufficiently small values for x_{ij}, M can be completed to a positive Q-matrix, implying that D_9 has positive Q-completion. However, in view of Proposition 5.7, D_9 does not have nonnegative Q-completion, since D_9 has all loops.



D_9

Figure 9. A digraph having positive Q-completion, but not nonnegative Q-completion

The following example shows that a digraph having Q-completion may fail to have nonnegative Q-completion.

Example 7.3. *Consider the digraph D_{10} in Figure 10. Let $M = [m_{ij}]$ be a partial Q-matrix specifying D_{10}. We write,*

$$M = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & m_{22} & m_{23} \\ m_{31} & x_{32} & m_{33} \end{bmatrix},$$

where x_{ij} are unspecified. We put $x_{12} = -x$ and all other unspecified entries as x. It is easy to see that with sufficiently large values for x, M can be completed to a Q-matrix, implying that D_{10} has Q-completion. However, in view of Proposition 5.7, D_{10} does not have nonnegative Q-completion, because it omits only one loop.

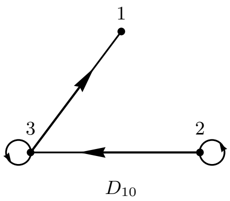


Figure 10. A digraph having Q-completion, but not nonnegative Q-completion

Suppose D is a digraph having nonnegative Q-completion. Then, $\overline{D}$ is stratified and omits at least two loops. For all small digraphs (including all digraph of order 4) having these properties are seen to have Q-completion. Whether a stratified digraph omitting a loop necessarily have Q-completion is not known (see Question 2.9 in [4]). We do not know whether there is a digraph having nonnegative Q completion, but not Q-completion.

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