Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences
https://dergipark.org.tr/en/pub/yyufbed

Research Article

Approximation Properties of a Class of Kantorovich Type Operators Associated with the Charlier Polynomials

Kerem GEZER, Mine MENEKŞE YILMAZ*

Gaziantep University, Science and Arts Faculty, Mathematics Department, 27310, Gaziantep, Türkiye
Kerem GEZER, ORCID No: 0000-0001-6715-845X, Mine MENEKŞE YILMAZ, ORCID No: 0000-0002-9242-8447
*Corresponding author e-mail: menekse@gantep.edu.tr

Article Info

Received: 11.10.2022
Accepted: 13.01.2023
Online August 2023

DOI:10.53433/yyufbed.1187512

Keywords
Charlier Polynomials,
Kantorovich type operators,
Uniform convergence,
Voronovskaya type theorem

Abstract: In this paper, we introduce a kind of Charlier polynomial-based Szász-Kantorovich type operator. We begin by using Korovkin's theorem to demonstrate the uniform convergence of these operators. Second, using mathematical techniques like Peetre's $\mathcal{K}$-functional notion and the common modulus of the operators, we evaluate the order of convergence of the operators. Third, we use the Voronovskaya type approximation theorem to derive an asymptotic formula for the operator we gave. Finally, we give a numerical example using Maple 2022.

Charlier Polinomlarıyla İlişkili Kantorovich Tipi Operatörler Sınıfının Yaklaşım Özellikleri

Makale Bilgileri

Geliş: 11.10.2022
Kabul: 13.01.2023
Online Ağustos 2023

DOI:10.53433/yyufbed.1187512

Anahtar Kelimeler
Charlier Polinomları,
Düzgün yakınsaklık
Kantorovich tipi operatörler,
Voronovskaya tipi teorem

Öz: Bu çalışmada, Charlier polinom tabanlı Szász-Kantorovich tipi bir operatör tanıtıyoruz. Bu operatörlerin düzgün yakınsamasını göstermek için Korovkin teoremini kullanarak başlıyoruz. İkinci olarak, Peetre' ın $\mathcal{K}$-fonksiyonel kavramı ve operatörlerin olağan süreklilik modülü gibi matematiksel teknikleri kullanarak, operatörlerin yakınsama oranını değerlendiriyoruz Üçüncüsü, verdiğimiz operatör için asimptotik bir formül türetmek için Voronovskaya tipi yaklaşım teoremini kullanıyoruz. Son olarak Maple 2022 kullanarak sayısal bir örnek veriyoruz.

1. Introduction

Otto Szász (1950) defined the following linear positive operators

$$P_n(u;f) = e^{-ux}\sum_{k=0}^{\infty}\frac{(ux)^k}{k!}f\left(\frac{k}{u}\right), u > 0. \quad (1)$$

He presented a uniform approximation for continuous functions on $[0,\infty)$. Several authors presented various types of these operators and gave their approximation properties in some functional spaces (Jakimovski & Leviatan, 1969; Păltănea, 2008; Aral et al., 2014; Atakut & Büyükyazıcı, 2010 and 2016, Ağyüz, 2021; Aslan, 2022; Aslan & Mursaleen, 2022).

Varma & Tasdelen (2012) presented a form of the Szász operators based on Charlier polynomials $C_k^{\alpha}(x)$ (Ismail, 2005) having the generating function of the form:

$$e^{-aw}(1+w)^x=\sum_{k=0}^{\infty}\frac{C_k^{(a)}(x)w^k}{k!}, a\neq 0, \tag{2}$$

and the explicit form of the Charlier polynomial is

$$C_k^{(a)}(x)=\sum_{n=0}^{k}\binom{k}{n}\binom{x}{n}n!\,(-a)^{k-n}. \tag{3}$$

They described a new sequence that involves Charlier polynomials as follows:

$$L_n(f;x,a)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}\ \ (-(a-1)nx)}{k!}f\left(\frac{k}{n}\right), \tag{4}$$

where $a>1$ and $x\geq 0$. And also, they gave a Kantorovich-type generalization of the operators L_n. They obtained some approximation properties and studied the order of approximation for the operators Equation 4. For some articles based on Charlier polynomials, see: Agrawal & İspir (2016); Wafi & Rao (2016); Kajla & Agrawal (2016); Çavdar (2017); Ayık (2018); Ansari et al. (2020); Al-Abied et al. (2021).
Based on Equation 4 for $n\in\mathbb{N}$, we define the operators, denoted by $K_n(f;x)$, as follows:

Definition 1 Let $K_n\colon C[0,\infty)\to C[0,\infty)$. For $(n\in\mathbb{N})$, $0\leq\ \alpha\leq\beta, n+\beta\geq 1$ and $f\in C[0,\infty)$, the operators K_n defined by

$$K_n(f;x)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}}f(s)ds, \tag{5}$$

where $\left\{C_k^{(a)}\right\}_{k=0}^{\infty}$ are Charlier polynomials that are positive for $a>1$ and $x\geq 0$.

The goal of the present study is to establish some approximation properties of the Kantorovich-type operators including Charlier polynomials defined by Equation 5.
The rest of this paper is structured as follows:

In the next section, we give some auxiliary lemmas for the operators $K_n(f;x)$. In the third section, we investigate the uniform approximation of the sequence $\{K_n(f;x)\}_{n=0}^{\infty}$, and then we estimate the order of convergence of the operators K_n with the help of the usual modulus of continuity and the Peetre's $\mathcal{K}$-functional. In the fourth section, we give a Voronovskaya-type theorem for the operators given in Equation 5.

2. Material and Methods Preliminaries

In this section, we give the following lemmas which are used in the theorems.

Lemma 1 For $a>1$ and the operators $K_n(f;x)$, we have the following equalities:

$$K_n(e_0;x)=1, \tag{6}$$

$$K_n(e_1;x) = \frac{n}{n+\beta}x + \frac{2\alpha+3}{2(n+\beta)}, \tag{7}$$

$$K_n(e_2;x) = \frac{n^2}{(n+\beta)^2}x^2 + \frac{nx}{(n+\beta)^2}\left(\frac{4a-3}{a-1} + 2\alpha\right) + \frac{3\alpha^2+9\alpha+10}{3(n+\beta)^2}, \tag{8}$$

$$\begin{aligned} K_n(e_3;x) = {} & \frac{n^3x^3}{(n+\beta)^3} + \frac{n^2x^2}{(n+\beta)^3}\left(\frac{15}{2} + \frac{3}{a-1} + 3\alpha\right) \\ & + \frac{nx}{(n+\beta)^3}\left(\frac{31}{2} + \frac{15+6\alpha}{2(a-1)} + \frac{2}{(a-1)^2} + 3\alpha^2 + 12\alpha\right) \\ & + \frac{2\alpha^3+18\alpha^2+20\alpha+37}{4(n+\beta)^3} \end{aligned} \tag{9}$$

$$\begin{aligned} K_n(e_4;x) = {} & \frac{n^4x^4}{(n+\beta)^4} + \frac{n^3x^3}{(n+\beta)^4}\left[12 + \frac{6}{a-1} + 4\alpha\right] \\ & + \frac{n^2x^2}{(n+\beta)^4}\left[46 + \frac{36+12\alpha}{a-1} + \frac{11}{(a-1)^2} + 30\alpha + 6\alpha^2\right] \\ & + \frac{nx}{(n+\beta)^4}\left[64 + \frac{46+30\alpha+6\alpha^2}{a-1} + \frac{24+8\alpha}{(a-1)^2} + \frac{6}{(a-1)^3} + 4\alpha^3 \right. \\ & \left. + 24\alpha^2 + 62\alpha\right] + \frac{5\alpha^4+30\alpha^3+100\alpha^2+185\alpha+151}{5(n+\beta)^4}, \end{aligned} \tag{10}$$

where $e_m(s) = s^m \in C[0,\infty)$, $m = 0,1,2,3,4$.

Proof One shall obtain the desired results by using the generating function of Charlier polynomials directly in the equations.

$$\begin{aligned} K_n(e_0;x) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)\sum_{k=0}^{\infty}\frac{C_k^{(a)}\ (-(a-1)nx)}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} ds \\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)e\left(1-\frac{1}{a}\right)^{-(a-1)nx}\frac{1}{n+\beta} = 1 \end{aligned} \tag{11}$$

$$K_n(e_1;x) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)\sum_{k=0}^{\infty}\frac{C_k^{(a)}\ \ (-(a-1)nx)}{k!}\int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} sds$$

$$= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)\sum_{k=0}^{\infty}\frac{C_k^{(a)}\ \ (-(a-1)nx)}{k!}\left[\frac{k}{(n+\beta)^2}+\frac{2\alpha+1}{2(n+\beta)^2}\right]$$

$$= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)\ e\left(1-\frac{1}{a}\right)^{-(a-1)nx}(nx+1)\frac{1}{(n+\beta)^2} \tag{12}$$

$$+ e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}(n+\beta)\ e\left(1-\frac{1}{a}\right)^{-(a-1)nx}\frac{2\alpha+1}{2(n+\beta)^2}$$

$$= \frac{n}{n+\beta}x + \frac{2\alpha+1}{2(n+\beta)}.$$

We may similarly prove other cases.

Lemma 2 For $n \in \mathbb{N}$, the central moments for the operators $K_n(f;x)$ are given as follows:

$$K_n(e_1 - x;x) = \left(\frac{n}{n+\beta}-1\right)x + \frac{2\alpha+3}{2(n+\beta)} \tag{13}$$

$$K_n((e_1-x)^2;x) = x^2\left(\frac{n}{n+\beta}-1\right)^2 + x\left(\frac{n}{(n+\beta)^2}\left(\frac{4a-3}{a-1}+2\alpha\right)-\frac{2\alpha+3}{(n+\beta)}\right) + \frac{3\alpha^2+9\alpha+10}{3(n+\beta)^2}. \tag{14}$$

$$\begin{aligned}K_n((e_1-x)^4;x) &= \left(\frac{n}{n+\beta}-1\right)^4 x^4 + \frac{n^3}{(n+\beta)^4}\left(\frac{12a-6}{a-1}+4\alpha\right)x^3 \\ &+\left(-\frac{4n^2}{(n+\beta)^3}\left(\frac{15a-9}{2a-2}+3\alpha\right)+\frac{6n}{(n+\beta)^2}\left(\frac{4a-3}{a-1}+2\alpha\right)-\frac{4\alpha+6}{n+\beta}\right)x^3 \\ &+\left(\frac{n^2}{(n+\beta)^4}\left[46+\frac{36+12\alpha}{a-1}+\frac{11}{(a-1)^2}+30\alpha+6\alpha^2\right]\right. \\ &-\frac{4n}{(n+\beta)^3}\left(\frac{31}{2}+\frac{15+6\alpha}{2(a-1)}+\frac{2}{(a-1)^2}+3\alpha^2+12\alpha\right) \\ &\left.+\frac{6\alpha^2+18\alpha+20}{(n+\beta)^2}\right)x^2 \\ &+\left(\frac{n}{(n+\beta)^4}\left[64+\frac{46+30\alpha+6\alpha^2}{a-1}+\frac{24+8\alpha}{(a-1)^2}+\frac{6}{(a-1)^3}+4\alpha^3+24\alpha^2+62\alpha\right]\right. \\ &\left.-\frac{2\alpha^3+18\alpha^2+20\alpha+37}{4(n+\beta)^3}\right)x \\ &+\frac{5\alpha^4+30\alpha^3+100\alpha^2+185\alpha+151}{5(n+\beta)^4}.\end{aligned} \tag{15}$$

Proof Due to the linearity property of the operators K_n provides the following equality:

$$K_n((e_1-x)^4;x) = K_n(e_4;x) - 4x\,K_n(e_3;x) \\ +6x^2\,K_n(e_2;x) - 4x^3\,K_n(e_1;x) + x^4\,K_n(e_0;x). \tag{16}$$

Using Equation 6-10, we get Equation 13. Similarly, using the linearity property of K_n , we obtain Equation 14 and Equation 15.

Theorem 1 Let $E:=\{f:[0,\infty)\to\mathbb{R}, |f(x)|\le ce^{bx}, c\in\mathbb{R}, b\in\mathbb{R}^+\}$.
If $f\in C[0,\infty)\cap E$, then

$$\lim_{n\to\infty} K_n(f;x) = f(x)\,, \tag{17}$$

and the operators K_n converge uniformly in each compact subset of $[0,\infty)$.

Proof Considering Lemma 1, the following holds uniformly in each compact subinterval of [0,∞):

$$\lim_{n\to\infty}\|K_n(e_m;x)-x^m\| = 0 \text{ for } m=0,1,2. \tag{18}$$

According to Korovkin's theorem (Korovkin, 1953), we obtain the desired result.

3. Results

3.1. Order of approximation

In this section, we establish the order of approximation of the operators K_n to f with the aid of the modulus of continuity and Peetre's $\mathcal{K}$-functional. We begin this section by defining the concepts we will use in the theorems.

Let $\tilde{C}[0,\infty)$ be the space of uniformly continuous functions on [0,∞). For $\delta>0$, the usual modulus of continuity $\omega(f;\delta):[0,\infty)\to\mathbb{R}$ defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} |f(x)-f(y)|, \tag{19}$$

with the following properties

$$|f(x)-f(y)| \le \omega(f;\delta)\left(1+\frac{|x-y|}{\delta}\right). \tag{20}$$

and $\lambda>0$,

$$\omega(f;\lambda\delta) \le (1+\lambda)\omega(f;\delta) \le (1+\lambda^2)\omega(f;\delta). \tag{21}$$

(Barbosu, 2004).

Let $C_B[0,\infty)$ be the space of real-valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup_{x\in[0,\infty)} |f(x)|$.

Peetre's $\mathcal{K}$-functional of the function $f\in C_B[0,\infty)$ is defined as

$$\mathcal{K}(f;\delta) := \inf_{g\in C_B^2[0,\infty)} \left\{\|f-g\|_{C_B} + \delta\|g\|_{C_B^2}\right\}, \tag{22}$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$ with the norm

$$\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}. \tag{23}$$

Teorem 2 If $f \in \tilde{C}[0,\infty) \cap E$, then

$$|K_n(s,x) - f(x)| \leq 2\omega(f; \sqrt{K_n((e_1 - x)^2; x)}). \tag{24}$$

Proof Assume that $I_n := |K_n(s,x) - f(x)|$. We know that $\left(1 - \frac{1}{a}\right)^{(a-1)nx} \geq 0$ and $\sum_{k=0}^{\infty} \frac{C_k^{(a)}\ (-(a-1)nx)}{k!} \geq 0$, and also from Lemma 1, we have $K_n(1;x) = 1$. The following inequality is clear from these facts:

$$I_n \leq (n+\beta)e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}\ (-(a-1)nx)}{k!} \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} |f(s) - f(x)| ds.$$

Using Equation 20 and Equation 21, we get the following:

$$|f(s) - f(x)| \leq \omega(f; \delta^{-1}|s - x|) \leq (1 + \delta^{-2}(s - x)^2)\omega(f; \delta). \tag{25}$$

Using Lemma 2, we obtained the following form:

$$I_n \leq e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} (n + \beta) \sum_{k=0}^{\infty} \frac{C_k^{(a)}\ (-(a-1)nx)}{k!} \int_{\frac{k+\alpha}{n+\beta}}^{\frac{k+\alpha+1}{n+\beta}} \left(1 + \frac{|s - x|^2}{\delta^2}\right) \omega(f; \delta) ds \tag{26}$$

$$\begin{aligned} I_n &\leq \{K_n(e_0; x) + \delta^{-2} K_n((e_1 - x)^2; x)\}\, \omega(f; \delta) \\ &\leq \{1 + \delta^{-2} K_n((e_1 - x)^2; x)\}\, \omega(f; \delta), \end{aligned} \tag{27}$$

for any $\delta > 0$ and each $x \in [0,\infty)$.

Since $K_n((e_1 - x)^2; x) \geq 0$ for each $x \in [0,\infty)$, we may choose $\delta := \sqrt{K_n((e_1 - x)^2; x)}$ then we have the desired result.

We give the following estimation theorem involving Peetre's $\mathcal{K}$-functional, similar to the theorem in Sucu (2022).

Theorem 3 If $f \in C_B^2[0,\infty)$, then

$$|K_n(f; x) - f(x)| \leq 2\mathcal{K}\big(x; \vartheta_n(x)\big), \tag{28}$$

where $\vartheta_n(x) := \frac{1}{2}\{K_n(e_1 - x; x) + K_n((e_1 - x)^2; x)\}$ and $\mathcal{K}$ is the Peetre's $\mathcal{K}$-functional of f.

Proof From the Taylor formula of g, the linearity property of operators K_n and Lemma 1, we may write the following:

$$K_n(g;x) - g(x) = g'(x)K_n(s-x;x) + \frac{g''(\eta)}{2}K_n((s-x)^2;x), \quad \eta \in (x,s). \tag{29}$$

Also, we may reach the following inequality using Equation 23 in Equation 29:

$$\begin{aligned} |K_n(g;x) - g(x)| &\leq \|g'\|_{C_B}|K_n(e_1-x;x)| + \frac{1}{2}\|g''\|_{C_B}|K_n((e_1-x)^2;x)| \\ &\leq \|g\|_{C_B^2}\left\{|K_n(e_1-x;x)| + \frac{1}{2}|K_n((e_1-x)^2;x)|\right\} \\ &\leq \|g\|_{C_B^2}\vartheta_n(x). \end{aligned} \tag{30}$$

By applying Equation 30, we get the following:

$$\begin{aligned} |K_n(f;x) - f(x)| &\leq |K_n(f-g;x)| + |K_n(g;x) - g(x)| + |f(x) - g(x)| \\ &\leq 2\|f-g\|_{C_B} + |K_n(g;x) - g(x)| \\ &\leq 2\{\|f-g\|_{C_B} + \|g\|_{C_B^2}\vartheta(x)\}. \end{aligned} \tag{31}$$

If we take the infimum over all $g \in C_B^2[0,\infty)$, Equation 31 implies the following result:

$$|K_n(f;x) - f(x)| \leq 2\mathcal{K}\big(x;\vartheta_n(x)\big) \tag{32}$$

which finishes the proof.

3.2. Voronovskaya-type theorem

Lemma 3 For fixed $x_0 \in [0,\infty)$, we have

$$\lim_{n\to\infty} nK_n(e_1-x_0;x) = -\beta x_0 + \frac{2\alpha+3}{2}, \tag{33}$$

$$\lim_{n\to\infty} nK_n((e_1-x_0)^2;x) = x_0\left(1+\frac{1}{a-1}\right). \tag{34}$$

Proof Applying Equation 13 and Equation 14, the desired result is achieved.

Lemma 4 For each fixed $x_0 \in [0,\infty)$, there is a positive constant $M(x_0)$, depending only on x_0 such that and

$$K_n((e_1-x_0)^4;x_0) \leq M(x_0)\frac{1}{(n+\beta)^2}, \tag{35}$$

for all $n \in \mathbb{N}$.

Proof By Equation 15, we can find a boundary for $K_n((s-x_0)^4;x_0)$**.**

$$K_n((s-x_0)^4;x_0) \le \left(\frac{1}{n+\beta}-1\right)^4 x_0^4 + \frac{1}{(n+\beta)^2}\left\{4+\frac{6}{a-1}+\frac{11}{(a-1)^2}\right\}x_0^2$$
$$+\frac{1}{(n+\beta)^3}\left\{64+\frac{46+30\alpha+6\alpha^2}{a-1}+\frac{24+8\alpha}{(a-1)^2}+\frac{6}{(a-1)^3}+4\alpha^3+24\alpha^2+62\alpha\right.$$
$$\left.-\frac{2\alpha^3+18\alpha^2+20\alpha+37}{4(n+\beta)^3}\right\}x_0$$
$$+\frac{1}{(n+\beta)^4}\left\{\frac{5\alpha^4+30\alpha^3+100\alpha^2+185\alpha+151}{5}\right\}$$
$$\le \frac{1}{(n+\beta)^2}\left[x_0^4+\left(4+\frac{6}{a-1}+\frac{11}{(a-1)^2}\right)x_0^2\right. \tag{36}$$
$$+\left(64+\frac{46+30\alpha+6\alpha^2}{a-1}+\frac{24+8\alpha}{(a-1)^2}+\frac{6}{(a-1)^3}+4\alpha^3+24\alpha^2\right.$$
$$\left.+62\alpha-\frac{2\alpha^3+18\alpha^2+20\alpha+37}{4}\right)x_0$$
$$\left.+\frac{5\alpha^4+30\alpha^3+100\alpha^2+185\alpha+151}{5}\right]$$
$$\le \frac{1}{(n+\beta)^2}\ M(x_0).$$

Theorem 4 If $f\in C_B^2[0,\infty)$, then

$$\lim_{n\to\infty} n[K_n(f;x)-f(x)] = \frac{x}{2}\left(\frac{a}{a-1}\right)f''(x)+f'(x)\left(-\beta x+\frac{2\alpha+3}{2}\right), \tag{37}$$

for every fixed $x\in[0,\infty)$.

Proof For a fixed point $x_0\in[0,\infty)$, we write the Taylor formula as follows:

$$f(s)=f(x_0)+(s-x)f'(x_0)+\frac{1}{2}(s-x_0)^2f''(x_0)+g(s;x_0)(s-x)^2, \tag{38}$$

where $g(s:x_0)$ is the Peano form of the remainder and $g(.;x_0)\in C[0,\infty)$ with $\lim_{s\to x_0} g(s;x_0)=0$. By Equation 3 and Equation 19, we get

$$n[K_n(f;x_0)-f(x_0)] = f'(x_0)nK_n(e_1-x_0;x_0)+\frac{1}{2}f''(x_0)nK_n((e_1-x_0)^2;x_0)$$
$$+nK_n(g(s;x_0)(e_1-x_0)^2;x_0), \tag{39}$$

for every $n\in\mathbb{N}$.

By applying the Cauchy-Schwarz inequality to the third term on the right-hand side of Equation 39, we have

$$nK_n(g(s;x_0)(e_1-x_0)^2;x_0) \le \{K_n(g^2(s;x_0);x_0)\}^{\frac{1}{2}}\{n^2K_n((e_1-x)^4;x_0\}^{\frac{1}{2}}. \tag{40}$$

The function $h(s,x_0) = g^2(s;x_0)$, $s \geq 0$, we get $h(s,x_0) \in C[0,\infty)$ and $\lim_{s\to x_0} h(s,x_0) = 0$.

Hence

$$\lim_{s\to x_0} K_n(g^2(s;x_0);x_0) = \lim_{s\to x_0} K_n(h(s;x_0);x_0) = \lim_{s\to x_0} h(x;x_0) = 0. \tag{41}$$

uniformly concerning $x_0 \in [0,\infty)$.

Using Equation 35 in Equation 40, we get

$$nK_n(g(s;x_0)(e_1-x_0)^2;x_0) \leq \left(K_n(g^2(s;x_0);x_0)\right)^{\frac{1}{2}} \left(n^2 M(x_0)\frac{1}{(n+\beta)^2}\right)^{\frac{1}{2}}. \tag{42}$$

It results that $\lim_{n\to\infty} nK_n(g(s;x_0)(e_1-x_0)^2;x_0) = 0$.

Finally, by Equation 33 and Equation 34, we have the desired result as follows:

$$\lim_{n\to\infty} n(K_n(f;x_0)-f(x_0)) = \frac{x_0}{2}\left(1+\frac{1}{a-1}\right)f''(x_0) + \left(-\beta x_0 + \frac{2\alpha+3}{2}\right)f'(x_0). \tag{43}$$

3.3. Numerical example

In this part, we support our results giving an example with the help of the modulus of continuity, $\omega(f;\delta)$, by using Maple 2022.

Example 5 The approximation of K_n to $f(x) = \frac{x^2}{\sqrt{1+x^2}}$ on $[0,\infty)$ for fixed $\alpha = 1$, $\beta = 1$ $a = 2$ is shown in Table 1. Let $E_n := |K_n\,(f;x) - f(x)|$.

Table 1. The error estimation of $f(x) = \frac{x^2}{\sqrt{1+x^2}}$ using $\omega(f;\delta)$

n	E_n
10^2	0.0574922114
10^3	0.0064708796
10^4	0.0011087936
10^5	0.0003054410
10^6	0.0000950418
10^7	0.0000300062

When we examine the Table 1, we notice that the approximation errors of the operators K_n decrease as *n* increases.

4. Discussion and Conclusion

In the article by Varma & Tasdelen (2012), the positive linear operator was defined using Charlier polynomials under certain conditions. With the idea of this study, we have defined a new operator based on Charlier polynomials and proved the uniform convergence of these operators with a Korovkin-type approximation. We have used the usual modulus of continuity, Petree's $\mathcal{K}$-functional to estimate the order of the convergence by operators K_n. We have obtained an asymptotic formula for the operators K_n. Finally, we have obtained error estimations for operators K_n with the help of the modulus of continuity $\omega(f;\delta)$ and then we have presented the numerical results via the table.

For further studies, one may obtain the order of convergence of these operators in terms of the modulus of smoothness in weighted spaces. In addition, the q analogue and convergence properties of these operators can also be a subject of study.

Acknowledgements

Some parts of this paper have been presented at the 4th International Conference on Pure and Applied Mathematics ICPAM-VAN 2022 June 22-23, 2022.

References

Agrawal, P. N., & İspir, N. (2016). Degree of approximation for bivariate Chlodowsky–Szász–Charlier type operators. *Results in Mathematics*, 69(3-4), 369-385. doi:10.1007/s00025-015-0495-6

Ağyüz, E. (2021). On the convergence properties of Kantorovich-Szasz type operators involving tangent polynomials. *Adıyaman University Journal of Science*, 11(2), 244-252. doi:10.37094/adyujsci.905311

Al-Abied, A. A. H., Mursaleen, A. M., & Mursaleen, M. (2021). Szász type operators involving Charlier polynomials and approximation properties. *Filomat*, 35(15), 5149-5159. doi:10.2298/FIL2115149A

Ansari, K. J., Mursaleen, M., Shareef Kp, M., & Ghouse, M. (2020). Approximation by modified Kantorovich–Szász type operators involving Charlier polynomials. *Advances in Difference Equations*, 2020, 192. doi:10.1186/s13662-020-02645-6

Aral, A., Inoan, D., & Raşa, I. (2014). On the generalized Szász–Mirakyan operators. *Results in Mathematics*, 65 (3), 441-452. doi:10.1007/s00025-013-0356-0

Aslan, R. (2022). On a Stancu form Szász-Mirakjan-Kantorovich operators based on shape parameter λ. *Advanced Studies: Euro-Tbilisi Mathematical Journal*, 15(1), 151-166. doi:10.32513/asetmj/19322008210

Aslan, R., & Mursaleen, M. (2022). Approximation by bivariate Chlodowsky type Szász–Durrmeyer operators and associated GBS operators on weighted spaces. J*ournal of Inequalities and Applications*, 2022(1), 1-19. doi:10.1186/s13660-022-02763-7

Atakut, Ç., & Büyükyazıcı, İ. (2010). Stancu type generalization of the Favard– Szász operators. *Applied Mathematics Letters*, 23(12), 1479-1482. doi:10.1016/j.aml.2010.08.017

Atakut, Ç., & Büyükyazıcı, İ. (2016). Approximation by Kantorovich-Szász type operators based on Brenke type polynomials. *Numerical Functional Analysis and Optimization*, 37(12), 1488-1502. doi:10.1080/01630563.2016.1216447

Ayık, A. (2018). *Charlier polinomlarını içeren genelleştirilmiş Szász operatörlerinin Kantrovich tipi genelleştirilmesi*. (Yüksek Lisans Tezi), Necmettin Erbakan Üniversitesi, Fen Bilimleri Enstitüsü, Konya, Türkiye.

Barbosu, D. (2004). Kantorovich-Stancu type operators. *Journal of Inequalities in Pure and Applied Mathematics*, 5(3), 1-6.

Çavdar, B. (2017). *Szász-Charlier tipi operatörlerin gama tipi genelleştirilmesi*. (Yüksek Lisans Tezi), Necmettin Erbakan Üniversitesi, Fen Bilimleri Enstitüsü, Konya, Türkiye.

Ismail, M. E. H. (2005). *Classical and Quantum Orthogonal Polynomials in One Variable* (Encyclopedia of Mathematics and its Applications). Cambridge, UK: Cambridge University Press. doi:10.1017/CBO9781107325982

Jakimovski, A., & Leviatan, D. (1969). Generalized Szász operators for the approximation in the infinite interval. *Mathematica (Cluj)*, 11(34), 97-103.

Kajla, A. & Agrawal, P. N. (2016). Szász-Kantorovich type operators based on Charlier polynomials. *Kyungpook Mathematical Journal*, 56(3), 877-897. doi:10.5666/kmj.2016.56.3.877

Korovkin, P. P. (1953). On convergence of linear positive operators in the space of continuous functions (Russian). *Doklady Akademii Nauk SSSR (NS)*, 90, 961–964.

Păltănea, R. (2008). Modified Szász-Mirakjan operators of integral form. *Carpathian Journal of Mathematics*, 24(3), 378-385.

Szasz, O. (1950). Generalization of S. Bernstein's polynomials to the infinite interval. *Journal of Research of the National Bureau of Standards*, 45(3), 239-245.

Sucu, S. (2022). Stancu type operators including generalized Brenke polynomials. *Filomat*, 36(7), 2381-2389. doi:10.2298/FIL2207381S

Varma, S., & Taşdelen, F. (2012). Szász type operators involving Charlier polynomials. Mathematical and Computer Modelling, 56(5-6), 118-122. doi:10.1016/j.mcm.2011.12.017

Wafi, A., & Rao, N. (2016). A generalization of Szász-type operators which preserves constant and quadratic test functions. Cogent Mathematics, 3(1), 1227023. doi:10.1080/23311835.2016.1227023