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On Error Analysis of Systems of Linear Equations with Hadamard Coefficients

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Abstract
In this study, error analysis for linear equation systems whose coefficients matrix is Hadamard matrix is discussed. In the Hx=f problem, the effect of the change in the elements in the f vector at the same rate and the change in the elements in the f vector at different rates were examined, and the relative error and absolute error equations for this system were given.

1. Introduction

A special type of matrix, Hadamard matrices, was first described by James Sylvester in 1867, but after Jacques Hadamard changed the definition of the size of these matrices in 1893, this matrix type was named after him. Looking at the literature, it can be said that the studies on Hadamard matrices have progressed in two directions. The first is the studies on the development of the theory of Hadamard matrices and generally includes the methods of obtaining these matrices, and the second is the studies on the use of Hadamard matrices in different fields. Numerous studies show that Hadamard matrices are used in many areas such as error detection, coding of audio and video signals, code correction and statistics. To get an idea the use of Hadamard matrices in areas such as error correction coding, signal processing and statistics, the book Hadamard Matrix Analysis and Synthesis can be reviewed [1]. Similarly, the book Hadamard Matrices and Their Applications contains sections on the applications of Hadamard matrices in signal processing, coding and cryptography [2].

In this study, in the event that the coefficient matrix of a linear equation system is Hadamard matrix, relative error and absolute error analysis were performed.

2. Introduction of Hadamard Matrices

Let H is the square matrix with all elements ±1, and I_n is the identity matrix of size nxn

$$H.H^T = n.I_n \tag{1}$$

the H matrix that provides the equality is called the n-dimensional Hadamard matrix and is denoted by H_n [1], [2], [3], [4]. Below are 1,2 and 4 dimensional Hadamard matrices

$$[1],\ \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{2}$$

In order for an n-dimensional Hadamard matrix to exist, it must be n=1, n=2, or n=4k (k∈N) [4]. That is, the size of Hadamard matrices is n=1,2,4,8,12,16,20,24,….

2.1. Norm and condition number for the Hx = f problem

Table 1. Condition number of Hadamard matrices in different norms

α	$\|H\|_\alpha$	$\|H^{-1}\|_\alpha$	$K_\alpha(H) = \|H\|_\alpha \|H^{-1}\|_\alpha$
1-norm	n	1	n
2-norm	$\sqrt{n}$	$1/\sqrt{n}$	1
Frobenius-norm	n	1	n
İnfinity-norm	n	1	n

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According to this table, as the size of the system increases (as the H matrix gets larger), the condition number for 1-norm, frobenius-norm andinfinity-norm will also increase. The fact that the condition number is large does not mean that our system is unsolvable or badly conditional. Systems of linear equations with Hadamard coefficients (SLEHC) are solvable systems and have only one solution. This is shown by the condition number according to the 2-norm ($K_2(H) = 1$). The 2-norm best characterizes the condition number for SLEHC.

If Q is an orthogonal matrix, since $K_2(H) = 1$ for the 2-norm, orthogonal matrices are called "perfectly conditioned" [5]. In this case, it can be said that SLEHC is "perfectly defined" since $K_2(H) = 1$.

If H is an n-dimensional Hadamard matrix, the following equations are obtained when 2-norm is used in the $Hx=f$ problem:

$$\|Hx\|_2 = \|H\|_2\|x\|_2 \quad (3)$$

$$\|x\|_2 = \frac{\|f\|_2}{\sqrt{n}} \quad (4)$$

3. Error Analysis for the Hx = f Problem

In error analysis in numerical calculations, there are two ways to measure the size of the error: Absolute error and Relative error [6].

For a vector $x \in \mathrm{R}^n$, where $\|.\|$ is any vector norm, the relative and absolute error are as follows [5]

$$\varepsilon_{absolute} = \|x - \hat{x}\| \quad (5)$$

$$\varepsilon_{relative} = \frac{\varepsilon_{absolute}}{\|x\|} = \frac{\|x-\hat{x}\|}{\|x\|} \;,\; x \neq 0 \quad (6)$$

3.1. Error evaluation 1

Since H is a fixed matrix in the $Hx = f$ system, errors in the solution of this problem would result from changes in the f vector. Consider the problem $H_4x = f$, whose real solution is given below for the 4-dimensional Hadamard

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1{,}1345 \\ 2{,}4567 \\ 3{,}5678 \\ 4{,}6789 \end{bmatrix}$$

$$x = \begin{bmatrix} 2{,}959475 \\ -0{,}052775 \\ -1{,}163875 \\ -0{,}608325 \end{bmatrix}$$

Now let's see how the errors in the f vector will be reflected in the result.

Let there be an error of 0.0001 in one element of f (on the first element).

$$\hat{f} = \begin{bmatrix} 1{,}1346 \\ 2{,}4567 \\ 3{,}5678 \\ 4{,}6789 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} 2{,}9595 \\ -0{,}05275 \\ -1{,}16385 \\ -0{,}6083 \end{bmatrix}$$

$$\hat{x} - x = \begin{bmatrix} 0{,}000025 \\ 0{,}000025 \\ 0{,}000025 \\ 0{,}000025 \end{bmatrix}$$

A difference of 10^{-4} in one element of f is reflected in each element of the result vector as a difference of $\mp 25.10^{-6}$. Absolute error here:

$\|\hat{x} - x\| = 50.10^{-6}$

The result is a change of 0.0001 in two elements of f (selected 2nd and 3rd):

$$\hat{f} = \begin{bmatrix} 1{,}1345 \\ 2{,}4568 \\ 3{,}5679 \\ 4{,}6789 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} 2{,}959525 \\ -0{,}052825 \\ -1{,}163875 \\ -0{,}608325 \end{bmatrix}$$

$$\hat{x} - x = \begin{bmatrix} 0{,}00005 \\ -0{,}00005 \\ 0{,}00000 \\ 0{,}00000 \end{bmatrix} \quad \|\hat{x} - x\| = \sqrt{2}.50.10^{-6}$$

A difference of 10^{-4} in the two elements of f is reflected as a difference of $\mp 50.10^{-6}$ in the two elements of the result vector.

With a change of 0.0001 in the three elements of f (selected 1,2 and 4), the result is:

$$\hat{f} = \begin{bmatrix} 1{,}1346 \\ 2{,}4568 \\ 3{,}5678 \\ 4{,}679 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} 2{,}95955 \\ -0{,}05275 \\ -1{,}16385 \\ -0{,}608385 \end{bmatrix}$$

$$\hat{x} - x = \begin{bmatrix} 0{,}000075 \\ 0{,}000025 \\ 0{,}000025 \\ -0{,}000025 \end{bmatrix} \quad \|\hat{x} - x\| = \sqrt{3}.50.10^{-6}$$

With a change of 0.0001 in the four elements of f, the result is:

$$\hat{f} = \begin{bmatrix} 1{,}1346 \\ 2{,}4568 \\ 3{,}5679 \\ 4{,}679 \end{bmatrix} \quad \hat{x} = \begin{bmatrix} 2{,}959575 \\ -0{,}052775 \\ -1{,}163875 \\ -0{,}608325 \end{bmatrix}$$

$$\hat{x} - x = \begin{bmatrix} 0{,}0001 \\ 0{,}0000 \\ 0{,}0000 \\ 0{,}0000 \end{bmatrix} \|\hat{x} - x\| = 0{,}000 = 2.50.10^{-6}$$

Now, Can we write the above results as follows?

$$\|\hat{x}-x\| = \sqrt{number\ of\ changing\ elements}.\frac{1}{\sqrt{n}}. \quad (7)$$

To answer this question, it was applied for H_8, similar to the work done for H_4 above and it was seen that the above formula is provided for the absolute error that will ocur when there are differences of $\varepsilon = 10^{-4}$, $\varepsilon = 10^{-3}$, $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-1}$, in one or more elements of the *f* vector [7].

Here, the reason why ε is chosen as $\varepsilon = 10^{-k}$ instead of any number is to give an idea about rounding errors. In the rounding of numbers, while the section after the decimal part to be rounded is discarded, the previous digit will either stay the same or increase by 1. Equality (7) is valid when the elements of vector f are changed at the same rate. If different changes are made in the elements of f, what can be said about how the system behaves and how these differences are reflected in the result? The answer to this question is covered in 3.2.

3.2. Error evaluation 2

Let $\hat{x}$ be the approximate solution of the linear equation system $Ax = f$. The cause of the error may be rounding errors in the operation process or incorrect input of the matrix *A* and the vector *f*.
e, solution error ($e = x - \hat{x}$) provides the following system

$$Ae = r \quad (8)$$

Here r is as follows and is called the remaining vector or residual vector [8]

$$r = f - A\hat{x} \quad (9)$$

There is the following inequality with $r = f - A\hat{x}$ and $e = x - \hat{x}$ [8]:

$$\frac{1}{K(A)}\frac{\|r\|}{\|f\|} \le \frac{\|e\|}{\|x\|} \le K(A)\frac{\|r\|}{\|f\|} \quad (10)$$

According to this theorem, the lower limit of the relative error in the solution of the Ax = f system is $\frac{1}{K(A)}\frac{\|r\|}{\|f\|}$ and the upper limit is $K(A)\frac{\|r\|}{\|f\|}$. Now let's apply this theorem to the Hadamard matrix:
Since $K_2(A)$=1 for SLEHC, inequality (10)

$$\frac{\|r\|_2}{\|f\|_2} \le \frac{\|e\|_2}{\|x\|_2} \le \frac{\|r\|_2}{\|f\|_2} \quad (11)$$

takes the form. Hence,

$$\|r\|_2\|x\|_2 = \|e\|_2\|f\|_2 \quad (12)$$

equality is achieved.

Since $r = f - \hat{f}$ and $e = x - \hat{x}$ as a result, according to the 2-norm for the relative error of the $Hx = f$ problem;

$$\frac{\|x-\hat{x}\|_2}{\|x\|_2} = \frac{\|f-\hat{f}\|_2}{\|f\|_2} \quad (13)$$

is obtained.

3.3. Error evaluation 3

There are two cases for errors in the f vector:

- ε the amount of error is the same or different in each element
- The error exists in every element or some elements

The case where all elements of *f* change at the same rate

$$Hx = f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad H\hat{x} = \hat{f} = \begin{bmatrix} f_1+\varepsilon \\ f_2+\varepsilon \\ \vdots \\ f_n+\varepsilon \end{bmatrix}, \hat{f}-f = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \end{bmatrix}$$

$$\|x-\hat{x}\|_2 = \frac{\|f-\hat{f}\|_2}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sqrt{\varepsilon^2+\varepsilon^2+\cdots+\varepsilon^2} = \frac{1}{\sqrt{n}}\sqrt{n\varepsilon^2} = \varepsilon$$

The case where some elements of f (k items) change at the same rate

$$Hx = f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad H\hat{x} = \hat{f} = \begin{bmatrix} f_1+\varepsilon \\ f_2+\varepsilon \\ \vdots \\ f_k+\varepsilon \\ f_{k+1} \\ \vdots \\ f_n \end{bmatrix},$$

$$\hat{f}-f = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\|x-\hat{x}\|_2 = \frac{\|f-\hat{f}\|_2}{\sqrt{n}} = \frac{1}{\sqrt{n}}\sqrt{\varepsilon^2+\varepsilon^2+\cdots+\varepsilon^2} = \frac{1}{\sqrt{n}}\sqrt{k\varepsilon^2} = \sqrt{\frac{k}{n}}\varepsilon$$

The case where all elements of f change at different rates

$$Hx = f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad H\hat{x} = \hat{f} = \begin{bmatrix} f_1+\varepsilon_1 \\ f_2+\varepsilon_2 \\ \vdots \\ f_n+\varepsilon_n \end{bmatrix},$$

$$\hat{f}-f=\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_n\end{bmatrix}$$

$$\|x-\hat{x}\|_2=\frac{\|f-\hat{f}\|_2}{\sqrt{n}}=\frac{1}{\sqrt{n}}\sqrt{\varepsilon_1^2+\varepsilon_2^2+\cdots+\varepsilon_n^2}$$
$$=\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}\varepsilon_i^2}$$

The case where some elements of f (k items) change at different rates

$$Hx=f=\begin{bmatrix}f_1\\ f_2\\ \vdots\\ f_n\end{bmatrix},\quad H\hat{x}=\hat{f}=\begin{bmatrix}f_1+\varepsilon_1\\ f_2+\varepsilon_2\\ \vdots\\ f_k+\varepsilon_k\\ f_{k+1}\\ \vdots\\ f_n\end{bmatrix},$$

$$\hat{f}-f=\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \vdots\\ \varepsilon_k\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$$\|x-\hat{x}\|_2=\frac{\|f-\hat{f}\|_2}{\sqrt{n}}=\frac{1}{\sqrt{n}}\sqrt{\varepsilon_1^2+\varepsilon_2^2+\cdots+\varepsilon_k^2}$$
$$=\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{k}\varepsilon_i^2}$$

So if we generalize the above four cases: using (13);

$$\frac{\|x-\hat{x}\|_2}{\|x\|_2}=\frac{\|f-\hat{f}\|_2}{\|f\|_2}\xrightarrow{\|x\|_2=\frac{\|f\|_2}{\sqrt{n}}}\|x-\hat{x}\|_2$$
$$=\frac{\|f\|_2}{\sqrt{n}}\frac{\|f-\hat{f}\|_2}{\|f\|_2}=\frac{\|f-\hat{f}\|_2}{\sqrt{n}}$$

$$\|x-\hat{x}\|_2=\frac{\|f-\hat{f}\|_2}{\sqrt{n}} \tag{14}$$

is obtained. This results in

$$\|x-\hat{x}\|_2=\frac{1}{\sqrt{n}}\sqrt{\varepsilon_1^2+\varepsilon_2^2+\cdots+\varepsilon_k^2}$$
$$=\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{k}\varepsilon_i^2} \tag{15}$$

If necessary, subheadings can be added under the main heading [5].

4. Results and Discussion

While the relative error and absolute error in the solution of the $Ax=f$ problem for any matrix A are evaluated according to the lower and upper limits, it can be found exactly how much error was made in the solution of the SLEHC. This error is due to the change in the f vector. That is, if the error in the *f* vector is known exactly, the relative error in the solution of the system can be found with the following equation:

$$\frac{\|x-\hat{x}\|_2}{\|x\|_2}=\frac{\|f-\hat{f}\|_2}{\|f\|_2} \tag{16}$$

Absolute error as a result of changes in the *f* vector can be calculated using the (following) equation

$$\|x-\hat{x}\|_2=\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{k}\varepsilon_i^2} \tag{17}$$

As a result of this equation, the following equations can be written:

For $\varepsilon_i=\varepsilon$; $\quad k=n\Rightarrow\|x-\hat{x}\|_2=\varepsilon$

$\quad k<n\Rightarrow\|x-\hat{x}\|_2=\frac{\sqrt{k}}{\sqrt{n}}\varepsilon$

For $\varepsilon_i\neq\varepsilon_j$; $\quad k=n\Rightarrow\|x-\hat{x}\|_2=\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}\varepsilon_i^2}$

$\quad k<n\Rightarrow\|x-\hat{x}\|_2=\frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{k}\varepsilon_i^2}$

Equation (17) shows that the absolute error in solving the problem $Hx=f$ is related to the size of the system (in other words, the size of the *H* matrix).

Theoretically, (15) and (17) are formulas that give the absolute error exactly, but if we consider that this calculation is in practice made in a computer environment, there will be a rounding error again due to the computer's ability to store numbers in a limited capacity. However, since the *Hx=f* system is a "perfectly conditioned" problem, the error in question will be quite small. That is the important thing is how sensitive the error is intended to be.

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Statement of Research and Publication Ethics

The study is complied with research and publication ethics

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