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On the metric dimension of rotationally-symmetric convex polytopes*

Research Article

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Abstract: Metric dimension is a generalization of affine dimension to arbitrary metric spaces (provided a resolving set exists). Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_n \geq 1$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim\limits_{n\to\infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $dim(G_n) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension, otherwise $\mathcal{F}$ has unbounded metric dimension. If all graphs in $\mathcal{F}$ have the same metric dimension, then $\mathcal{F}$ is called a family of graphs with constant metric dimension.
In this paper, we study the metric dimension of some classes of convex polytopes which are rotationally-symmetric. It is shown that these classes of convex polytoes have the constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these classes of convex polytopes. It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension.

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1. Notation and preliminary results

Slater refereed to the metric dimension of a graph as its *location number* and motivated the study of this invariant by its application to the placement of a minimum number of sonar/loran detecting devices

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in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set ([18],[19]). These concepts have also some applications in chemistry for representing chemical compounds ([5],[12]) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [16].

If G is a connected graph, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$. W is called a *resolving set* [5] or *locating set* [18] if every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is the *metric dimension* or *location number* of G, denoted by $dim(G)$ [3]. The concepts of resolving set and metric basis have previously appeared in the literature (see [3-6, 8-12, 15-21]).

For a given ordered set of vertices $W = \{w_1, w_2, \ldots, w_k\}$ of a graph G, the ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.

A useful property in finding $dim(G)$ is the following lemma:

Lemma 1.1. *[20] Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.*

By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [3] determined the dimension of *wheel* W_n, Caceres *et al.* [8] the dimension of *fan* f_n and Tomescu and Javaid [21] the dimension of *Jahangir graph* J_{2n}.

Theorem 1.2. *([3], [8], [21]) Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*
(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

The metric dimension of all these plane graphs depends upon the number of vertices in the graph. On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$. In [5] it was shown that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices n. Javaid *et al.* proved in [11] that the plane graph *antiprism* A_n constitute a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [13]. The metric dimension of cartesian product of graphs has been discussed in [4, 17].

The metric dimension of some classes of *convex polytopes* has been determined in [9] and [10] where it was shown that these classes of convex polytopes have constant metric dimension 3 and following open problems were raised in [9] and [10].
Open problem [9]: Is it the case that the graph of every convex polytope has constant metric dimension?
Open problem [10]: Let G' be the graph of a convex polytope obtained from the graph of convex polytope G by adding extra edges in G such that $V(G') = V(G)$. Is it the case that G' and G will always have the same metric dimension?

Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [6]. Some bounds for this invariant, in terms of the diameter of the graph, are given in [15] and it was shown in [5, 15–17] that the metric dimension of *trees* can be determined efficiently. It appears unlikely that significant progress can be made in determining the dimension of a graph unless it belongs to a class for which the distances between vertices can be described in some systematic manner.

Bača defined in [2] the graph of convex polytope R_n which is obtained as a combination of the graph of a prism and the graph of an antiprism. The prism and antiprism have constant metric dimension [4, 11] and it was proved in [9] that the graph of convex polytope R_n also has constant metric dimension. In this paper, we extend this study to some classes of convex polytopes which are obtained by combination of two different graph of convex polytopes. We prove that these classes of convex polytopes have constant metric dimension and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of convex polytopes. In what follows all indices i which do not satisfy the given inequalities will be taken modolu n.

2. The graph of convex polytope B_n

The graph of convex polytope B_n (Fig. 1) consisting of $2n$ 4-sided faces, n 3-sided faces, n 5-sided faces and a pair of n-sided faces is obtained by the combination of the graph of convex polytope Q_n [2] and graph of a prism D_n. We have

$$V(B_n) = \{a_i; b_i; c_i; d_i, e_i : 1 \leq i \leq n\}$$

and

$$\begin{aligned} E(B_n) = &\{a_ia_{i+1}; b_ib_{i+1}; d_id_{i+1}; e_ie_{i+1} : 1 \leq i \leq n\} \\ &\cup\{a_ib_i; b_ic_i; b_{i+1}c_i; c_id_i; d_ie_i : 1 \leq i \leq n\}. \end{aligned}$$

For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle, cycle induced by

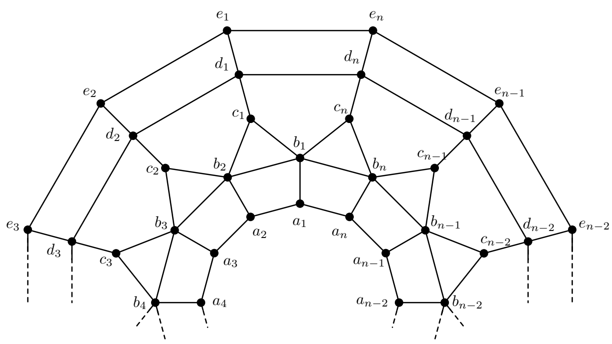

Figure 1. The graph of convex polytope B_n

$\{b_i : 1 \leq i \leq n\}$, the interior cycle, cycle induced by $\{d_i : 1 \leq i \leq n\}$, the exterior cycle, cycle induced by $\{e_i : 1 \leq i \leq n\}$, the outer cycle and set of vertices $\{c_i : 1 \leq i \leq n\}$, the set of interior vertices.
The metric dimension of graph of convex polytope Q_n and graph of a prism D_n have been studied in [9] and [4]. In the next theorem, we show that the metric dimension of the graph of convex polytope B_n is 3. Note that the choice of appropriate basis vertices (also referred to as landmarks in [14]) is core of the problem.

Theorem 2.1. *For $n \geq 6$, let the graph of convex polytopes be B_n; then $dim(B_n) = 3$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.
Case(i) When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(B_n)$, we show that W is a resolving set for B_n in this case. For this we give representations of any vertex of $V(B_n)\backslash W$ with

respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \le i \le k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations of the set of interior vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \le i \le k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \le i \le k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \le i \le 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4, 4, k+3), & i = 1; \\ (i+3, i+2, k-i+4), & 2 \le i \le k; \\ (k+3, k+3, 4), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+3), & k+2 \le i \le 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(B_n) \le 3$. On the other hand, we show that $dim(B_n) \ge 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(B_n) = 2$, then there are following possibilities to be discussed.
(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1, a_t\}) = r(b_1|\{a_1, a_t\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.
(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(b_n|\{b_1, b_t\}) = r(c_n|\{b_1, b_t\}) = (1, t)$ and for $t = k+1$, $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\}) = (1, k-1)$, a contradiction.
(3) Both vertices are in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(b_1|\{c_1, c_t\}) = r(d_1|\{c_1, c_t\}) = (1, t)$ and for $t = k+1$, $r(d_2|\{c_1, c_{k+1}\}) = r(d_n|\{c_1, c_{k+1}\}) = (1, k-1)$, a contradiction.
(4) Both vertices are in the exterior cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(c_1|\{d_1, d_t\}) = r(d_n|\{d_1, d_t\}) = (1, t)$ and for $t = k+1$, $r(d_2|\{d_1, d_{k+1}\}) = r(d_n|\{d_1, d_{k+1}\}) = (1, k-1)$, a contradiction.
(5) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(d_1|\{e_1, e_t\}) = r(e_n|\{e_1, e_t\}) = (1, t)$ and for $t = k+1$, $r(e_2|\{e_1, e_{k+1}\}) = r(e_n|\{e_1, e_{k+1}\}) = (1, k-1)$, a contradiction.
(6) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose

that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \le t \le k+1)$. Then for $t=1$, we have $r(a_2|\{a_1,b_1\}) = r(a_n|\{a_1,b_1\}) = (1,2)$ and when $2 \le t \le k+1$, $r(a_2|\{a_1,b_{k+1}\}) = r(b_1|\{a_1,b_{k+1}\}) = (1,t-1)$, a contradiction.
(7) One vertex is in the inner cycle and other in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(b_n|\{a_1,c_t\}) = r(c_n|\{a_1,c_t\}) = (2,t+1)$ and when $t = k+1$, $r(a_3|\{a_1,b_{k+1}\}) = r(c_n|\{a_1,b_{k+1}\}) = (2,k)$, a contradiction.
(8) One vertex is in the inner cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $t=1$, we have $r(d_2|\{a_1,d_1\}) = r(e_1|\{a_1,d_1\}) = (4,1)$. If $2 \le t \le k$, $r(a_2|\{a_1,d_t\}) = r(b_1|\{a_1,d_t\}) = (1,t+1)$ and when $t=k+1$, $r(a_n|\{a_1,d_{k+1}\}) = r(b_1|\{a_1,$
$d_{k+1}\}) = (1,k+1)$, a contradiction.
(9) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $t=1$, we have $r(b_2|\{a_1,e_1\}) = r(c_n|\{a_1,e_1\}) = (2,3)$ and when $2 \le t \le k+1$, $r(b_2|\{a_1,e_t\}) = r(c_1|\{a_1,e_t\}) = (2,t+1)$, a contradiction.
(10) One vertex is in the interior cycle and other in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(b_n|\{b_1,c_t\}) = r(c_n|\{b_1,c_t\}) = (1,t+1)$. For $t=k$, we have $r(c_1|\{b_1,c_k\}) = r(b_n|\{b_1,c_k\}) = (1,k)$ and when $t=k+1$, $r(b_2|\{b_1,c_{k+1}\}) = r(c_n|\{b_1,c_{k+1}\}) = (1,k)$, a contradiction.
(11) One vertex is in the interior cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(a_1|\{b_1,d_t\}) = r(b_n|\{b_1,d_t\}) = (1,t+2)$. For $t=k$, we have $r(b_n|\{b_1,d_k\}) = r(c_1|\{b_1,d_k\}) = (1,k+1)$ and when $t=k+1$, $r(b_2|\{b_1,d_{k+1}\}) = r(c_n|\{b_1,d_{k+1}\}) = (1,k+1)$, a contradiction.
(12) One vertex is in the interior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $1 \le t \le k-1$, we have $r(a_1|\{b_1,e_t\}) = r(b_n|\{b_1,e_t\}) = (1,t+3)$. For $t=k$, we have $r(b_n|\{b_1,e_k\}) = r(c_1|\{b_1,e_k\}) = (1,k+2)$ and when $t=k+1$, $r(b_2|\{b_1,e_{k+1}\}) = r(c_n|\{b_1,e_{k+1}\}) = (1,k+2)$, a contradiction.
(13) One vertex is in the set of interior vertices and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is d_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, $r(a_1|\{c_1,d_t\}) = r(b_n|\{c_1,d_t\}) = (2,t+2)$ and when $t=k+1$, we have $r(d_2|\{c_1,d_{k+1}\}) = r(d_n|\{c_1,d_{k+1}\}) = (2,k)$, a contradiction.
(14) One vertex is in the set of interior vertices and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(a_1|\{c_1,e_t\}) = r(b_n|\{c_1,e_t\}) = (2,t+3)$ and when $t=k+1$, $r(e_2|\{c_1,e_{k+1}\}) = r(e_n|\{c_1,e_{k+1}\}) = (3,k-1)$, a contradiction.
(15) One vertex is in the set of exterior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(c_1|\{d_1,e_t\}) = r(d_n|\{d_1,e_t\}) = (1,t+1)$ and when $t=k+1$, we have $r(e_2|\{d_1,e_{k+1}\}) = r(e_n|\{d_1,e_{k+1}\}) = (2,k)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(B_n)$ implying that $dim(B_n) = 3$ in this case.
Case(ii) When n is odd.
In this case, we can write $n = 2k+1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Again we show that $W = \{a_1, a_2, a_{k+1}\} \subset V(B_n)$ is a resolving set for B_n in this case. For this we give representations of any vertex of $V(B_n)\backslash W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (k,k,1), & i = k+2 \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i=1; \\ (i,i-1,k-i+2), & 2 \le i \le k+1; \\ (k+1,k+1,2), & i=k+2; \\ (2k-i+3,2k-i+4,i-k), & k+3 \le i \le 2k+1. \end{cases}$$

Representations of set of interior vertices are

$$r(c_i|W) = \begin{cases} (2,2,k+1), & i=1; \\ (i+1,i,k-i+2), & 2 \le i \le k; \\ (k+2,k+1,2), & i=k+1; \\ (2k-i+3,2k-i+4,i-k+1), & k+2 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(d_i|W) = \begin{cases} (3,3,k+2), & i=1; \\ (i+2,i+1,k-i+3), & 2 \le i \le k; \\ (k+3,k+2,3), & i=k+1; \\ (2k-i+4,2k-i+5,i-k+2), & k+2 \le i \le 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+3), & i=1; \\ (i+3,i+2,k-i+4), & 2 \le i \le k; \\ (k+4,k+3,4), & i=k+1; \\ (2k-i+5,2k-i+6,i-k+3), & k+2 \le i \le 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(B_n) \le 3$.
On the other hand, suppose that $dim(B_n) = 2$, then there are the same possibilities as in case (**i**) and contradiction can be deduced analogously. This implies that $dim(B_n) = 3$ in this case, which completes the proof. □

3. The graph of convex polytope $\mathbb{C}_n$

The graph of convex polytope $\mathbb{C}_n$ (Fig. 2) consisting of $3n$ 3-sided faces, n 4-sided faces, n 5-sided faces and a pair of n-sided faces is obtained by the combination of the graph of convex polytope Q_n [2] and graph of an antiprism A_n [1]. We have

$$V(\mathbb{C}_n) = \{a_i; b_i; c_i; d_i; e_i : 1 \le i \le n\}$$

and

$$\begin{aligned} E(\mathbb{C}_n) = &\{a_i a_{i+1}; b_i b_{i+1}; d_i d_{i+1}; e_i e_{i+1} : 1 \le i \le n\} \\ &\cup \{a_i b_i; b_i c_i; c_i d_i; d_i e_i; b_{i+1} c_i; d_{i+1} e_i : 1 \le i \le n\}. \end{aligned}$$

The graph of convex polytope $\mathbb{C}_n$ can also be obtained from the graph of convex polytope B_n by adding new edges $d_{i+1}e_i$ and having the same vertex set. i.e. $V(\mathbb{C}_n) = V(B_n)$ and $E(\mathbb{C}_n) = E(B_n) \cup \{d_{i+1}e_i : 1 \le i \le n\}$.
For our purpose, we call the cycle induced by $\{a_i : 1 \le i \le n\}$, the inner cycle, cycle induced by $\{b_i : 1 \le i \le n\}$, the interior cycle, cycle induced by $\{d_i : 1 \le i \le n\}$, the exterior cycle, cycle induced by $\{e_i : 1 \le i \le n\}$, the outer cycle and set of vertices $\{c_i : 1 \le i \le n\}$, the set of interior vertices.
The metric dimension of graph of convex polytope Q_n and graph of an antiprism A_n have been studied in [9] and [11]. In the next theorem, we show that the metric dimension of the graph of convex polytope $\mathbb{C}_n$ is 3. Again, choice of appropriate landmarks is crucial.

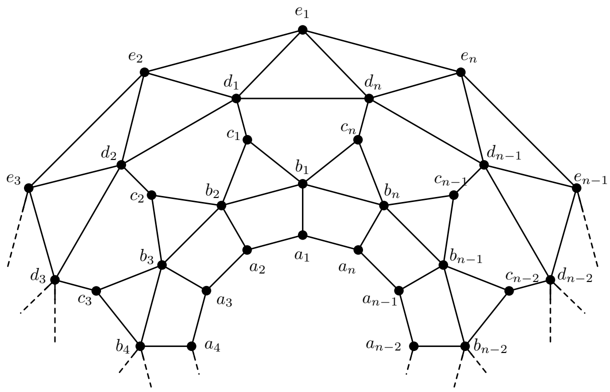

Figure 2. The graph of convex polytope $\mathbb{C}_n$

Theorem 3.1. *Let $\mathbb{C}_n$ denotes the graph of convex polytope; then $dim(\mathbb{C}_n) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.
Case(i) When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(\mathbb{C}_n)$, we show that W is a resolving set for $\mathbb{C}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{C}_n)\backslash W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (2k-i+2, 2k-i+3, i-k), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the set of interior vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+1, k+1, 2), & i = k+1; \\ (2k-i+2, 2k-i+3, i-k+1), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4, 4, k+2), & i = 1; \\ (i+3, i+2, k-i+3), & 2 \leq i \leq k-1; \\ (k+3, k+2, 4), & i = k; \\ (2k-i+3, 2k-i+4, i-k+3), & k+1 \leq i \leq 2k-1; \\ (4, 4, k+3), & i = 2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(\mathbb{C}_n) \leq 3$. On the other hand, we show that $dim(\mathbb{C}_n) \geq 3$ by proving that there is no resolving set W such that $|W| = 2$. Suppose on contrary that $dim(\mathbb{C}_n) = 2$, then there are following possibilities to be discussed.
(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(a_n|\{a_1, a_t\}) = r(b_1|\{a_1, a_t\}) = (1, t)$ and for $t = k+1$, $r(a_2|\{a_1, a_{k+1}\}) = r(a_n|\{a_1, a_{k+1}\}) = (1, k-1)$, a contradiction.
(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(b_n|\{b_1, b_t\}) = r(c_n|\{b_1, b_t\}) = (1, t)$ and for $t = k+1$, $r(b_2|\{b_1, b_{k+1}\}) = r(b_n|\{b_1, b_{k+1}\}) = (1, k-1)$, a contradiction.
(3) Both vertices are in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(b_1|\{c_1, c_t\}) = r(d_1|\{c_1, c_t\}) = (1, t)$ and for $t = k+1$, $r(d_2|\{c_1, c_{k+1}\}) = r(d_n|\{c_1, c_{k+1}\}) = (1, k-1)$, a contradiction.
(4) Both vertices are in the exterior cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(c_1|\{d_1, d_t\}) = r(d_n|\{d_1, d_t\}) = (1, t)$ and for $t = k+1$, $r(d_2|\{d_1, d_{k+1}\}) = r(d_n|\{d_1, d_{k+1}\}) = (1, k-1)$, a contradiction.
(5) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(d_1|\{e_1, e_t\}) = r(e_n|\{e_1, e_t\}) = (1, t)$ and for $t = k+1$, $r(e_2|\{e_1, e_{k+1}\}) = r(e_n|\{e_1, e_{k+1}\}) = (1, k-1)$, a contradiction.
(6) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, b_1\}) = r(a_n|\{a_1, b_1\}) = (1, 2)$ and when $2 \leq t \leq k+1$, $r(a_2|\{a_1, b_t\}) = r(b_1|\{a_1, b_t\}) = (1, t-1)$, a contradiction.
(7) One vertex is in the inner cycle and other in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(b_n|\{a_1, c_t\}) = r(c_n|\{a_1, c_t\}) = (2, t+1)$ and when $t = k+1$, $r(a_3|\{a_1, b_{k+1}\}) = r(c_n|\{a_1, b_{k+1}\}) = (2, k)$, a contradiction.
(8) One vertex is in the inner cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(d_2|\{a_1, d_1\}) = r(e_1|\{a_1, d_1\}) = (4, 1)$. If $2 \leq t \leq k$, $r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_n|\{a_1, d_{k+1}\}) = r(b_1|\{a_1, d_{k+1}\}) = (1, k+1)$, a contradiction.
(9) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(b_2|\{a_1, e_t\}) = r(c_n|\{a_1, e_t\}) = (2, 3)$ and when $2 \leq t \leq k+1$, $r(b_2|\{a_1, e_t\}) = r(c_1|\{a_1, e_t\}) = (2, t+1)$, a contradiction.
(10) One vertex is in the interior cycle and other in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(b_n|\{b_1, c_t\}) = r(c_n|\{b_1, c_t\}) = (1, t+1)$. For $t = k$, we have $r(c_1|\{b_1, c_k\}) = r(b_n|\{b_1, c_k\}) = (1, k)$ and when $t = k+1$, $r(b_2|\{b_1, c_{k+1}\}) = r(c_n|\{b_1, c_{k+1}\}) = (1, k)$, a contradiction.
(11) One vertex is in the interior cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1, d_t\}) = r(b_n|\{b_1, d_t\}) = (1, t+2)$. For $t = k$, we have $r(b_n|\{b_1, d_k\}) = r(c_1|\{b_1, d_k\}) = (1, k+1)$ and when $t = k+1$, $r(b_2|\{b_1, d_{k+1}\}) = r(c_n|\{b_1, d_{k+1}\}) = (1, k+1)$, a contradiction.
(12) One vertex is in the interior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1, e_t\}) = r(b_n|\{b_1, e_t\}) = (1, t+3)$. For $t = k$, we have $r(b_n|\{b_1, e_k\}) = r(c_1|\{b_1, e_k\}) = (1, k+2)$ and when $t = k+1$, $r(b_2|\{b_1, e_{k+1}\}) = r(c_n|\{b_1, e_{k+1}\}) = (1, k+1)$, a contradiction.

(13) One vertex is in the set of interior vertices and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(a_1|\{c_1, d_t\}) = r(b_n|\{c_1, d_t\}) = (2, t+2)$, and when $t = k+1$, $r(d_2|\{c_1, d_{k+1}\}) = r(d_n|\{c_1, d_{k+1}\}) = (2, k)$, a contradiction.
(14) One vertex is in the set of interior vertices and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(a_1|\{c_1, e_t\}) = r(b_n|\{c_1, e_t\}) = (2, t+3)$ and when $t = k+1$, $r(c_n|\{c_1, e_{k+1}\}) = r(e_1|\{c_1, e_{k+1}\}) = (2, k)$, a contradiction.
(15) One vertex is in the set of exterior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k$, we have $r(c_1|\{d_1, e_t\}) = r(d_n|\{d_1, e_t\}) = (1, t+1)$ and when $t = k+1$, $r(d_n|\{d_1, e_{k+1}\}) = r(e_n|\{d_1, e_{k+1}\}) = (1, k-1)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(\mathbb{C}_n)$ implying that $dim(\mathbb{C}_n) = 3$ in this case.
Case(ii) When n is odd.
In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(\mathbb{C}_n)$, we show that W is a resolving set for $\mathbb{C}_n$ in this case. For this we give representations of any vertex of $V(\mathbb{C}_n) \backslash W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & i = 1; \\ (i, i-1, k-i+2), & 2 \leq i \leq k+1; \\ (k+1, k+1, 2), & i = k+2; \\ (2k-i+3, 2k-i+4, i-k), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of interior vertices are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(d_i|W) = \begin{cases} (3, 3, k+2), & i = 1; \\ (i+2, i+1, k-i+3), & 2 \leq i \leq k; \\ (k+3, k+2, 3), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+2), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4, 4, k+2), & i = 1; \\ (i+3, i+2, k-i+3), & 2 \leq i \leq k-1; \\ (k+3, k+2, 4), & i = k; \\ (k+3, k+3, 4), & i = k+1; \\ (2k-i+4, 2k-i+5, i-k+3), & k+2 \leq i \leq 2k; \\ (4, 4, k+3), & i = 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(\mathbb{C}_n) \leq 3$ in this case.
On the other hand, suppose that $dim(\mathbb{C}_n) = 2$, then there are the same subcases as in case (**i**) and contradiction can be obtained analogously. This implies that $dim(\mathbb{C}_n) = 3$ in this case, which completes the proof. □

Note that the result in above theorem gives the positive answer to the open problem raised in [10] in this case.

4. The graph of convex polytope E_n

The graph of convex polytope E_n is obtained as a combination of graph of convex polytope T_n [10] and graph of an antiprism A_n [1]. The graph of convex polytope E_n can also be obtained from the graph

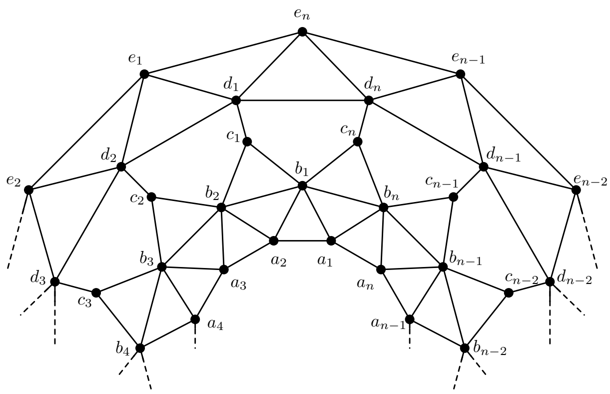

Figure 3. The graph of convex polytope E_n

of convex polytope $\mathbb{C}_n$ by adding new edges $a_{i+1}b_i$ and having the same vertex set. i.e. $V(E_n) = V(\mathbb{C}_n)$ and $E(E_n) = E(\mathbb{C}_n) \cup \{a_{i+1}b_i : 1 \leq i \leq n\}$.
For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle, cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle, cycle induced by $\{d_i : 1 \leq i \leq n\}$, the exterior cycle, cycle induced by $\{e_i : 1 \leq i \leq n\}$, the outer cycle and set of vertices $\{c_i : 1 \leq i \leq n\}$, the set of interior vertices.

The metric dimension of graph of convex polytope T_n and graph of a prism D_n have been studied in [10] and [4]. In the next theorem, we show that the metric dimension of the graph of convex polytope E_n is 3. Once again, the choice of appropriate landmarks is crucial.

Theorem 4.1. *Let E_n denotes the graph of convex polytope; then $dim(E_n) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities. We consider the two cases.
Case(i) When n is even.
In this case, we can write $n = 2k$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_3, a_{k+1}\} \subset V(E_n)$, we show that W is a resolving set for E_n in this case. For this we give representations of any vertex of $V(E_n)\backslash W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (1, 1, k-1), & i = 2; \\ (i-1, i-3, k-i+1), & 4 \leq i \leq k; \\ (k-1, k-1, 1), & i = k+2; \\ (2k-i+1, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i=1; \\ (2,1,k-1), & i=2; \\ (i,i-2,k-i+1), & 3 \le i \le k; \\ (k,k-1,1), & i=k+1; \\ (k-1,k,2), & i=k+2; \\ (2k-i+1,2k-i+3,i-k), & k+3 \le i \le 2k. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2,2,k), & i=1; \\ (3,2,k-1), & i=2; \\ (i+1,i-1,k-i+1), & 3 \le i \le k-1; \\ (k+1,k-1,2), & i=k; \\ (k,k,2), & i=k+1; \\ (2k-i+1,2k-i+3,i-k+1), & k+2 \le i \le 2k-1; \\ (2,3,k+1), & i=2k. \end{cases}$$

and

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i=1; \\ (4,3,k), & i=2; \\ (i+2,i,k-i+1), & 3 \le i \le k-1; \\ (k+2,k,3), & i=k; \\ (k+1,k+1,3), & i=k+1; \\ (2k-i+2,2k-i+4,i-k+2), & k+2 \le i \le 2k-1; \\ (3,3,k+2), & i=2k. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+1), & i=1; \\ (5,4,k), & i=2; \\ (i+3,i+1,k-i+2), & 3 \le i \le k-2; \\ (k+2,k,4), & i=k-1; \\ (k+2,k+1,4), & i=k; \\ (k+1,k+2,4), & i=k+1; \\ (2k-i+2,2k-i+4,i-k+3), & k+2 \le i \le 2k-2; \\ (4,5,k+2), & i=2k-1; \\ (4,4,k+2), & i=2k. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(E_n) \le 3$. On the other hand, we show that $dim(E_n) \ge 3$. Suppose on contrary that $dim(E_n) = 2$, then there are following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{a_1,a_t\}) = r(b_n|\{a_1,a_t\}) = (1,t)$ and for $t = k+1$, $r(a_2|\{a_1,a_{k+1}\}) = r(a_n|\{a_1,a_{k+1}\}) = (1,k-1)$, a contradiction.

(2) Both vertices are in the interior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_1|\{b_1,b_t\}) = r(b_n|\{b_1,b_t\}) = (1,t)$ and for $t = k+1$, $r(b_2|\{b_1,b_{k+1}\}) = r(b_n|\{b_1,b_{k+1}\}) = (1,k-1)$, a contradiction.

(3) Both vertices are in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(a_n|\{c_1,c_t\}) = r(b_n|\{c_1,c_t\}) = (2,t+1)$ and for $t = k+1$, $r(a_1|\{c_1,c_{k+1}\}) = r(b_1|\{c_1,c_{k+1}\}) = (1,k)$, a contradiction.

(4) Both vertices are in the exterior cycle. Without loss of generality we suppose that one resolving

vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(c_1|\{d_1, d_t\}) = r(d_n|\{d_1, d_t\}) = (1, t)$ and for $t = k+1$, $r(d_2|\{d_1, d_{k+1}\}) = r(d_n|\{d_1, d_{k+1}\}) = (1, k-1)$, a contradiction.
(5) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \leq t \leq k+1)$. Then for $2 \leq t \leq k$, we have $r(d_1|\{e_1, e_t\}) = r(e_n|\{e_1, e_t\}) = (1, t)$ and for $t = k+1$, $r(e_2|\{e_1, e_{k+1}\}) = r(e_n|\{e_1, e_{k+1}\}) = (1, k-1)$, a contradiction.
(6) One vertex is in the inner cycle and other in the interior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, b_1\}) = r(b_n|\{a_1, b_1\}) = (1, 1)$ and when $2 \leq t \leq k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(b_1|\{a_1, b_{k+1}\}) = (1, t-1)$, a contradiction.
(7) One vertex is in the inner cycle and other in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, c_1\}) = r(b_n|\{a_1, c_1\}) = (1, 2)$ and when $2 \leq t \leq k+1$, $r(a_2|\{a_1, b_{k+1}\}) = r(b_1|\{a_1, b_{k+1}\}) = (1, t)$, a contradiction.
(8) One vertex is in the inner cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(d_2|\{a_1, d_1\}) = r(e_1|\{a_1, d_1\}) = (4, 1)$. If $2 \leq t \leq k$, $r(a_2|\{a_1, d_t\}) = r(b_1|\{a_1, d_t\}) = (1, t+1)$ and when $t = k+1$, $r(a_n|\{a_1, d_{k+1}\}) = r(b_n|\{a_1, d_{k+1}\}) = (1, k)$, a contradiction.
(9) One vertex is in the inner cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $t = 1$, we have $r(a_2|\{a_1, e_1\}) = r(b_n|\{a_1, e_1\}) = (1, 4)$. For $2 \leq t \leq k$, $r(a_2|\{a_1, e_t\}) = r(b_1|\{a_1, e_t\}) = (1, t+2)$ and when $t = k+1$, $r(d_2|\{a_1, e_{k+1}\}) = r(e_1|\{a_1, e_{k+1}\}) = (1, k)$, a contradiction.
(10) One vertex is in the interior cycle and other in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1, c_t\}) = r(b_n|\{b_1, c_t\}) = (1, t+1)$. For $t = k$, we have $r(a_2|\{b_1, c_k\}) = r(b_n|\{b_1, c_k\}) = (1, k)$ and when $t = k+1$, $r(a_1|\{b_1, c_{k+1}\}) = r(c_n|\{b_1, c_{k+1}\}) = (1, k)$, a contradiction.
(11) One vertex is in the interior cycle and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1, d_t\}) = r(b_n|\{b_1, d_t\}) = (1, t+2)$. For $t = k$, we have $r(a_2|\{b_1, d_k\}) = r(b_n|\{b_1, d_k\}) = (1, k+1)$ and when $t = k+1$, $r(a_1|\{b_1, d_{k+1}\}) = r(c_n|\{b_1, d_{k+1}\}) = (1, k+1)$, a contradiction.
(12) One vertex is in the interior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{b_1, e_t\}) = r(b_n|\{b_1, e_t\}) = (1, t+3)$. For $t = k$, we have $r(a_n|\{b_1, e_k\}) = r(c_n|\{b_1, e_k\}) = (1, k+2)$ and when $t = k+1$, $r(a_2|\{b_1, e_{k+1}\}) = r(c_1|\{b_1, e_{k+1}\}) = (1, k+2)$, a contradiction.
(13) One vertex is in the set of interior vertices and other in the exterior cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is d_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{c_1, d_t\}) = r(b_n|\{c_1, d_t\}) = (2, t+2)$. For $t = k$, we have $r(d_n|\{c_1, d_k\}) = r(e_n|\{c_1, d_k\}) = (2, k)$ and when $t = k+1$, $r(d_2|\{c_1, d_{k+1}\}) = r(d_n|\{c_1, d_{k+1}\}) = (1, k-1)$, a contradiction.
(14) One vertex is in the set of interior vertices and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k-1)$. Then for $1 \leq t \leq k-1$, we have $r(a_1|\{c_1, e_t\}) = r(b_n|\{c_1, e_t\}) = (2, t+3)$. For $t = k$, we have $r(d_n|\{c_1, e_k\}) = r(e_n|\{c_1, e_k\}) = (2, k)$ and when $t = k+1$, $r(d_n|\{c_1, e_{k+1}\}) = r(e_n|\{c_1, e_{k+1}\}) = (2, k-1)$, a contradiction.
(15) One vertex is in the set of exterior cycle and other in the outer cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is e_t $(1 \leq t \leq k+1)$. Then for $1 \leq t \leq k-1$, we have $r(c_1|\{d_1, e_t\}) = r(d_n|\{d_1, e_t\}) = (1, t+1)$. For $t = k$, we have $r(d_n|\{d_1, e_k\}) = r(e_n|\{d_1, e_k\}) = (1, k)$ and when $t = k+1$, $r(d_n|\{d_1, e_{k+1}\}) = r(e_n|\{d_1, e_{k+1}\}) = (1, k-1)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(E_n)$ implying that

$dim(E_n) = 3$ in this case.
Case(ii) When n is odd.
In this case, we can write $n = 2k+1$, $k \geq 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_3, a_{k+1}\} \subset V(E_n)$, again we show that W is a resolving set for E_n in this case. For this we give representations of any vertex of $V(E_n)\backslash W$ with respect to W.
Representations of the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (1,1,k-1), & i = 2; \\ (i-1, i-3, k-i+1), & 4 \leq i \leq k; \\ (k, k-1, 1), & i = k+2; \\ (k-1, k, 2), & i = k+3; \\ (2k-i+2, 2k-i+4, i-k-1), & k+4 \leq i \leq 2k+1. \end{cases}$$

Representations of the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,2,k), & i = 1; \\ (2,1,k-1), & i = 2; \\ (i, i-2, k-i+1), & 3 \leq i \leq k; \\ (k+1, k-1, 1), & i = k+1; \\ (2k-i+2, 2k-i+4, i-k), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations of the set of interior vertices are

$$r(c_i|W) = \begin{cases} (2,2,k), & i = 1; \\ (3,2,k-1), & i = 2; \\ (i+1, i-1, k-i+1), & 3 \leq i \leq k-1; \\ (k+1, k-1, 2), & i = k; \\ (k+1, k, 2), & i = k+1; \\ (k, k+1, 3), & i = k+2; \\ (2k-i+2, 2k-i+4, i-k+1), & k+3 \leq i \leq 2k; \\ (2, 3, k+1), & i = 2k+1. \end{cases}$$

Representations of the vertices on exterior cycle are

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i = 1; \\ (4,3,k), & i = 2; \\ (i+2, i, k-i+2), & 3 \leq i \leq k-1; \\ (k+2, k, 3), & i = k; \\ (k+2, k+1, 3), & i = k+1; \\ (k+1, k+2, 4), & i = k+2; \\ (2k-i+3, 2k-i+5, i-k+2), & k+3 \leq i \leq 2k; \\ (3, 4, k+2), & i = 2k+1. \end{cases}$$

Representations of the vertices on outer cycle are

$$r(e_i|W) = \begin{cases} (4,4,k+1), & i = 1; \\ (5,4,k), & i = 2; \\ (i+3, i+1, k-i+2), & 3 \leq i \leq k-1; \\ (k+3, k+1, 4), & i = k; \\ (k+2, k+2, 4), & i = k+1; \\ (2k-i+3, 2k-i+5, i-k+3), & k+2 \leq i \leq 2k-1; \\ (4, 5, k+3), & i = 2k; \\ (4, 4, k+2), & i = 2k+1. \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $dim(E_n) \leq 3$ in this case.
On the other hand, suppose that $dim(E_n) = 2$, then there are the same subcases as in case (**i**) and contradiction can be obtained analogously. This implies that $dim(E_n) = 3$ in this case, which completes the proof. □

This result also supports the open problem raised in [10] positively.

5. Concluding remarks

In this paper, we have studied the metric dimension of some classes of convex polytopes which are obtained by the combination of two different graph of convex polytopes. We see that the metric dimension of these classes of convex polytopes is finite and does not depend upon the number of vertices in these graphs and only three vertices appropriately chosen suffice to resolve all the vertices of these classes of convex polytopes. It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension. We also note that the results proved in Theorem 3 and 4 give the answer to the open problem raised in [10] positively.

Note that in [16] Melter and Tomescu gave an example of infinite regular plane graph (namely the digital plane endowed with city-block distance) having no finite metric basis. We close this section by raising a question as an open problem that naturally arises from the text.

Open problem: *Let G be the graph of a convex polytope which is obtained by the combination of graph of two different convex polytopes G_1 and G_2 both having constant metric dimension. Is it the case that G will always have the constant metric dimension?*

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