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Laplace Transform of nested analytic functions via Bell's polynomials

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Abstract

Bell's polynomials have been used in many different fields, ranging from number theory to operators theory. In this article we show a method to compute the Laplace Transform (LT) of nested analytic functions. To this aim, we provide a table of the first few values of the complete Bell's polynomials, which are then used to evaluate the LT of composite exponential functions. Furthermore a code for approximating the Laplace Transform of general analytic composite functions is created and presented. A graphical verification of the proposed technique is illustrated in the last section.

Keywords: Laplace transform Bell's polynomials Composite functions.
2010 MSC: 44A10, 05A10, 11B65, 11B83.

1. Introduction

The common view that there is no formula for the Laplace Transform (LT) of composite analytic functions is disproved using Bell's polynomials [1, 5, 7, 10, 13], as in the case of the derivative of nested functions, for which Bell's polynomials are exploited.

The Bell's polynomials are exploited in very different fields, ranging from number theory [11, 15, 16] to operators theory [14], and from differential equations [10] to integral transforms [3, 12].

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The importance of the LT [8], and of the even more useful Fourier Transform [2], is well known and it is not necessary to remind it here.

We use the classic definition

$$\mathcal{L}(f) := \int_0^\infty \exp(-s\,t) f(t)\,dt = F(s)\,.$$

The LT converts a function of a real variable t (often representing the time) to a function of a complex variable s (representing the complex frequency). The LT can be applied to locally integrable functions on $[0, +\infty)$. It converges in each half plane $Re(s) > a$, where a is a constant (the so-called convergence abscissa), depending on the behavior at infinity of $f(t)$.

Exploiting the Taylor's expansion of the considered analytic function, and expressing the coefficients in terms of Bell's polynomials computed at the initial point, we approximate the LT of nested functions by a series expansion, which is certainly convergent, if the considered LT exists.
We start from the easier case of the LT of a nested exponential function. To this aim, we show the first few values of the complete Bell's polynomials which are applied in this case. The result is a Laurent expansion approximating the relevant LT.

Then we consider the case of the LT of general nested functions. The main problem is to provide a table of Bell's polynomials, which exhibit a highly increasing number of addends, but their evaluation at a fixed point is an easy matter, using a suitable computer code.

Only in very few cases our results can be compared with the LT of nested functions appearing in the literature. This is shown in equations (17) and (20).

In the last section, the proposed technique was verified also graphically, in the two cases of composed functions whose transform and anti-transform are known (see [9]). All the numerical results were obtained using the computer algebra program Mathematica©.

The second-order Bell's polynomials $Y_n^{[2]}$, representing the derivatives of nested functions of the type $f(g(h(t))$ are then introduced, and two LT examples of this type of function are given.

In the last Section the computer program used in the applications is shown, and a table of second-order Bell's polynomials is reported.

2. Definition of Bell's polynomials

The n-th derivative of the composite (differentiable) function $\Phi(t) := f(g(t))$, as computed by using the chain rule, can be expressed in terms of Bell's polynomials as follows

$$\Phi_n := D_t^n \Phi(t) = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\, f_k, \tag{1}$$

where

$$f_h := D_x^h f(x)|_{x=g(t)}, \quad g_k := D_t^k g(t). \tag{2}$$

The coefficients $B_{n,k}$, for all $k = 1, \ldots, n$, are polynomials of the variables $g_1, g_2, \ldots, g_{n-k+1}$, that are homogeneous of degree k and *isobaric* of weight n (i.e. they are a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$).

The Bell's polynomials satisfy the recursion

$$\begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \ldots; f_n, g_n; f_{n+1}, g_{n+1}) = \\ = \displaystyle\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}) g_{k+1}. \end{cases} \tag{3}$$

and are given explicitly by Faà di Bruno's formula (which has a high computational complexity)

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \ldots r_n!} f_r \left[\frac{g_1}{1!}\right]^{r_1} \left[\frac{g_2}{2!}\right]^{r_2} \cdots \left[\frac{g_n}{n!}\right]^{r_n}, \tag{4}$$

where the sum runs over all the partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ is the number of parts of the considered partition [13].

The $B_{n,k}$ coefficients satisfy the recursion $\forall\, n$

$$\begin{gathered} B_{n,1} = g_n\,, \quad B_{n,n} = g_1^n\,, \\ B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \ldots, g_{n-k-h+1})\, g_{h+1}\,. \end{gathered} \tag{5}$$

3. Laplace transform of composed functions

Let $f(g(t))$ be a composite function analytic in a neighborhood of the origin, whose Taylor's expansion is given by

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\,, \quad a_n = D_t^n[f(g(t))]_{t=0}\,. \tag{6}$$

According to the preceding equations, it results

$$\begin{aligned} &a_0 = f(\mathring{g}_0)\,, \\ &a_n = D_t^n[f(g(t))]_{t=0} = \sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\, \mathring{f}_k\,, \quad (n \geq 1)\,, \end{aligned} \tag{7}$$

where

$$\mathring{f}_k := D_x^k f(x)|_{x=g(0)}, \qquad \mathring{g}_h := D_t^h g(t)|_{t=0}. \tag{8}$$

This expansion can be used in computing the LT of analytic composite functions.

Theorem 3.1. *Consider a composed function $f(g(t))$, analytic in a neighborhood of the origin, and such that its growth to infinity is such that its LT exists. Let its Taylor series expansion be expressed by the equation* (6).
Then, for its LT the following equation holds

$$\int_0^{+\infty} f(g(t)) e^{-ts} dt = \frac{f(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\, \mathring{f}_k \right) \frac{1}{s^{n+1}}\,. \tag{9}$$

Proof. – We firstly note that the convergence of the series in the second member of equation (9) is a direct consequence of the existence of the integral in the first member, which is a prerequisite for our computational approach. Furthermore, using the uniform convergence of Taylor's expansion, we can write

$$\begin{aligned} \int_0^{+\infty} f(g(t)) e^{-ts} dt &= \frac{f(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} \int_0^{+\infty} \sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\, \mathring{f}_k\, \frac{t^n}{n!}\, e^{-ts} dt = \\ &= \frac{f(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\, \mathring{f}_k \right) \int_0^{+\infty} \frac{t^n}{n!}\, e^{-ts} dt\,, \end{aligned} \tag{10}$$

so that the conclusion follows by using the LT of powers.

4. The particular case of the exponential function

In the particular case where $f(x) = e^x$, that is equivalent to considering the function $e^{g(t)}$ and assuming $g(0) = 0$; we then have the more simple form

$$\sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \dots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k = \sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \dots, \overset{\circ}{g}_{n-k+1}) = B_n(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \dots, \overset{\circ}{g}_n)\,, \tag{11}$$

where B_n are the *complete Bell's polynomials.* It is $B_0(g_0) := f(g_0)$, and the first few values of B_n, for $n = 1, 2, \dots, 10$, are given by

$$\begin{aligned}
B_1 &= g_1,\\
B_2 &= g_1^2 + g_2,\\
B_3 &= g_1^3 + 3g_1g_2 + g_3,\\
B_4 &= g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4,\\
B_5 &= g_1^5 + 10g_1^3g_2 + 15g_1g_2^2 + 10g_1^2g_3 + 10g_2g_3 + 5g_1g_4 + g_5,\\
B_6 &= g_1^6 + 15g_1^4g_2 + 45g_1^2g_2^2 + 15g_2^3 + 20g_1^3g_3 + 60g_1g_2g_3 + 10g_3^2 + 15g_1^2g_4 + 15g_2g_4 + 6g_1g_5 + g_6,\\
B_7 &= g_1^7 + 21g_1^5g_2 + 105g_1^3g_2^2 + 105g_1g_2^3 + 35g_1^4g_3 + 210g_1^2g_2g_3 + 105g_2^2g_3 + 70g_1g_3^2 + 35g_1^3g_4 +\\
&\quad 105g_1g_2g_4 + 35g_3g_4 + 21g_1^2g_5 + 21g_2g_5 + 7g_1g_6 + g_7,\\
B_8 &= g_1^8 + 28g_1^6g_2 + 210g_1^4g_2^2 + 420g_1^2g_2^3 + 105g_2^4 + 56g_1^5g_3 + 560g_1^3g_2g_3 + 840g_1g_2^2g_3 + 280g_1^2g_3^2 +\\
&\quad 280g_2g_3^2 + 70g_1^4g_4 + 420g_1^2g_2g_4 + 210g_2^2g_4 + 280g_1g_3g_4 + 35g_4^2 + 56g_1^3g_5 + 168g_1g_2g_5 +\\
&\quad 56g_3g_5 + 28g_1^2g_6 + 28g_2g_6 + 8g_1g_7 + g_8,\\
B_9 &= g_1^9 + 36g_1^7g_2 + 378g_1^5g_2^2 + 1260g_1^3g_2^3 + 945g_1g_2^4 + 84g_1^6g_3 + 1260g_1^4g_2g_3 + 3780g_1^2g_2^2g_3 +\\
&\quad 1260g_2^3g_3 + 840g_1^3g_3^2 + 2520g_1g_2g_3^2 + 280g_3^3 + 126g_1^5g_4 + 1260g_1^3g_2g_4 + 1890g_1g_2^2g_4 +\\
&\quad 1260g_1^2g_3g_4 + 1260g_2g_3g_4 + 315g_1g_4^2 + 126g_1^4g_5 + 756g_1^2g_2g_5 + 378g_2^2g_5 + 504g_1g_3g_5 +\\
&\quad 126g_4g_5 + 84g_1^3g_6 + 252g_1g_2g_6 + 84g_3g_6 + 36g_1^2g_7 + 36g_2g_7 + 9g_1g_8 + g_9,\\
B_{10} &= g_1^{10} + 45g_1^8g_2 + 630g_1^6g_2^2 + 3150g_1^4g_2^3 + 4725g_1^2g_2^4 + 945g_2^5 + 120g_1^7g_3 + 2520g_1^5g_2g_3 +\\
&\quad 12600g_1^3g_2^2g_3 + 12600g_1g_2^3g_3 + 2100g_1^4g_3^2 + 12600g_1^2g_2g_3^2 + 6300g_2^2g_3^2 + 2800g_1g_3^3 +\\
&\quad 210g_1^6g_4 + 3150g_1^4g_2g_4 + 9450g_1^2g_2^2g_4 + 3150g_2^3g_4 + 4200g_1^3g_3g_4 + 12600g_1g_2g_3g_4 +\\
&\quad 2100g_3^2g_4 + 1575g_1^2g_4^2 + 1575g_2g_4^2 + 252g_1^5g_5 + 2520g_1^3g_2g_5 + 3780g_1g_2^2g_5 + 2520g_1^2g_3g_5 +\\
&\quad 2520g_2g_3g_5 + 1260g_1g_4g_5 + 126g_5^2 + 210g_1^4g_6 + 1260g_1^2g_2g_6 + 630g_2^2g_6 + 840g_1g_3g_6 +\\
&\quad 210g_4g_6 + 120g_1^3g_7 + 360g_1g_2g_7 + 120g_3g_7 + 45g_1^2g_8 + 45g_2g_8 + 10g_1g_9 + g_{10}\,.
\end{aligned}$$

The values of the complete Bell's polynomials for particular parameter choices can be found in [11].
The complete Bell's polynomials satisfy the identity (see e.g. [10])

$$B_{n+1}(g_1, \dots, g_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(g_1, \dots, g_{n-k})\, g_{k+1}. \tag{12}$$

In this case equation (9) reduces to

$$\int_0^{+\infty} \exp(g(t))\, e^{-ts} dt = \frac{\exp(\overset{\circ}{g}_0)}{s} + \sum_{n=1}^{\infty} B_n(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \dots, \overset{\circ}{g}_n)\, \frac{1}{s^{n+1}}\ . \tag{13}$$

In what follows we evaluate the approximation of the LT of nested functions. The reported results have been obtained using the computer algebra program Mathematica©.

4.1. Examples

We start considering the case of the LT of nested exponential functions

• Let $f(x) = e^x$ and $g(t) = \sin t$. Then $g_1 = 1, g_2 = 0, g_3 = -1, g_4 = 0$, and in general $g_{2h} = 0, g_{2h+1} = (-1)^h$, $h = 1, 2, 3, \dots$.

According to the above table of B_n, it results

$$B_1(1) = 1,\ B_2(1, 0) = 1,\ B_3(1, 0, -1) = 0,\ B_4(1, 0, -1, 0) = -3,\ B_5(1, 0, -1, 0, 1) = -8\,.$$

Then

$$\int_0^{+\infty} \exp(\sin t)\, e^{-ts} dt = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{3}{s^5} - \frac{8}{s^6} + O\left(\frac{1}{s^7}\right) . \tag{14}$$

• Consider the complete elliptic integral of the second kind $g(t) := E(t)$ and the LT of the corresponding exponential function. We find

$$\begin{aligned}\int_0^{+\infty} \exp(E(t))\, e^{-ts} dt = \frac{e^{\pi/2}}{s} - \frac{\pi}{8s^2} + \frac{\pi^2 - 3\pi}{64s^3} - \frac{\pi^3 - 9\pi^2 + 30\pi}{512s^4} + \\ \frac{\pi^4 - 18\pi^3 + 147\pi^2 - 525\pi}{4096s^5} + O\left(\frac{1}{s^6}\right) .\end{aligned} \tag{15}$$

5. The general case

Examples of the proposed method for approximating the LT of general nested functions are reported in what follows.

• Assuming $f(x) = \arctan(x)$, $g(t) = \log(1+t)$, it results

$$\begin{aligned}\int_0^{+\infty} \arctan[\log(1+t)]\, e^{-ts} dt = \frac{1}{s^2} - \frac{1}{s^3} + \frac{6}{s^5} - \frac{22}{s^6} - \frac{30}{s^7} + \frac{952}{s^8} - \frac{5656}{s^9} + \\ \frac{9952}{s^{10}} - \frac{508320}{s^{11}} + O\left(\frac{1}{s^{12}}\right) .\end{aligned} \tag{16}$$

A lot of further examples can be constructed using the above method and the most of them have not a close expression in terms of special functions.

5.1. Graphical display in two known cases

• Test case #1

Considering the composed function $\cosh(\nu \mathrm{arcsinh}(t))$, it results [9]

$$L(s) = \int_0^{+\infty} \cosh(\nu \mathrm{arcsinh}(t))\, e^{-ts} dt = \frac{S_{1,\nu}(s)}{s}\,,\ \Re s > 0\,. \tag{17}$$

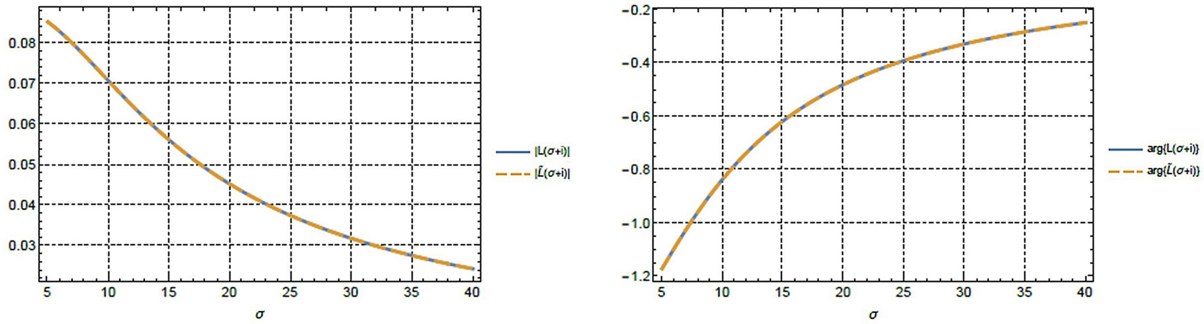
where $S_{1,\nu}$ denotes a special case of the Lommel function $S_{\mu,\nu}$ [6].
Assuming $\nu = \pi$, and using our approximation we have found

$$\begin{aligned}\int_0^{+\infty} \cosh[\pi \mathrm{arcsinh}(t)]\, e^{-ts} dt = \frac{1}{s} + \frac{\pi^2}{s^3} + \frac{\pi^2(\pi^2-4)}{s^5} + \\ \frac{\pi^2(\pi^4 - 20\pi^2 + 64)}{s^7} \frac{\pi^2(\pi^6 - 56\pi^4 + 784\pi^2 - 2304)}{s^9} + \\ \frac{\pi^2(\pi^8 - 120\pi^6 + 4368\pi^4 - 52480\pi^2 + 147456)}{s^{11}} + O\left(\frac{1}{s^{13}}\right) .\end{aligned} \tag{18}$$

so that, by inverse Laplace transformation, one can readily conclude that

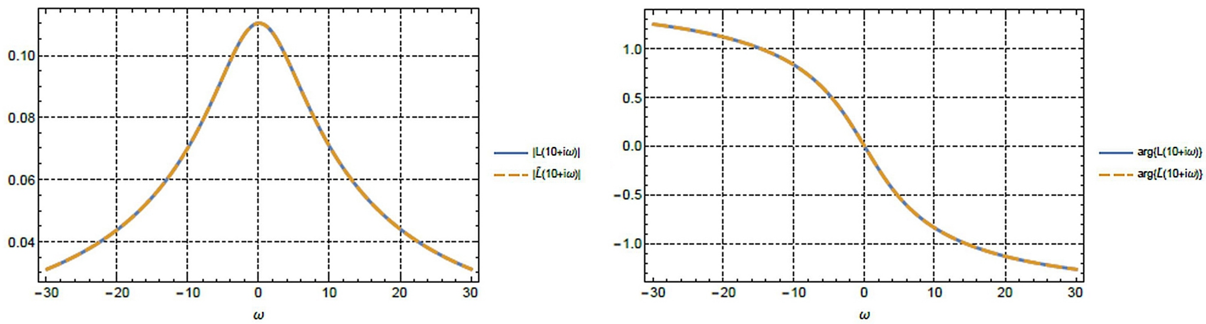
$$\begin{aligned}\tilde{l}(t) \simeq \left(1 + \frac{\pi^2}{2!}t^2 + \frac{\pi^2(\pi^2-4)}{4!}t^4 + \frac{\pi^2(\pi^4 - 20\pi^2 + 64)}{6!}t^6 + \right. \\ \frac{\pi^2(\pi^6 - 56\pi^4 + 784\pi^2 - 2304)}{8!}t^8 + \frac{\pi^2(\pi^8 - 120\pi^6 + 4368\pi^4 - 52480\pi^2 + 147456)}{10!} \\ , t^{10} H(t)\,,\end{aligned} \tag{19}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.
The distributions of $L(s)$ and $\tilde{L}(s)$ along the cut sections $\omega = \Im s = 1$ and $\sigma = \Re s = 5$ are reported in Figures 1 and 2, respectively. As it can be noticed, the agreement between the exact transform (17) (for $\nu = \pi$) and the relevant approximation (18) is very good especially as $s \to +\infty$. Conversely, the functions $l(t)$ and $\tilde{l}(t)$ tend to match for $t \to 0^+$ as one would expect from theory (see Figure 3).



(a) (b)

Figure 1: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \cosh[\pi \mathrm{arcsinh}(t)]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\omega = 1$.



(a) (b)

Figure 2: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \cosh[\pi \mathrm{arcsinh}(t)]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\sigma = 5$.

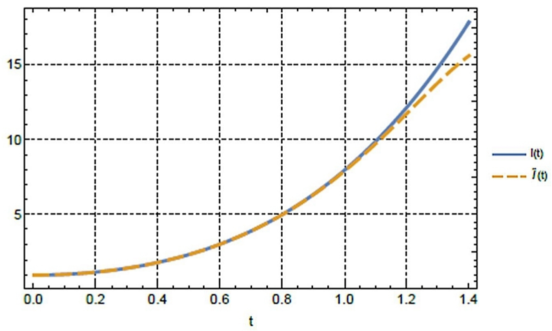


Figure 3: Distribution of $l(t) = \cosh[\pi \mathrm{arcsinh}(t)]$ and the relevant approximant $\tilde{l}(t)$.

• Test case #2

Considering the composed function $J_\nu(a\sinh(t))$ with $\Re a > 0, \Re\nu > -1$, it results [9]

$$L(s) = \int_0^{+\infty} J_\nu(a\sinh(t))\, e^{-ts} dt = J_{\frac{\nu+s}{2}}\left(\tfrac{a}{2}\right) K_{\frac{\nu-s}{2}}\left(\tfrac{a}{2}\right), \;\; \Re s > -\tfrac{1}{2}, \tag{20}$$

where J_ν and K_ν are Bessel functions.
Assuming $\nu = 0$, and $a = 1$, we find the LT

$$L(s) = \int_0^{+\infty} J_0(\sinh(t))\, e^{-ts} dt = J_{\frac{s}{2}}\left(\tfrac{1}{2}\right) K_{-\frac{s}{2}}\left(\tfrac{1}{2}\right), \;\; \Re s > -\tfrac{1}{2}. \tag{21}$$

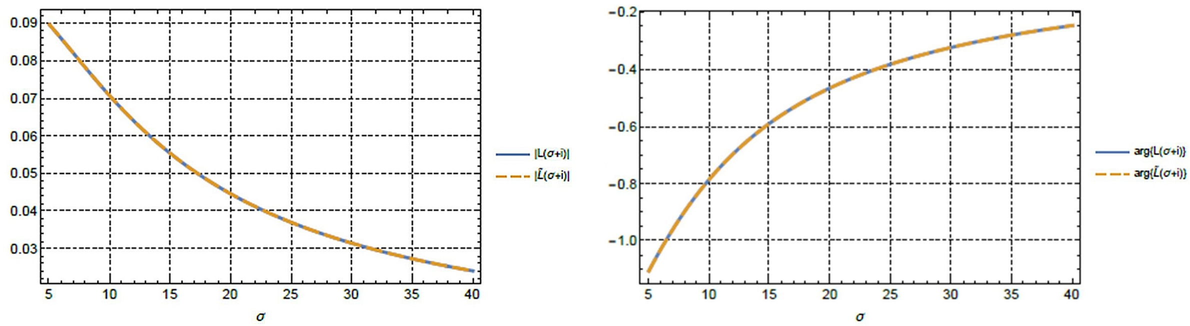
Using our approximation, we have found

$$\begin{aligned} L(s) \simeq \tilde{L}(s) = \int_0^{+\infty} J_0(\sinh(t)) e^{-ts} dt = \frac{1}{s} - \frac{1}{2s^3} - \frac{13}{8s^5} - \frac{13}{16s^7} + \\ \frac{9827}{128s^9} + \frac{309649}{256s^{11}} + O\left(\frac{1}{s^{13}}\right), \end{aligned} \tag{22}$$

so that, by inverse Laplace transformation, one can readily conclude that:

$$\tilde{l}(t) \simeq \left(1 - \frac{1}{4}t^2 - \frac{13}{192}t^4 - \frac{13}{11520}t^6 + \frac{9827}{5160960}t^8 + \frac{309649}{928972800}t^{10}\right) H(t)\;, \tag{23}$$

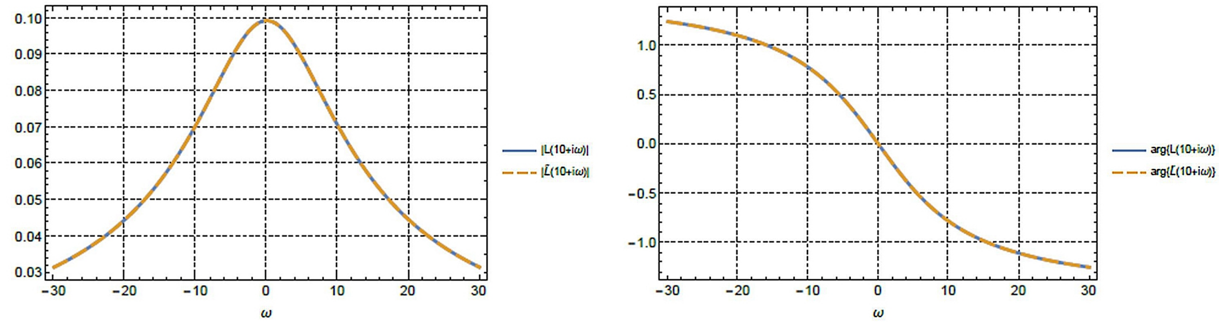
with $H(\cdot)$ denoting the classical Heaviside distribution.



(a) (b)

Figure 4: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = J_0(\sinh(t))$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\omega = 1$.

The distributions of $L(s)$ and $\tilde{L}(s)$ along the cut sections $\omega = \Im s = 1$ and $\sigma = \Re s = 5$ are reported in Figures 4 and 5, respectively. As it can be noticed, the agreement between the exact transform (21) and the relevant approximation (22) is very good especially as $s \to +\infty$. Conversely, the functions $l(t)$ and $\tilde{l}(t)$ tend to match for $t \to 0^+$ as one would expect from theory (see Figure 6).



(a) (b)

Figure 5: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = J_0(\sinh(t))$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical exp

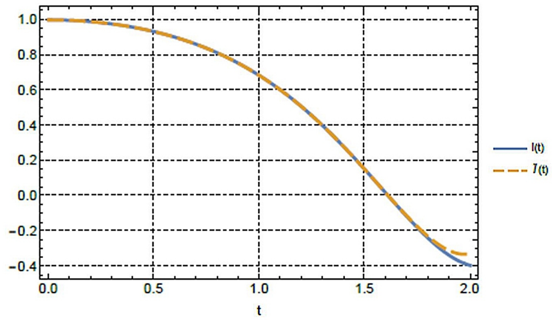


Figure 6: Distribution of $l(t) = J_0(\sinh(t))$ and the relevant approximant $\tilde{l}(t)$.

6. An extension of the Bell's polynomials

We consider the second-order Bell's polynomials, $Y_n^{[2]}(f_1, g_1, h_1; f_1, g_1, h_1; \dots; f_n, g_n, h_n)$, defined by the n-th derivative of the composite function $\Phi(t) := f(g(h(t)))$.

Consider the functions $x = h(t)$, $z = g(x)$, and $y = f(z)$, and suppose that $h(t)$, $g(x)$, and $f(z)$ are n times differentiable with respect to their variables, so that the composite function $\Phi(t) := f(g(h(t)))$ can be differentiated n times with respect to t, by using the chain rule.

We use, as before, the following notations:

$$\Phi_j := D_t^j \Phi(t), \quad f_h := D_y^h f(y)|_{y=g(x)}, \quad g_k := D_x^k g(x)|_{x=h(t)}, \quad h_r := D_t^r h(t).$$

Then the n-th derivative can be represented by

$$\Phi_n = Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \dots; f_n, g_n, h_n) = Y_n^{[2]}([f, g, h]_n)\,,$$

where the $Y_n^{[2]}$ are defined as the second order Bell's polynomials.

The first few polynomials are as follows.

$$\begin{aligned}
&Y_1^{[2]}([f,g,h]_1) = f_1g_1h_1;\\
&Y_2^{[2]}([f,g,h]_2) = f_1g_1h_2 + f_1g_2h_1^2 + f_2g_1^2h_1^2;\\
&Y_3^{[2]}([f,g,h]_3) = f_1g_1h_3 + f_1g_3h_1^3 + 3f_1g_2h_1h_2 + 3f_2g_1g_2h_1^3 + f_3g_1^3h_1^3;\\
&Y_4^{[2]}([f,g,h]_4) = f_4g_1^4h_1^4 + 6f_3g_1^2g_2h_1^4 + 3f_2g_2^2h_1^4 + 4f_2g_1g_3h_1^4 + f_1g_4h_1^4 + 6f_3g_1^3h_1^2h_2 +\\
&+ 18f_2g_1g_2h_1^2h_2 + 6f_1g_3h_1^2h_2 + 3f_2g_1^2h_2^2 + 3f_1g_2h_2^2 + 4f_2g_1^2h_1h_3 + 4f_1g_2h_1h_3 + f_1g_1h_4;\\
&Y_5^{[2]}([f,g,h]_5) = f_5g_1^5h_1^5 + 10f_4g_1^3g_2h_1^5 + 15f_3g_1g_2^2h_1^5 + 10f_3g_1^2g_3h_1^5 + 10f_2g_2g_3h_1^5 +\\
&+ 5f_2g_1g_4h_1^5 + f_1g_5h_1^5 + 10f_4g_1^4h_1^3h_2 + 60f_3g_1^2g_2h_1^3h_2 + 30f_2g_2^2h_1^3h_2 + 40f_2g_1g_3h_1^3h_2 +\\
&+ 10f_1g_4h_1^3h_2 + 15f_3g_1^3h_1h_2^2 + 45f_2g_1g_2h_1h_2^2 + 15f_1g_3h_1h_2^2 + 10f_3g_1^3h_1^2h_3 + 30f_2g_1g_2h_1^2h_3 +\\
&+ 10f_1g_3h_1^2h_3 + 10f_2g_1^2h_2h_3 + 10f_1g_2h_2h_3 + 5f_2g_1^2h_1h_4 + 5f_1g_2h_1h_4 + f_1g_1h_5\,.
\end{aligned} \tag{24}$$

A more extended table is given in the last Section.
The connections to the ordinary Bell's polynomials are expressed below.

Theorem 6.1. *For every integer n, the polynomials $Y_n^{[2]}$ are represented in terms of the ordinary Bell's ones by the following equation*

$$\begin{aligned}
&Y_n^{[2]}(f_1,g_1,h_1;\dots;f_n,g_n,h_n) =\\
&= Y_n\left(f_1, Y_1(g_1,h_1); f_2, Y_2(g_1,h_1;g_2,h_2);\dots;f_n, Y_n(g_1,h_1;g_2,h_2;\dots;g_n,h_n)\right)
\end{aligned} \tag{25}$$

Proof. – Using induction, we have that (28) is true for $n=1$, since

$$Y_1^{[2]}(f_1,g_1,h_1) = f_1\; g_1\; h_1 = f_1\; Y_1(g_1,h_1) = Y_1\left(f_1, Y_1(g_1,h_1)\right).$$

Then assuming that equation (28) is true for n, it follows that

$$\begin{aligned}
&Y_{n+1}^{[2]}(f_1,g_1,h_1;\dots;f_{n+1},g_{n+1},h_{n+1}) = D_t\; Y_n^{[2]}(f_1,g_1,h_1;\dots;f_n,g_n,h_n) =\\
&= D_t\; Y_n\left(f_1, Y_1(g_1,h_1);\dots;f_n, Y_n(g_1,h_1;g_2,h_2;\dots;g_n,h_n)\right) =\\
&= Y_{n+1}\left(f_1, Y_1(g_1,h_1);\dots;f_{n+1}, Y_{n+1}(g_1,h_1;g_2,h_2;\dots;g_{n+1},h_{n+1})\right)\;.
\end{aligned} \tag{26}$$

Consequently, we have the following theorem:

Theorem 6.2. *The second-order Bell's polynomials verify the recursion*

$$\begin{aligned}
&Y_0^{[2]} = f_1;\\
&Y_{n+1}^{[2]}(f_1,g_1,h_1;\dots;f_{n+1},g_{n+1},h_{n+1}) = \sum_{k=0}^{n}\binom{n}{k} Y_{n-k}^{[2]}\left(f_2,g_1,h_1;f_3,g_2,h_2;\;\dots\right.\\
&\qquad\qquad \left.\dots;f_{n-k+1},g_{n-k},h_{n-k}\right) Y_{k+1}(g_1,h_1;\dots;g_{k+1},h_{k+1}).
\end{aligned} \tag{27}$$

Proof. – By means of (28) we express $Y_{n+1}^{[2]}(f_1,g_1,h_1;\dots;f_{n+1},g_{n+1},h_{n+1})$ in terms of $Y_{n+1}\left(f_1, Y_1(g_1,h_1);\dots;f_{n+1}, Y_{n+1}(g_1,h_1;\dots;g_{n+1},h_{n+1})\right)$. Then, by using the recurrence relation (3) and again (28), we obtain the expansion (30).

7. Laplace transform of nested functions

Let be $f(g(h(t)))$ be a composite function analytic in a neighborhood of the origin, so that it is expressed by the Taylor's expansion

$$f(g((h(t))) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\,, \quad a_n = D_t^n[f(g((h(t)))]_{t=0}\,. \tag{28}$$

According to the preceding equations, it results

$$\begin{aligned} &a_0 = \overset{\circ}{f}_0 = f(g(h(0))\,, \\ &a_n = D_t^n[f(g((h(t)))]_{t=0} = Y_n^{[2]}(\overset{\circ}{f}_1, \overset{\circ}{g}_1, \overset{\circ}{h}_1; \ldots; \overset{\circ}{f}_n, \overset{\circ}{g}_n, \overset{\circ}{h}_n)\,, \quad (n \geq 1)\,, \end{aligned} \tag{29}$$

where

$$\overset{\circ}{f}_h := D_x^h f(y)|_{y=g(0)}, \quad \overset{\circ}{g}_k := D_t^k g(x)|_{x=h(0)}, \quad \overset{\circ}{h}_r := D_t^r h(t)|_{t=0} \tag{30}$$

This expansion can be used in computing the LT of analytic nested functions.

Theorem 7.1. *Considering a nested function $f(g((h(t)))$, which is analytic in a neighborhood of the origin, and can be represented by the Taylor's expansion in* (31)*. For its LT the following expression holds*

$$\begin{aligned} \int_0^{+\infty} f(g((h(t)))e^{-ts}dt &= \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{\infty} Y_n^{[2]}(\overset{\circ}{f}_1, \overset{\circ}{g}_1, \overset{\circ}{h}_1; \ldots; \overset{\circ}{f}_n, \overset{\circ}{g}_n, \overset{\circ}{h}_n)\, \frac{t^n}{n!}\, e^{-ts}dt = \\ &= \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{\infty} Y_n^{[2]}(\overset{\circ}{f}_1, \overset{\circ}{g}_1, \overset{\circ}{h}_1; \ldots; \overset{\circ}{f}_n, \overset{\circ}{g}_n, \overset{\circ}{h}_n)\, \frac{1}{s^{n+1}}\,. \end{aligned} \tag{31}$$

Proof. – It is a straightforward application of the definition of second-order Bell's polynomials.

7.1. Example 1

- Assuming $f(x) = e^{x-1}$, $g(y) = \cos(y)$, $h(t) = \sin(t)$, it results

$$\int_0^{+\infty} \exp[\cos(\sin(t)) - 1]\, e^{-ts}dt = \frac{1}{s} - \frac{1}{s^3} + \frac{8}{s^5} - \frac{127}{s^7} + \frac{3523}{s^9} - \frac{146964}{s^{11}} + O\left(\frac{1}{s^{13}}\right)\,. \tag{32}$$

The corresponding inverse LT is approximated by

$$\tilde{l}(t) \simeq \left(1 - \frac{1}{2}\,t^2 + \frac{1}{3}\,t^4 - \frac{127}{720}\,t^6 + \frac{3523}{40320}\,t^8 - \frac{12247}{302400}\,t^{10}\right) H(t)\,, \tag{33}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

7.2. Example 2

- Assuming $f(x) = \log\left(1 + \frac{x}{2}\right)$, $g(y) = \cosh(y) - 1$, $h(t) = \sin(t)$, it results

$$\int_0^{+\infty} \log\left[1 + \frac{\cosh(\sin(t)) - 1}{2}\right]\, e^{-ts}dt = \frac{1}{2s^3} - \frac{9}{4s^5} - \frac{27}{2s^7} + \frac{1169}{8s^9} - \frac{5869}{2s^{11}} + O\left(\frac{1}{s^{13}}\right)\,. \tag{34}$$

The corresponding inverse LT is approximated by

$$\tilde{l}(t) \simeq \left(\frac{1}{4}\,t^2 - \frac{3}{32}\,t^4 + \frac{3}{160}\,t^6 - \frac{167}{46080}\,t^8 + \frac{5869}{7257600}\,t^{10}\right) H(t)\,, \tag{35}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

7.3. Example 3

- Assuming $f(x) = e^x$, $g(y) = J_1(y)$, $h(t) = \sin(t)$, it results

$$\int_0^{+\infty} \exp[J_1(\sin(t)]\, e^{-ts} dt = \frac{1}{s} - \frac{1}{2s^2} + \frac{1}{4s^3} - \frac{3}{4s^4} - \frac{27}{16s^5} + \frac{77}{32s^6} + \\ + \frac{1227}{64s^7} + \frac{385}{128s^8} - \frac{82663}{256s^9} - \frac{439229}{512s^{10}} + \frac{6754489}{1024s^{11}} + O\left(\frac{1}{s^{12}}\right) . \tag{36}$$

The corresponding inverse LT is approximated by

$$\tilde{l}(t) \simeq \left(1 + \frac{1}{2}\,t + \frac{1}{8}\,t^2 - \frac{1}{8}\,t^3 - \frac{9}{128}\,t^4 + \frac{77}{3840}\,t^5 + \frac{409}{15360}\,t^6 + \frac{11}{18432}\,t^7 - \right. \\ \left. \frac{11809}{1474560}\,t^8 - \frac{62747}{26542080}\,t^9 + \frac{964927}{530841600}\,t^{10}\right) H(t)\ , \tag{37}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

7.4. Example 4

- Assuming $f(x) = \arctan(x)$, $g(y) = y^{1/3}$, $h(t) = \cosh(t)$, it results

$$\int_0^{+\infty} \arctan[(\cosh(t))^{1/3}]\, e^{-ts} dt = \frac{\pi}{4s} + \frac{1}{6s^3} - \frac{1}{3s^5} + \frac{43}{18s^7} - \frac{338}{9s^9} + \frac{18523}{18s^{11}} + O\left(\frac{1}{s^{13}}\right) . \tag{38}$$

The corresponding inverse LT is approximated by

$$\tilde{l}(t) \simeq \left(\frac{\pi}{4} + \frac{1}{12}\,t^2 - \frac{1}{72}\,t^4 + \frac{43}{12960}\,t^6 - \frac{169}{181440}\,t^8 + \frac{18523}{65318400}\,t^{10}\right) H(t)\ , \tag{39}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

8. The used Mathematica© code and $Y_n^{[2]}$ polynomials up to $n = 9$

8.1. An example of the used Mathematica© code

Here we report the Mathematica© code used to evaluate the approximation of the LT of composite functions through Bell's polynomials.

```
In[ ]:= g[t_] = Cosh[t];

In[ ]:= f[x_] = Log[x];

In[ ]:= g0[n_] = SeriesCoefficient[n! g[t], {t, 0, n}]

Out[ ]= { 1  Mod[n, 2] == 0 && n ≥ 0
        { 0  True

In[ ]:= f0[n_] = SeriesCoefficient[n! f[x], {x, g[0], n}]

Out[ ]= { (-1)^(1+n) Gamma[n]   n ≥ 1
        { 0                     True
```

In[ ]:= $\mathcal{A}$[n_] := $\sum_{k=1}^{n}$ BellY[n, k, Table[g0[m], {m, 1, n - k + 1}]] × f0[k]

```
In[ ]:= N = 10;

In[ ]:= Table[g0[n], {n, 0, N}]

Out[ ]= {1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1}

In[ ]:= Table[f0[n], {n, 0, N}]

Out[ ]= {0, 1, -1, 2, -6, 24, -120, 720, -5040, 40320, -362880}

In[ ]:= Table[A[n], {n, 0, N}]

Out[ ]= {0, 0, 1, 0, -2, 0, 16, 0, -272, 0, 7936}
```

In[ ]:= $\mathcal{L}[s\_] = \dfrac{f[g[0]]}{s} + \sum_{n=1}^{N} \dfrac{\mathcal{A}[n]}{s^{n+1}}$

Out[ ]= $\dfrac{7936}{s^{11}} - \dfrac{272}{s^{9}} + \dfrac{16}{s^{7}} - \dfrac{2}{s^{5}} + \dfrac{1}{s^{3}}$

8.2. Table of second-order Bell's polynomials

In[◦]:= $$\texttt{Y[n\_] :=} \sum_{k=1}^{n} (\texttt{BellY[n, k, Table[}h_m\texttt{, \{m, 1, n - k + 1\}]]}\; g_k)$$

In[◦]:= $$\texttt{Y2[n\_] :=} \sum_{k=1}^{n} (\texttt{BellY[n, k, Table[Y[m], \{m, 1, n - k + 1\}]]}\; f_k)$$

In[◦]:= `Y2[1] // FullSimplify // Expand`

Out[◦]= $f_1 g_1 h_1$

In[◦]:= `Y2[2] // FullSimplify // Expand`

Out[◦]= $f_2 g_1^2 h_1^2 + f_1 g_2 h_1^2 + f_1 g_1 h_2$

In[◦]:= `Y2[3] // FullSimplify // Expand`

Out[◦]= $f_3 g_1^3 h_1^3 + 3 f_2 g_1 g_2 h_1^3 + f_1 g_3 h_1^3 + 3 f_2 g_1^2 h_1 h_2 + 3 f_1 g_2 h_1 h_2 + f_1 g_1 h_3$

In[◦]:= `Y2[4] // FullSimplify // Expand`

Out[◦]= $f_4 g_1^4 h_1^4 + 6 f_3 g_1^2 g_2 h_1^4 + 3 f_2 g_2^2 h_1^4 + 4 f_2 g_1 g_3 h_1^4 + f_1 g_4 h_1^4 + 6 f_3 g_1^3 h_1^2 h_2 + 18 f_2 g_1 g_2 h_1^2 h_2 + 6 f_1 g_3 h_1^2 h_2 + 3 f_2 g_1^2 h_2^2 + 3 f_1 g_2 h_2^2 + 4 f_2 g_1^2 h_1 h_3 + 4 f_1 g_2 h_1 h_3 + f_1 g_1 h_4$

In[◦]:= `Y2[5] // FullSimplify // Expand`

Out[◦]= $f_5 g_1^5 h_1^5 + 10 f_4 g_1^3 g_2 h_1^5 + 15 f_3 g_1 g_2^2 h_1^5 + 10 f_3 g_1^2 g_3 h_1^5 + 10 f_2 g_2 g_3 h_1^5 + 5 f_2 g_1 g_4 h_1^5 + f_1 g_5 h_1^5 + 10 f_4 g_1^4 h_1^3 h_2 + 60 f_3 g_1^2 g_2 h_1^3 h_2 + 30 f_2 g_2^2 h_1^3 h_2 + 40 f_2 g_1 g_3 h_1^3 h_2 + 10 f_1 g_4 h_1^3 h_2 + 15 f_3 g_1^3 h_1 h_2^2 + 45 f_2 g_1 g_2 h_1 h_2^2 + 15 f_1 g_3 h_1 h_2^2 + 10 f_3 g_1^3 h_1^2 h_3 + 30 f_2 g_1 g_2 h_1^2 h_3 + 10 f_1 g_3 h_1^2 h_3 + 10 f_2 g_1^2 h_2 h_3 + 10 f_1 g_2 h_2 h_3 + 5 f_2 g_1^2 h_1 h_4 + 5 f_1 g_2 h_1 h_4 + f_1 g_1 h_5$

In[◦]:= `Y2[6] // FullSimplify // Expand`

Out[◦]= $f_6 g_1^6 h_1^6 + 15 f_5 g_1^4 g_2 h_1^6 + 45 f_4 g_1^2 g_2^2 h_1^6 + 15 f_3 g_2^3 h_1^6 + 20 f_4 g_1^3 g_3 h_1^6 + 60 f_3 g_1 g_2 g_3 h_1^6 + 10 f_2 g_3^2 h_1^6 + 15 f_3 g_1^2 g_4 h_1^6 + 15 f_2 g_2 g_4 h_1^6 + 6 f_2 g_1 g_5 h_1^6 + f_1 g_6 h_1^6 + 15 f_5 g_1^5 h_1^4 h_2 + 150 f_4 g_1^3 g_2 h_1^4 h_2 + 225 f_3 g_1 g_2^2 h_1^4 h_2 + 150 f_3 g_1^2 g_3 h_1^4 h_2 + 150 f_2 g_2 g_3 h_1^4 h_2 + 75 f_2 g_1 g_4 h_1^4 h_2 + 15 f_1 g_5 h_1^4 h_2 + 45 f_4 g_1^4 h_1^2 h_2^2 + 270 f_3 g_1^2 g_2 h_1^2 h_2^2 + 135 f_2 g_2^2 h_1^2 h_2^2 + 180 f_2 g_1 g_3 h_1^2 h_2^2 + 45 f_1 g_4 h_1^2 h_2^2 + 15 f_3 g_1^3 h_2^3 + 45 f_2 g_1 g_2 h_2^3 + 15 f_1 g_3 h_2^3 + 20 f_4 g_1^4 h_1^3 h_3 + 120 f_3 g_1^2 g_2 h_1^3 h_3 + 60 f_2 g_2^2 h_1^3 h_3 + 80 f_2 g_1 g_3 h_1^3 h_3 + 20 f_1 g_4 h_1^3 h_3 + 60 f_3 g_1^3 h_1 h_2 h_3 + 180 f_2 g_1 g_2 h_1 h_2 h_3 + 60 f_1 g_3 h_1 h_2 h_3 + 10 f_2 g_1^2 h_3^2 + 10 f_1 g_2 h_3^2 + 15 f_3 g_1^3 h_1^2 h_4 + 45 f_2 g_1 g_2 h_1^2 h_4 + 15 f_1 g_3 h_1^2 h_4 + 15 f_2 g_1^2 h_2 h_4 + 15 f_1 g_2 h_2 h_4 + 6 f_2 g_1^2 h_1 h_5 + 6 f_1 g_2 h_1 h_5 + f_1 g_1 h_6$

In[∘]:= `Y2[7] // FullSimplify // Expand`

Out[∘]=
$$\begin{aligned}
&f_7 g_1^7 h_1^7 + 21 f_6 g_1^5 g_2 h_1^7 + 105 f_5 g_1^3 g_2^2 h_1^7 + 105 f_4 g_1 g_2^3 h_1^7 + 35 f_5 g_1^4 g_3 h_1^7 + 210 f_4 g_1^2 g_2 g_3 h_1^7 + \\
&105 f_3 g_2^2 g_3 h_1^7 + 70 f_3 g_1 g_3^2 h_1^7 + 35 f_4 g_1^3 g_4 h_1^7 + 105 f_3 g_1 g_2 g_4 h_1^7 + 35 f_2 g_3 g_4 h_1^7 + \\
&21 f_3 g_1^2 g_5 h_1^7 + 21 f_2 g_2 g_5 h_1^7 + 7 f_2 g_1 g_6 h_1^7 + f_1 g_7 h_1^7 + 21 f_6 g_1^6 h_1^5 h_2 + 315 f_5 g_1^4 g_2 h_1^5 h_2 + \\
&945 f_4 g_1^2 g_2^2 h_1^5 h_2 + 315 f_3 g_2^3 h_1^5 h_2 + 420 f_4 g_1^3 g_3 h_1^5 h_2 + 1260 f_3 g_1 g_2 g_3 h_1^5 h_2 + \\
&210 f_2 g_3^2 h_1^5 h_2 + 315 f_3 g_1^2 g_4 h_1^5 h_2 + 315 f_2 g_2 g_4 h_1^5 h_2 + 126 f_2 g_1 g_5 h_1^5 h_2 + 21 f_1 g_6 h_1^5 h_2 + \\
&105 f_5 g_1^5 h_1^3 h_2^2 + 1050 f_4 g_1^3 g_2 h_1^3 h_2^2 + 1575 f_3 g_1 g_2^2 h_1^3 h_2^2 + 1050 f_3 g_1^2 g_3 h_1^3 h_2^2 + \\
&1050 f_2 g_2 g_3 h_1^3 h_2^2 + 525 f_2 g_1 g_4 h_1^3 h_2^2 + 105 f_1 g_5 h_1^3 h_2^2 + 105 f_4 g_1^4 h_1 h_2^3 + 630 f_3 g_1^2 g_2 h_1 h_2^3 + \\
&315 f_2 g_2^2 h_1 h_2^3 + 420 f_2 g_1 g_3 h_1 h_2^3 + 105 f_1 g_4 h_1 h_2^3 + 35 f_5 g_1^5 h_1^4 h_3 + 350 f_4 g_1^3 g_2 h_1^4 h_3 + \\
&525 f_3 g_1 g_2^2 h_1^4 h_3 + 350 f_3 g_1^2 g_3 h_1^4 h_3 + 350 f_2 g_2 g_3 h_1^4 h_3 + 175 f_2 g_1 g_4 h_1^4 h_3 + 35 f_1 g_5 h_1^4 h_3 + \\
&210 f_4 g_1^4 h_1^2 h_2 h_3 + 1260 f_3 g_1^2 g_2 h_1^2 h_2 h_3 + 630 f_2 g_2^2 h_1^2 h_2 h_3 + 840 f_2 g_1 g_3 h_1^2 h_2 h_3 + \\
&210 f_1 g_4 h_1^2 h_2 h_3 + 105 f_3 g_1^3 h_2^2 h_3 + 315 f_2 g_1 g_2 h_2^2 h_3 + 105 f_1 g_3 h_2^2 h_3 + 70 f_3 g_1^3 h_1 h_3^2 + \\
&210 f_2 g_1 g_2 h_1 h_3^2 + 70 f_1 g_3 h_1 h_3^2 + 35 f_4 g_1^4 h_1^3 h_4 + 210 f_3 g_1^2 g_2 h_1^3 h_4 + 105 f_2 g_2^2 h_1^3 h_4 + \\
&140 f_2 g_1 g_3 h_1^3 h_4 + 35 f_1 g_4 h_1^3 h_4 + 105 f_3 g_1^3 h_1 h_2 h_4 + 315 f_2 g_1 g_2 h_1 h_2 h_4 + \\
&105 f_1 g_3 h_1 h_2 h_4 + 35 f_2 g_1^2 h_3 h_4 + 35 f_1 g_2 h_3 h_4 + 21 f_3 g_1^3 h_1^2 h_5 + 63 f_2 g_1 g_2 h_1^2 h_5 + \\
&21 f_1 g_3 h_1^2 h_5 + 21 f_2 g_1^2 h_2 h_5 + 21 f_1 g_2 h_2 h_5 + 7 f_2 g_1^2 h_1 h_6 + 7 f_1 g_2 h_1 h_6 + f_1 g_1 h_7
\end{aligned}$$

In[∘]:= `Y2[8] // FullSimplify // Expand`

Out[∘]=
$$\begin{aligned}
&f_8 g_1^8 h_1^8 + 28 f_7 g_1^6 g_2 h_1^8 + 210 f_6 g_1^4 g_2^2 h_1^8 + 420 f_5 g_1^2 g_2^3 h_1^8 + 105 f_4 g_2^4 h_1^8 + 56 f_6 g_1^5 g_3 h_1^8 + \\
&560 f_5 g_1^3 g_2 g_3 h_1^8 + 840 f_4 g_1 g_2^2 g_3 h_1^8 + 280 f_4 g_1^2 g_3^2 h_1^8 + 280 f_3 g_2 g_3^2 h_1^8 + 70 f_5 g_1^4 g_4 h_1^8 + \\
&420 f_4 g_1^2 g_2 g_4 h_1^8 + 210 f_3 g_2^2 g_4 h_1^8 + 280 f_3 g_1 g_3 g_4 h_1^8 + 35 f_2 g_4^2 h_1^8 + 56 f_4 g_1^3 g_5 h_1^8 + \\
&168 f_3 g_1 g_2 g_5 h_1^8 + 56 f_2 g_3 g_5 h_1^8 + 28 f_3 g_1^2 g_6 h_1^8 + 28 f_2 g_2 g_6 h_1^8 + 8 f_2 g_1 g_7 h_1^8 + f_1 g_8 h_1^8 + \\
&28 f_7 g_1^7 h_1^6 h_2 + 588 f_6 g_1^5 g_2 h_1^6 h_2 + 2940 f_5 g_1^3 g_2^2 h_1^6 h_2 + 2940 f_4 g_1 g_2^3 h_1^6 h_2 + 980 f_5 g_1^4 g_3 h_1^6 h_2 + \\
&5880 f_4 g_1^2 g_2 g_3 h_1^6 h_2 + 2940 f_3 g_2^2 g_3 h_1^6 h_2 + 1960 f_3 g_1 g_3^2 h_1^6 h_2 + 980 f_4 g_1^3 g_4 h_1^6 h_2 + \\
&2940 f_3 g_1 g_2 g_4 h_1^6 h_2 + 980 f_2 g_3 g_4 h_1^6 h_2 + 588 f_3 g_1^2 g_5 h_1^6 h_2 + 588 f_2 g_2 g_5 h_1^6 h_2 + \\
&196 f_2 g_1 g_6 h_1^6 h_2 + 28 f_1 g_7 h_1^6 h_2 + 210 f_6 g_1^6 h_1^4 h_2^2 + 3150 f_5 g_1^4 g_2 h_1^4 h_2^2 + 9450 f_4 g_1^2 g_2^2 h_1^4 h_2^2 + \\
&3150 f_3 g_2^3 h_1^4 h_2^2 + 4200 f_4 g_1^3 g_3 h_1^4 h_2^2 + 12\,600 f_3 g_1 g_2 g_3 h_1^4 h_2^2 + 2100 f_2 g_3^2 h_1^4 h_2^2 + \\
&3150 f_3 g_1^2 g_4 h_1^4 h_2^2 + 3150 f_2 g_2 g_4 h_1^4 h_2^2 + 1260 f_2 g_1 g_5 h_1^4 h_2^2 + 210 f_1 g_6 h_1^4 h_2^2 + 420 f_5 g_1^5 h_1^2 h_2^3 + \\
&4200 f_4 g_1^3 g_2 h_1^2 h_2^3 + 6300 f_3 g_1 g_2^2 h_1^2 h_2^3 + 4200 f_3 g_1^2 g_3 h_1^2 h_2^3 + 4200 f_2 g_2 g_3 h_1^2 h_2^3 + \\
&2100 f_2 g_1 g_4 h_1^2 h_2^3 + 420 f_1 g_5 h_1^2 h_2^3 + 105 f_4 g_1^4 h_2^4 + 630 f_3 g_1^2 g_2 h_2^4 + 315 f_2 g_2^2 h_2^4 + \\
&420 f_2 g_1 g_3 h_2^4 + 105 f_1 g_4 h_2^4 + 56 f_6 g_1^6 h_1^5 h_3 + 840 f_5 g_1^4 g_2 h_1^5 h_3 + 2520 f_4 g_1^2 g_2^2 h_1^5 h_3 + \\
&840 f_3 g_2^3 h_1^5 h_3 + 1120 f_4 g_1^3 g_3 h_1^5 h_3 + 3360 f_3 g_1 g_2 g_3 h_1^5 h_3 + 560 f_2 g_3^2 h_1^5 h_3 + \\
&840 f_3 g_1^2 g_4 h_1^5 h_3 + 840 f_2 g_2 g_4 h_1^5 h_3 + 336 f_2 g_1 g_5 h_1^5 h_3 + 56 f_1 g_6 h_1^5 h_3 + 560 f_5 g_1^5 h_1^3 h_2 h_3 + \\
&5600 f_4 g_1^3 g_2 h_1^3 h_2 h_3 + 8400 f_3 g_1 g_2^2 h_1^3 h_2 h_3 + 5600 f_3 g_1^2 g_3 h_1^3 h_2 h_3 + 5600 f_2 g_2 g_3 h_1^3 h_2 h_3 + \\
&2800 f_2 g_1 g_4 h_1^3 h_2 h_3 + 560 f_1 g_5 h_1^3 h_2 h_3 + 840 f_4 g_1^4 h_1 h_2^2 h_3 + 5040 f_3 g_1^2 g_2 h_1 h_2^2 h_3 + \\
&2520 f_2 g_2^2 h_1 h_2^2 h_3 + 3360 f_2 g_1 g_3 h_1 h_2^2 h_3 + 840 f_1 g_4 h_1 h_2^2 h_3 + 280 f_4 g_1^4 h_1^2 h_3^2 + \\
&1680 f_3 g_1^2 g_2 h_1^2 h_3^2 + 840 f_2 g_2^2 h_1^2 h_3^2 + 1120 f_2 g_1 g_3 h_1^2 h_3^2 + 280 f_1 g_4 h_1^2 h_3^2 + 280 f_3 g_1^3 h_2 h_3^2 + \\
&840 f_2 g_1 g_2 h_2 h_3^2 + 280 f_1 g_3 h_2 h_3^2 + 70 f_5 g_1^5 h_1^4 h_4 + 700 f_4 g_1^3 g_2 h_1^4 h_4 + 1050 f_3 g_1 g_2^2 h_1^4 h_4 + \\
&700 f_3 g_1^2 g_3 h_1^4 h_4 + 700 f_2 g_2 g_3 h_1^4 h_4 + 350 f_2 g_1 g_4 h_1^4 h_4 + 70 f_1 g_5 h_1^4 h_4 + 420 f_4 g_1^4 h_1^2 h_2 h_4 + \\
&2520 f_3 g_1^2 g_2 h_1^2 h_2 h_4 + 1260 f_2 g_2^2 h_1^2 h_2 h_4 + 1680 f_2 g_1 g_3 h_1^2 h_2 h_4 + 420 f_1 g_4 h_1^2 h_2 h_4 + \\
&210 f_3 g_1^3 h_2^2 h_4 + 630 f_2 g_1 g_2 h_2^2 h_4 + 210 f_1 g_3 h_2^2 h_4 + 280 f_3 g_1^3 h_1 h_3 h_4 + 840 f_2 g_1 g_2 h_1 h_3 h_4 + \\
&280 f_1 g_3 h_1 h_3 h_4 + 35 f_2 g_1^2 h_4^2 + 35 f_1 g_2 h_4^2 + 56 f_4 g_1^4 h_1^3 h_5 + 336 f_3 g_1^2 g_2 h_1^3 h_5 + \\
&168 f_2 g_2^2 h_1^3 h_5 + 224 f_2 g_1 g_3 h_1^3 h_5 + 56 f_1 g_4 h_1^3 h_5 + 168 f_3 g_1^3 h_1 h_2 h_5 + 504 f_2 g_1 g_2 h_1 h_2 h_5 + \\
&168 f_1 g_3 h_1 h_2 h_5 + 56 f_2 g_1^2 h_3 h_5 + 56 f_1 g_2 h_3 h_5 + 28 f_3 g_1^3 h_1^2 h_6 + 84 f_2 g_1 g_2 h_1^2 h_6 + \\
&28 f_1 g_3 h_1^2 h_6 + 28 f_2 g_1^2 h_2 h_6 + 28 f_1 g_2 h_2 h_6 + 8 f_2 g_1^2 h_1 h_7 + 8 f_1 g_2 h_1 h_7 + f_1 g_1 h_8
\end{aligned}$$

In[◦]:= `Y2[9] // FullSimplify // Expand`

Out[◦]=

$$
\begin{aligned}
& f_9 g_1^9 h_1^9 + 36 f_8 g_1^7 g_2 h_1^9 + 378 f_7 g_1^5 g_2^2 h_1^9 + 1260 f_6 g_1^3 g_2^3 h_1^9 + 945 f_5 g_1 g_2^4 h_1^9 + \\
& 84 f_7 g_1^6 g_3 h_1^9 + 1260 f_6 g_1^4 g_2 g_3 h_1^9 + 3780 f_5 g_1^2 g_2^2 g_3 h_1^9 + 1260 f_4 g_2^3 g_3 h_1^9 + \\
& 840 f_5 g_1^3 g_3^2 h_1^9 + 2520 f_4 g_1 g_2 g_3^2 h_1^9 + 280 f_3 g_3^3 h_1^9 + 126 f_6 g_1^5 g_4 h_1^9 + 1260 f_5 g_1^3 g_2 g_4 h_1^9 + \\
& 1890 f_4 g_1 g_2^2 g_4 h_1^9 + 1260 f_4 g_1^2 g_3 g_4 h_1^9 + 1260 f_3 g_2 g_3 g_4 h_1^9 + 315 f_3 g_1 g_4^2 h_1^9 + \\
& 126 f_5 g_1^4 g_5 h_1^9 + 756 f_4 g_1^2 g_2 g_5 h_1^9 + 378 f_3 g_2^2 g_5 h_1^9 + 504 f_3 g_1 g_3 g_5 h_1^9 + 126 f_2 g_4 g_5 h_1^9 + \\
& 84 f_4 g_1^3 g_6 h_1^9 + 252 f_3 g_1 g_2 g_6 h_1^9 + 84 f_2 g_3 g_6 h_1^9 + 36 f_3 g_1^2 g_7 h_1^9 + 36 f_2 g_2 g_7 h_1^9 + \\
& 9 f_2 g_1 g_8 h_1^9 + f_1 g_9 h_1^9 + 36 f_8 g_1^8 h_1^7 h_2 + 1008 f_7 g_1^6 g_2 h_1^7 h_2 + 7560 f_6 g_1^4 g_2^2 h_1^7 h_2 + \\
& 15\,120 f_5 g_1^2 g_2^3 h_1^7 h_2 + 3780 f_4 g_2^4 h_1^7 h_2 + 2016 f_6 g_1^5 g_3 h_1^7 h_2 + 20\,160 f_5 g_1^3 g_2 g_3 h_1^7 h_2 + \\
& 30\,240 f_4 g_1 g_2^2 g_3 h_1^7 h_2 + 10\,080 f_4 g_1^2 g_3^2 h_1^7 h_2 + 10\,080 f_3 g_2 g_3^2 h_1^7 h_2 + 2520 f_5 g_1^4 g_4 h_1^7 h_2 + \\
& 15\,120 f_4 g_1^2 g_2 g_4 h_1^7 h_2 + 7560 f_3 g_2^2 g_4 h_1^7 h_2 + 10\,080 f_3 g_1 g_3 g_4 h_1^7 h_2 + 1260 f_2 g_4^2 h_1^7 h_2 + \\
& 2016 f_4 g_1^3 g_5 h_1^7 h_2 + 6048 f_3 g_1 g_2 g_5 h_1^7 h_2 + 2016 f_2 g_3 g_5 h_1^7 h_2 + 1008 f_3 g_1^2 g_6 h_1^7 h_2 + \\
& 1008 f_2 g_2 g_6 h_1^7 h_2 + 288 f_2 g_1 g_7 h_1^7 h_2 + 36 f_1 g_8 h_1^7 h_2 + 378 f_7 g_1^7 h_1^5 h_2^2 + 7938 f_6 g_1^5 g_2 h_1^5 h_2^2 + \\
& 39\,690 f_5 g_1^3 g_2^2 h_1^5 h_2^2 + 39\,690 f_4 g_1 g_2^3 h_1^5 h_2^2 + 13\,230 f_5 g_1^4 g_3 h_1^5 h_2^2 + 79\,380 f_4 g_1^2 g_2 g_3 h_1^5 h_2^2 + \\
& 39\,690 f_3 g_2^2 g_3 h_1^5 h_2^2 + 26\,460 f_3 g_1 g_3^2 h_1^5 h_2^2 + 13\,230 f_4 g_1^3 g_4 h_1^5 h_2^2 + 39\,690 f_3 g_1 g_2 g_4 h_1^5 h_2^2 + \\
& 13\,230 f_2 g_3 g_4 h_1^5 h_2^2 + 7938 f_3 g_1^2 g_5 h_1^5 h_2^2 + 7938 f_2 g_2 g_5 h_1^5 h_2^2 + 2646 f_2 g_1 g_6 h_1^5 h_2^2 + \\
& 378 f_1 g_7 h_1^5 h_2^2 + 1260 f_6 g_1^6 h_1^3 h_2^3 + 18\,900 f_5 g_1^4 g_2 h_1^3 h_2^3 + 56\,700 f_4 g_1^2 g_2^2 h_1^3 h_2^3 + \\
& 18\,900 f_3 g_2^3 h_1^3 h_2^3 + 25\,200 f_4 g_1^3 g_3 h_1^3 h_2^3 + 75\,600 f_3 g_1 g_2 g_3 h_1^3 h_2^3 + 12\,600 f_2 g_3^2 h_1^3 h_2^3 + \\
& 18\,900 f_3 g_1^2 g_4 h_1^3 h_2^3 + 18\,900 f_2 g_2 g_4 h_1^3 h_2^3 + 7560 f_2 g_1 g_5 h_1^3 h_2^3 + 1260 f_1 g_6 h_1^3 h_2^3 + \\
& 945 f_5 g_1^5 h_1 h_2^4 + 9450 f_4 g_1^3 g_2 h_1 h_2^4 + 14\,175 f_3 g_1 g_2^2 h_1 h_2^4 + 9450 f_3 g_1^2 g_3 h_1 h_2^4 + \\
& 9450 f_2 g_2 g_3 h_1 h_2^4 + 4725 f_2 g_1 g_4 h_1 h_2^4 + 945 f_1 g_5 h_1 h_2^4 + 84 f_7 g_1^7 h_1^6 h_3 + 1764 f_6 g_1^5 g_2 h_1^6 h_3 + \\
& 8820 f_5 g_1^3 g_2^2 h_1^6 h_3 + 8820 f_4 g_1 g_2^3 h_1^6 h_3 + 2940 f_5 g_1^4 g_3 h_1^6 h_3 + 17\,640 f_4 g_1^2 g_2 g_3 h_1^6 h_3 + \\
& 8820 f_3 g_2^2 g_3 h_1^6 h_3 + 5880 f_3 g_1 g_3^2 h_1^6 h_3 + 2940 f_4 g_1^3 g_4 h_1^6 h_3 + 8820 f_3 g_1 g_2 g_4 h_1^6 h_3 + \\
& 2940 f_2 g_3 g_4 h_1^6 h_3 + 1764 f_3 g_1^2 g_5 h_1^6 h_3 + 1764 f_2 g_2 g_5 h_1^6 h_3 + 588 f_2 g_1 g_6 h_1^6 h_3 + \\
& 84 f_1 g_7 h_1^6 h_3 + 1260 f_6 g_1^6 h_1^4 h_2 h_3 + 18\,900 f_5 g_1^4 g_2 h_1^4 h_2 h_3 + 56\,700 f_4 g_1^2 g_2^2 h_1^4 h_2 h_3 + \\
& 18\,900 f_3 g_2^3 h_1^4 h_2 h_3 + 25\,200 f_4 g_1^3 g_3 h_1^4 h_2 h_3 + 75\,600 f_3 g_1 g_2 g_3 h_1^4 h_2 h_3 + \\
& 12\,600 f_2 g_3^2 h_1^4 h_2 h_3 + 18\,900 f_3 g_1^2 g_4 h_1^4 h_2 h_3 + 18\,900 f_2 g_2 g_4 h_1^4 h_2 h_3 + 7560 f_2 g_1 g_5 h_1^4 h_2 h_3 + \\
& 1260 f_1 g_6 h_1^4 h_2 h_3 + 3780 f_5 g_1^5 h_1^2 h_2^2 h_3 + 37\,800 f_4 g_1^3 g_2 h_1^2 h_2^2 h_3 + 56\,700 f_3 g_1 g_2^2 h_1^2 h_2^2 h_3 + \\
& 37\,800 f_3 g_1^2 g_3 h_1^2 h_2^2 h_3 + 37\,800 f_2 g_2 g_3 h_1^2 h_2^2 h_3 + 18\,900 f_2 g_1 g_4 h_1^2 h_2^2 h_3 + 3780 f_1 g_5 h_1^2 h_2^2 h_3 + \\
& 1260 f_4 g_1^4 h_2^3 h_3 + 7560 f_3 g_1^2 g_2 h_2^3 h_3 + 3780 f_2 g_2^2 h_2^3 h_3 + 5040 f_2 g_1 g_3 h_2^3 h_3 + 1260 f_1 g_4 h_2^3 h_3 + \\
& 840 f_5 g_1^5 h_1^3 h_3^2 + 8400 f_4 g_1^3 g_2 h_1^3 h_3^2 + 12\,600 f_3 g_1 g_2^2 h_1^3 h_3^2 + 8400 f_3 g_1^2 g_3 h_1^3 h_3^2 + \\
& 8400 f_2 g_2 g_3 h_1^3 h_3^2 + 4200 f_2 g_1 g_4 h_1^3 h_3^2 + 840 f_1 g_5 h_1^3 h_3^2 + 2520 f_4 g_1^4 h_1 h_2 h_3^2 + \\
& 15\,120 f_3 g_1^2 g_2 h_1 h_2 h_3^2 + 7560 f_2 g_2^2 h_1 h_2 h_3^2 + 10\,080 f_2 g_1 g_3 h_1 h_2 h_3^2 + 2520 f_1 g_4 h_1 h_2 h_3^2 + \\
& 280 f_3 g_1^3 h_3^3 + 840 f_2 g_1 g_2 h_3^3 + 280 f_1 g_3 h_3^3 + 126 f_6 g_1^6 h_1^5 h_4 + 1890 f_5 g_1^4 g_2 h_1^5 h_4 + \\
& 5670 f_4 g_1^2 g_2^2 h_1^5 h_4 + 1890 f_3 g_2^3 h_1^5 h_4 + 2520 f_4 g_1^3 g_3 h_1^5 h_4 + 7560 f_3 g_1 g_2 g_3 h_1^5 h_4 + \\
& 1260 f_2 g_3^2 h_1^5 h_4 + 1890 f_3 g_1^2 g_4 h_1^5 h_4 + 1890 f_2 g_2 g_4 h_1^5 h_4 + 756 f_2 g_1 g_5 h_1^5 h_4 + \\
& 126 f_1 g_6 h_1^5 h_4 + 1260 f_5 g_1^5 h_1^3 h_2 h_4 + 12\,600 f_4 g_1^3 g_2 h_1^3 h_2 h_4 + 18\,900 f_3 g_1 g_2^2 h_1^3 h_2 h_4 + \\
& 12\,600 f_3 g_1^2 g_3 h_1^3 h_2 h_4 + 12\,600 f_2 g_2 g_3 h_1^3 h_2 h_4 + 6300 f_2 g_1 g_4 h_1^3 h_2 h_4 + 1260 f_1 g_5 h_1^3 h_2 h_4 + \\
& 1890 f_4 g_1^4 h_1 h_2^2 h_4 + 11\,340 f_3 g_1^2 g_2 h_1 h_2^2 h_4 + 5670 f_2 g_2^2 h_1 h_2^2 h_4 + 7560 f_2 g_1 g_3 h_1 h_2^2 h_4 + \\
& 1890 f_1 g_4 h_1 h_2^2 h_4 + 1260 f_4 g_1^4 h_1^2 h_3 h_4 + 7560 f_3 g_1^2 g_2 h_1^2 h_3 h_4 + 3780 f_2 g_2^2 h_1^2 h_3 h_4 + \\
& 5040 f_2 g_1 g_3 h_1^2 h_3 h_4 + 1260 f_1 g_4 h_1^2 h_3 h_4 + 1260 f_3 g_1^3 h_2 h_3 h_4 + 3780 f_2 g_1 g_2 h_2 h_3 h_4 + \\
& 1260 f_1 g_3 h_2 h_3 h_4 + 315 f_3 g_1^3 h_1 h_4^2 + 945 f_2 g_1 g_2 h_1 h_4^2 + 315 f_1 g_3 h_1 h_4^2 + 126 f_5 g_1^5 h_1^4 h_5 + \\
& 1260 f_4 g_1^3 g_2 h_1^4 h_5 + 1890 f_3 g_1 g_2^2 h_1^4 h_5 + 1260 f_3 g_1^2 g_3 h_1^4 h_5 + 1260 f_2 g_2 g_3 h_1^4 h_5 + \\
& 630 f_2 g_1 g_4 h_1^4 h_5 + 126 f_1 g_5 h_1^4 h_5 + 756 f_4 g_1^4 h_1^2 h_2 h_5 + 4536 f_3 g_1^2 g_2 h_1^2 h_2 h_5 + \\
& 2268 f_2 g_2^2 h_1^2 h_2 h_5 + 3024 f_2 g_1 g_3 h_1^2 h_2 h_5 + 756 f_1 g_4 h_1^2 h_2 h_5 + 378 f_3 g_1^3 h_2^2 h_5 + \\
& 1134 f_2 g_1 g_2 h_2^2 h_5 + 378 f_1 g_3 h_2^2 h_5 + 504 f_3 g_1^3 h_1 h_3 h_5 + 1512 f_2 g_1 g_2 h_1 h_3 h_5 + \\
& 504 f_1 g_3 h_1 h_3 h_5 + 126 f_2 g_1^2 h_4 h_5 + 126 f_1 g_2 h_4 h_5 + 84 f_4 g_1^4 h_1^3 h_6 + 504 f_3 g_1^2 g_2 h_1^3 h_6 + \\
& 252 f_2 g_2^2 h_1^3 h_6 + 336 f_2 g_1 g_3 h_1^3 h_6 + 84 f_1 g_4 h_1^3 h_6 + 252 f_3 g_1^3 h_1 h_2 h_6 + 756 f_2 g_1 g_2 h_1 h_2 h_6 + \\
& 252 f_1 g_3 h_1 h_2 h_6 + 84 f_2 g_1^2 h_3 h_6 + 84 f_1 g_2 h_3 h_6 + 36 f_3 g_1^3 h_1^2 h_7 + 108 f_2 g_1 g_2 h_1^2 h_7 + \\
& 36 f_1 g_3 h_1^2 h_7 + 36 f_2 g_1^2 h_2 h_7 + 36 f_1 g_2 h_2 h_7 + 9 f_2 g_1^2 h_1 h_8 + 9 f_1 g_2 h_1 h_8 + f_1 g_1 h_9
\end{aligned}
$$

9. Conclusion

We have presented a method for approximating the integral of analytic composite functions. We started from the Taylor expansion of the considered function in a neighborhood of the origin. Since the coefficients con be expressed in terms of Bell's polynomials, the integral is reduced to the computation of an approximating series, which obviously converges if the integral is convergent. Then this methodology has been applied to the case of the LT of an analytic composite function, starting from the case of analytic nested exponential functions. The relevant Mathematica© code is provided. In the last Section the LT of analytic nested functions is considered, and the second-order Bell's polynomials used in this approach are reported. We want to stress that, even if we dealt with a basic subject, we have not found in the literature any general method for approximating this type of LTs, a gap which, in our opinion, has been now filled up. A graphical verification of the proposed technique, performed in the case when both the analytical forms of the transform and anti-transform are known, proved the correctness of our results.

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