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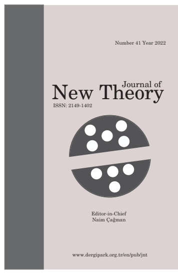

Cesàro Summability Involving δ-Quasi-Monotone and Almost Increasing Sequences

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Abstract — This paper generalises a well-known theorem on $\mid C,\rho \mid_\kappa$ summability to the $\varphi - \mid C,\rho;\beta \mid_\kappa$ summability of an infinite series using an almost increasing and a δ-quasi monotone sequence.

Keywords — *Cesàro summability, δ-quasi-monotone sequence, summability factors, almost increasing sequence, infinite series*

Mathematics Subject Classification (2020) — 40F05, 40G05

1. Introduction

The absolute summability of some infinite series (ISs) is an interesting topic. Especially, absolute Cesàro and absolute Riesz summability methods have different applications dealing with some well-known classes of sequence such as δ-quasi monotone, (ϕ,δ) monotone, almost increasing and quasi power increasing sequences. In [1], Bor and Özarslan proved two theorems on $\mid C,\rho;\beta \mid_\kappa$ and $\mid \bar{N},p_n;\beta \mid_\kappa$ summability methods. In [2–4], the authors obtained theorems on absolute Cesàro and absolute Riesz summability via almost increasing and δ-quasi monotone sequences. Özarslan [5–9], Bor [10], Kartal [11] proved theorems on absolute Cesàro summability factors. Kartal [12, 13] used almost increasing sequences to absolute Riesz summability, Bor and Agarwal [14], Kartal [15] applied almost increasing sequences to absolute Cesàro summability, also Bor et al. [16], Özarslan [17] operated quasi power increasing sequences. In [18], Özarslan and Şakar used (ϕ,δ) monotone sequences to get sufficient conditions for absolute Riesz summability of an ISs.

This article is organized as following: preliminaries on some sequences and the absolute Cesàro summability methods are given in Section 2, a known result about absolute Cesàro summability of a factored ISs is stated in Section 3, a generalisation of the theorem stated in Section 3 is proved in Section 4.

2. Preliminaries

In this section, several fundamental notions which will be used throughout this paper are defined. Throughout this paper, let $\sum a_n$ be an ISs with the partial sums (s_n) and (t_n^ρ) be the nth Cesàro

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mean of order ρ, with $\rho > -1$, of the sequence (na_n), that is [19]

$$t_n^\rho = \frac{1}{A_n^\rho} \sum_{r=1}^{n} A_{n-r}^{\rho-1} r a_r$$

where

$$A_n^\rho \simeq \frac{n^\rho}{\Gamma(\rho+1)}, \quad A_0^\rho = 1, \text{ and } A_{-n}^\rho = 0 \text{ for } n > 0$$

Here, Γ is gamma function defined by $\Gamma(\rho) = \int_0^\infty x^{\rho-1} e^{-x} dx$.
Let (ω_n^ρ) be a sequence defined as below [20]

$$\omega_n^\rho = \begin{cases} |t_n^\rho|, & \rho = 1 \\ \max_{1 \le r \le n} |t_r^\rho|, & 0 < \rho < 1 \end{cases} \tag{1}$$

Definition 2.1. [21] Let (φ_n) be any positive sequence. The series $\sum a_n$ is said to be summable $\varphi - | C, \rho; \beta |_\kappa$, $\kappa \geq 1$, $\rho > -1$, $\beta \geq 0$, if

$$\sum_{n=1}^{\infty} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid t_n^\rho \mid^\kappa < \infty$$

Remark 2.2. $\varphi - | C, \rho; \beta |_\kappa$ summability reduces to $| C, \rho |_\kappa$ summability [22] in case of $\varphi_n = n$ and $\beta = 0$.

Definition 2.3. [23] A sequence (G_n) is said to be δ-quasi-monotone if $G_n \to 0$, $G_n > 0$ ultimately and $\Delta G_n = G_n - G_{n+1} \geq -\delta_n$ where $\delta = (\delta_n)$ is a sequence of positive numbers.

Definition 2.4. [24] A positive sequence (c_n) is said to be almost increasing if there exist a positive increasing sequence (d_n) and two positive constants M and N such that $M d_n \leq c_n \leq N d_n$.

Lemma 2.5. [25] If $0 < \rho \leq 1$ and $1 \leq v \leq n$, then

$$\left| \sum_{r=0}^{v} A_{n-r}^{\rho-1} a_r \right| \leq \max_{1 \leq m \leq v} \left| \sum_{r=0}^{m} A_{m-r}^{\rho-1} a_r \right| \tag{2}$$

Lemma 2.6. [26] Let (H_n) be an almost increasing sequence such that $n|\Delta H_n| = O(H_n)$. If (G_n) is a δ-quasi-monotone, and $\sum n\delta_n H_n < \infty$, $\sum G_n H_n$ is convergent, then

$$n H_n G_n = O(1) \quad as \quad n \to \infty \tag{3}$$

$$\sum_{n=1}^{\infty} n H_n |\Delta G_n| < \infty \tag{4}$$

3. Known Result

In [27], $| C, \rho |_\kappa$ method has been used to obtain following theorem.

Theorem 3.1. Let (H_n) be an almost increasing sequence and (γ_n) be any sequence with $|\Delta H_n| = O(H_n/n)$ such that

$$|\gamma_n| H_n = O(1) \quad as \qquad n \to \infty \tag{5}$$

Assuming also that there exists a sequence of numbers (G_n) such that it is δ-quasi-monotone such that $\sum n\delta_n H_n < \infty$, $\sum G_n H_n$ is convergent, and $|\Delta \gamma_n| \leq |G_n|$ for all n. If the sequence (ω_n^ρ) satisfies the condition

$$\sum_{n=1}^{u} \frac{(\omega_n^\rho)^\kappa}{n H_n^{\kappa-1}} = O(H_u) \quad \text{as} \quad u \to \infty \tag{6}$$

then the series $\sum a_n \gamma_n$ is summable $| C, \rho |_\kappa$, $0 < \rho \leq 1$, $\kappa \geq 1$.

4. Main Result

The main concern of the article is to generalise Theorem 3.1 for the more general $\varphi - \mid C,\rho;\beta\mid_{\kappa}$ method.

Theorem 4.1. Let (H_n), (G_n) and (γ_n) be the sequences satisfying the same conditions as given in Theorem 3.1. Assuming that there is an $\epsilon > 0$ such that the sequence $(n^{\epsilon-\kappa}\varphi_n^{\beta\kappa+\kappa-1})$ is non-increasing. If

$$\sum_{n=1}^{u} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \frac{(\omega_n^{\rho})^{\kappa}}{H_n^{\kappa-1}} = O(H_u) \quad \text{as} \quad u \to \infty \tag{7}$$

then the series $\sum a_n\gamma_n$ is summable $\varphi - \mid C,\rho;\beta\mid_{\kappa}$, $\beta \geq 0$, $0 < \rho \leq 1$, $\epsilon + (\rho-1)\kappa > 0$, $\kappa \geq 1$.

Proof. Let $0 < \rho \leq 1$. Let (I_n^{ρ}) be the nth (C,ρ) mean of the sequence $(na_n\gamma_n)$. Using Abel's formula, we write

$$\begin{aligned} I_n^{\rho} &= \frac{1}{A_n^{\rho}} \sum_{i=1}^{n} A_{n-i}^{\rho-1} i a_i \gamma_i \\ &= \frac{1}{A_n^{\rho}} \sum_{i=1}^{n-1} \Delta\gamma_i \sum_{r=1}^{i} A_{n-r}^{\rho-1} r a_r + \frac{\gamma_n}{A_n^{\rho}} \sum_{i=1}^{n} A_{n-i}^{\rho-1} i a_i \end{aligned}$$

By Lemma 2.5, we achieve

$$\begin{aligned} |I_n^{\rho}| &\leq \frac{1}{A_n^{\rho}} \sum_{i=1}^{n-1} \mid \Delta\gamma_i \mid \left| \sum_{r=1}^{i} A_{n-r}^{\rho-1} r a_r \right| + \frac{\mid \gamma_n \mid}{A_n^{\rho}} \left| \sum_{i=1}^{n} A_{n-i}^{\rho-1} i a_i \right| \\ &\leq \frac{1}{A_n^{\rho}} \sum_{i=1}^{n-1} A_i^{\rho} \omega_i^{\rho} \mid \Delta\gamma_i \mid + \mid \gamma_n \mid \omega_n^{\rho} \\ &= I_{n,1}^{\rho} + I_{n,2}^{\rho} \end{aligned}$$

To prove Theorem 4.1, we need to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,j}^{\rho} \mid^{\kappa} < \infty \quad \text{for} \quad j = 1,2$$

First, for $j = 1$, we have

$$\sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^{\rho} \mid^{\kappa} \leq \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} (A_n^{\rho})^{-\kappa} \left\{ \sum_{i=1}^{n-1} A_i^{\rho} \omega_i^{\rho} |\Delta\gamma_i| \right\}^{\kappa}$$

Using the fact that $|\Delta\gamma_n| \leq |G_n|$ and Hölder's inequality, we achieve

$$\begin{aligned} \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^{\rho} \mid^{\kappa} &\leq \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} (A_n^{\rho})^{-\kappa} \sum_{i=1}^{n-1} (A_i^{\rho})^{\kappa} (\omega_i^{\rho})^{\kappa} |G_i|^{\kappa} \left\{ \sum_{i=1}^{n-1} 1 \right\}^{\kappa-1} \\ &= O(1) \sum_{i=1}^{u} i^{\rho\kappa} (\omega_i^{\rho})^{\kappa} |G_i| \, |G_i|^{\kappa-1} \sum_{n=i+1}^{u+1} \frac{\varphi_n^{\beta\kappa+\kappa-1} n^{\epsilon-\kappa}}{n^{1+\epsilon+(\rho-1)\kappa}} \end{aligned}$$

Here, using (3), we get $|G_i| = O(1/iH_i)$, therefore

$$\begin{aligned} \sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^{\rho} \mid^{\kappa} &= O(1) \sum_{i=1}^{u} i^{\rho\kappa} (\omega_i^{\rho})^{\kappa} |G_i| \frac{1}{(iH_i)^{\kappa-1}} \varphi_i^{\beta\kappa+\kappa-1} i^{\epsilon-\kappa} \int_i^{\infty} \frac{dx}{x^{1+\epsilon+(\rho-1)\kappa}} \\ &= O(1) \sum_{i=1}^{u} i \, |G_i| \, \varphi_i^{\beta\kappa+\kappa-1} i^{-\kappa} \frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}} \end{aligned}$$

Now, by applying Abel's formula, and by the hypotheses of Theorem 4.1 and Lemma 2.6, we achieve

$$\begin{aligned}
\sum_{n=2}^{u+1} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,1}^{\rho} \mid^{\kappa} &= O(1) \sum_{i=1}^{u-1} \Delta(i\,|G_i|) \sum_{r=1}^{i} \varphi_r^{\beta\kappa+\kappa-1} r^{-\kappa} \frac{(\omega_r^{\rho})^{\kappa}}{H_r^{\kappa-1}} \\
&\quad + O(1) u\,|G_u| \sum_{i=1}^{u} \varphi_i^{\beta\kappa+\kappa-1} i^{-\kappa} \frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}} \\
&= O(1) \left(\sum_{i=1}^{u-1} i|\Delta G_i| H_i + \sum_{i=1}^{u-1} |G_i|\, H_i + u\,|G_u|\, H_u \right) \\
&= O(1) \quad as \quad u \to \infty
\end{aligned}$$

Now, we write that $|\gamma_n| = O(1/H_n)$ from (5). Therefore, for $j = 2$, we get

$$\begin{aligned}
\sum_{n=1}^{u} \varphi_n^{\beta\kappa+k\kappa-1} n^{-\kappa} \mid I_{n,2}^{\rho} \mid^{\kappa} &= \sum_{n=1}^{u} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} |\gamma_n|\, |\gamma_n|^{\kappa-1} (\omega_n^{\rho})^{\kappa} \\
&= O(1) \sum_{n=1}^{u} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} |\gamma_n| \frac{1}{H_n^{\kappa-1}} (\omega_n^{\rho})^{\kappa}
\end{aligned}$$

From Abel's formula, we get

$$\begin{aligned}
\sum_{n=1}^{u} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,2}^{\rho} \mid^{\kappa} &= O(1) \sum_{n=1}^{u-1} \Delta\,|\gamma_n| \sum_{i=1}^{n} \varphi_i^{\beta\kappa+\kappa-1} i^{-\kappa} \frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}} \\
&\quad + O(1)\,|\gamma_u| \sum_{i=1}^{u} \varphi_i^{\beta\kappa+\kappa-1} i^{-\kappa} \frac{(\omega_i^{\rho})^{\kappa}}{H_i^{\kappa-1}}
\end{aligned}$$

Then, we achieve

$$\begin{aligned}
\sum_{n=1}^{u} \varphi_n^{\beta\kappa+\kappa-1} n^{-\kappa} \mid I_{n,2}^{\rho} \mid^{\kappa} &= O(1) \left(\sum_{n=1}^{u-1} |\Delta\gamma_n|\, H_n + |\gamma_u|\, H_u \right) \\
&= O(1) \left(\sum_{n=1}^{u-1} |G_n|\, H_n + |\gamma_u|\, H_u \right) \\
&= O(1) \quad as \quad u \to \infty
\end{aligned}$$

□

5. Conclusion

In this paper, a theorem dealing with generalised absolute Cesàro summability has been introduced which reduces to Theorem 3.1 for $\varphi_n = n$, $\beta = 0$ and $\epsilon = 1$. Hence, the equality (7) reduces to the equality (6). Furthermore, a known result on $|C,1|_{\kappa}$ summability can be deducted whenever $\varphi_n = n$, $\beta = 0$, $\rho = 1$ and $\epsilon = 1$ [27]. In the light of this study, one can generalise these results for using either different summability methods or different sequence classes.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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