Anadolu Üniversitesi Bilim ve Teknoloji Dergisi B- Teorik Bilimler
Anadolu University Journal of Science and Technology B - Theoretical Sciences

ANADOLU UNIVERSITY

2017 - Volume: 5 Number: 1
Page: 70 - 76
DOI: 10.20290/aubtdb.289635
Received: 27 January 2017 Revised: 06 March 2017 Accepted: 08 March 2017

CALCULATION OF CUTOFF FREQUENCY FOR POLYNOMIAL FAMILIES

Taner BÜYÜKKÖROĞLU [1,*], Gökhan ÇELEBİ [2]

[1] Department of Mathematics, Faculty of Science, Anadolu University, Eskişehir, Turkey
[2] Department of Mathematics, Faculty of Arts and Sciences, Bozok University, Yozgat, Turkey

ABSTRACT

In many stability problems, the investigation of pure imaginary roots for a polynomial family is very important. Given a pure imaginary complex number, the set of all images of uncertainty vectors is called the value set corresponding to this pure imaginary complex number. The question whether these sets contain the origin is very important from robust stability point of view of a polynomial family. Cutoff frequency guarantees the noninclusion of the origin to the value set for large frequencies. In this paper, we give a procedure for more strict estimation of cutoff frequency and applications of the obtained result to the constant inertia problem of a polynomial family.

Keywords: Cutoff frequency, Constant regular inertia, Multilinear polynomial families, Hurwitz stability

POLİNOM AİLELERİ İÇİN KESME FREKANSI HESABI

ÖZET

Kararlılık problemlerinde, polinom ailesinin pür imajiner köklerinin araştırılması oldukça önemlidir. Pür imajiner bir kompleks sayı verildiğinde, belirsizlik vektörlerinin görüntülerinin kümesi bu kompleks sayıya karşılık gelen değer kümesi olarak adlandırılır. Değer kümelerinin orijini kapsayıp kapsamadığı sorusu, polinom ailesinin gürbüz kararlılığı açısından çok önemlidir. Kesme frekansı, daha büyük frekanslar için değer kümesinin orijini içermemesini garanti eder. Bu çalışmada, kesme frekansını daha iyi belirlemek için bir prosedür verilmiştir ve elde edilen sonuçlar polinom ailesinin sabit inersiyon problemine uygulanmıştır.

Anahtar Kelimeler: Kesme frekansı, Sabit regüler inersiyon, Multilineer polinom aileleri, Hurwitz kararlılık

1. INTRODUCTION

The robust stability and root clustering of polynomials have attracted much attention in control theory (see [1-3] and reference therein). We consider a more general case where the coefficients of a polynomial depend on uncertainty parameters. A powerful tool for analyzing stability of a polynomial family in the frequency domain is the value set approach. The value set depends on the chosen frequency which varies in the positive real axis. In this paper our aim is determine an upper bound for the active frequencies.

Let

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{1}$$

be a fixed polynomial. $p(s)$ is said to be Hurwitz stable if all its roots lie in the open left half plane.

Let polynomial family be defined by

$$p(s,q) = a_n(q) s^n + a_{n-1}(q) s^{n-1} + \cdots + a_1(q) s + a_0(q) \tag{2}$$

*Corresponding Author: tbuyukkoroglu@anadolu.edu.tr

where the uncertainty vector q belongs to a box Q, where

$$Q = \{(q_1, q_2, \ldots, q_l) \in \mathbb{R}: q_i^- \leq q_i \leq q_i^+,\ i = 1,2, \ldots, l\} \tag{3}$$

and the functions $a_i: Q \to \mathbb{R}$ $(i = 1,2, \ldots, l)$ are continuous.

Denote the set of all polynomials $p(s, q)$ by $\mathcal{P}$, that is

$$\mathcal{P} = \{p(s, q): q \in Q\}\ . \tag{4}$$

$\mathcal{P}$ is said to be multilinear family if each coefficient function $a_i(q)$ is an affine linear with respect to each component of $q \in Q$. The family $\mathcal{P}$ is said to be robust Hurwitz stable if for each $q \in Q$ the polynomial $p(s, q)$ is Hurwitz stable.

Theorem 1 (Zero Exclusion Condition, [1, p. 113]) Suppose that the polynomial family $\mathcal{P}$ given by (4) has invariant degree and at least has one stable member $p(s, q^0)$. Then $\mathcal{P}$ is robustly stable if and only if $z = 0$ is excluded from the value set $A(\omega) = \{p(j\omega, q):\ q \in\ Q\}$ at all nonnegative frequencies, i.e.,

$$0 \notin p(j\omega, Q)$$

for all frequencies $\omega \geq 0$.

Theorem 1 gives a graphical test for robust stability of the family $\mathcal{P}$. By watching the value set $p(j\omega, Q)$ as ω varies from 0 to $+\infty$, we can check, by inspection, the condition $0 \notin p(j\omega, Q)$. This raises the following question: Can we find possible small cutoff frequency $\omega_c > 0$ such that $0 \notin p(j\omega, Q)$ for all $\geq \omega_c$?

One estimation for ω_c comes from classical bounds on the roots of a polynomial. It is well-known that the roots of a fixed positive coefficient polynomial $p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$ lie in the disc of radius

$$R = 1 + \frac{\max\{a_0, a_1, \ldots, a_{n-1}\}}{a_n}$$

with center at the origin (see [4, p. 123], Theorem (27.2)). Hence for any $\omega > R$, $s = j\omega$ can not be a root of the polynomial $p(s)$. From this it follows that for the polynomial family $p(s, q)$ with $a_n(q) > 0$ for all $q \in Q$, an appropriate cutoff frequency may be given as

$$\omega_c = 1 + \frac{\max\left\{\max\limits_q a_0(q), \max\limits_q a_1(q), \ldots, \max\limits_q a_{n-1}(q)\right\}}{\min\limits_q a_n(q)}. \tag{5}$$

Calculation of the cutoff frequency by (5) gives large value as usually. Therefore, the minimization of ω_c becomes an important problem the consideration of which is the subject of the next section.

2. MINIMIZATION OF ω_c

Consider family (2), where $a_i(q)$ $(i = 0,1, \ldots, n)$ are continuous in q. Consider pure imaginary root $s = j\omega$. We show that the nonexistence of such a root is equivalent to the nonexistence of a common solution of two polynomial equations defined on a box. Similar system of two equations for discrete polynomial families has been obtained in [5].

Suppose that the number $s = j\ \omega$ is a root of $\mathcal{P}$ and $\omega \neq 0$. Then $s = -j\ \omega$ is also a root, and there exist $\alpha_0(q), \alpha_1(q), \ldots, \alpha_{n-2}(q)$ such that

$$\begin{aligned} a_0(q)+a_1(q)s+\cdots+a_n(q)s^n &= (s-j\omega)(s+j\omega)(\alpha_0(q)+\alpha_1(q)s+\cdots+\alpha_{n-2}(q)s^{n-2}) \\ &= (s^2+\omega^2)(\alpha_0(q)+\alpha_1(q)s+\cdots+\alpha_{n-2}(q)s^{n-2}) \end{aligned} \quad (6)$$

is valid. Taking $t=\omega^2$ in (6), it follows that the equalities

$$\begin{array}{rcl} t\alpha_0(q) & = & a_0(q) \\ t\alpha_1(q) & = & a_1(q) \\ \alpha_0(q)+t\alpha_2(q) & = & a_2(q) \\ \alpha_1(q)+t\alpha_3(q) & = & a_3(q) \\ \vdots & \vdots & \vdots \\ \alpha_{k-2}(q)+t\alpha_k(q) & = & a_k(q) \\ \vdots & \vdots & \vdots \\ \alpha_{n-4}(q)+t\alpha_{n-2}(q) & = & a_{n-2}(q) \\ \alpha_{n-3}(q) & = & a_{n-1}(q) \\ \alpha_{n-2}(q) & = & a_n(q) \end{array} \quad (7)$$

are satisfied.

Elimination of $\alpha_0(q),\alpha_1(q),\ldots,\alpha_{n-2}(q)$ from (7) reduces the system of equations (7) into the system

$$\begin{aligned} f_1(t,q)=0, \\ f_2(t,q)=0. \end{aligned} \quad (8)$$

If n is odd, this system has the form

$$f_1(t,q):=a_0(q)-ta_2(q)+t^2a_4(q)+\cdots+t^{\frac{n-1}{2}}a_{n-1}(q)=0$$
$$f_2(t,q)\coloneqq a_1(q)-ta_3(q)+t^2a_5(q)+\cdots+t^{\frac{n-1}{2}}a_n(q)=0$$

If n is even, then

$$f_1(t,q):=a_0(q)-ta_2(q)+t^2a_4(q)+\cdots+t^{\frac{n}{2}}a_n(q)=0$$
$$f_2(t,q)\coloneqq a_1(q)-ta_3(q)+t^2a_5(q)+\cdots+t^{\frac{n-2}{2}}a_{n-1}(q)=0$$

For example, assume that $n=5$. Then the system (8) becomes

$$\begin{aligned} a_0(q)-ta_2(q)+t^2a_4(q)=0, \\ a_1(q)-ta_3(q)+t^2a_5(q)=0. \end{aligned} \quad (9)$$

Thus we obtain the following result:

Theorem 2 $(t_*,q_*)\in(0,\infty)\times Q$ is a solution of the system (8) if and only if $p(j\,\omega_*,q_*)=0$, where $t_*=w_*^2$. Equivalently, $p(j\,\omega,q)\neq 0$ for all $(t,q)\in[\omega_1,\omega_2]\times Q$ if and only if equtaion (8) has no solution on $[t_1,t_2]\times Q$ where $t_1=\omega_1^2$, $t_2=\omega_2^2$.

Proof: Assume that $(t_*,q_*)\in(0,\infty)\times Q$ is a solution of (8):

$$f_1(t,q)=0,$$
$$f_2(t,q)=0.$$

Then by (6), (7) the polynomial $a_0(q_*)+a_1(q_*)s+\cdots+a_n(q_*)s^n$ has roots $s=\pm j\omega_*$, where $t_*=\omega_*^2$, that is $p(j\,\omega_*,q_*)=p(-j\,\omega_*,q_*)=0$.
Conversely, if $p(j\,\omega_*,q_*)=0$ then from (6), (7) it follows that $f_1(t_*,q_*)=f_2(t_*,q_*)=0$, where $t_*=\omega_*^2$. Consequently, (t_*,q_*) is a solution of (8).

From now on we consider multilinear polynomial family (see Section 1). The following extremal property of a scalar multilinear function defined on a box is well-known.

Theorem 3 (Extremal property, [1, p. 245]) Suppose Q is a box in $\mathbb{R}^l$ with the set of extreme points $\{q^i\}$ and $f: Q \to \mathbb{R}$ is multilinear. Then both the maximum and minimum of $f(q)$ are attained at extreme points. That is

$$\max_{q\in Q} f(q) = \max_i f(q^i),$$
$$\min_{q\in Q} f(q) = \min_i f(q^i).$$

Let the family (2) be given, where the functions $a_i(q)$ are multilinear ($i = 0,1,\dots,n$). In system (8) both f_1 and f_2 are multilinear on q and polynomially dependent on t.
The system (8) is almost multilinear and the variables (t, q) vary on the box $[\omega_1^2, \omega_2^2] \times Q$. This system can be multilinearized by introducing new variables (see [5]). Indeed, if system (8) contains t^k as the greatest power of t, we can replace t^k by the product $t_1 t_2 \dots t_k$ where t_1, t_2, ..., t_k, are new variables and add new equations $t_2 - t_1 = 0$, $t_3 - t_1 = 0$, ..., $t_k - t_1 = 0$ to the system (8) (we set $t_1 = t$). This new extended system will then be multilinear and Theorem 3 will be applicable. For example, assume that $n = 5$. Then the system (9) is transformed into

$$\begin{aligned} a_0(q) - t_1 a_2(q) + t_1 t_2 a_4(q) &= 0 \\ a_1(q) - t_1 a_3(q) + t_1 t_2 a_5(q) &= 0 \\ t_2 - t_1 &= 0 \end{aligned}$$

where $(t_1, t_2, q) \in [\omega_1^2, \omega_2^2] \times [\omega_1^2, \omega_2^2] \times Q$.

Given a multilinear family $p(s, q)$, our aim is to find possible small ω_c. To this end we suggest the following algorithm.

Algorithm 1

i) Given family (2). Write the corresponding system (8) and multilinearize it.

ii) Find ω_c from (5).

iii) Construct the intervals $\left[\frac{\omega_c^2}{2}, \omega_c^2\right]$, $\left[\frac{\omega_c^2}{4}, \frac{\omega_c^2}{2}\right]$, $\left[\frac{\omega_c^2}{8}, \frac{\omega_c^2}{4}\right]$, ...

iv) Take the first interval $\left[\frac{\omega_c^2}{2}, \omega_c^2\right]$ and check for nonexistence of a root of the obtained multilinear system on the extended box $\left[\frac{\omega_c^2}{2}, \omega_c^2\right] \times \dots \times \left[\frac{\omega_c^2}{2}, \omega_c^2\right] \times Q$ by using Theorem 2. If the nonexistence of a root is satisfied then replace $\left[\frac{\omega_c^2}{2}, \omega_c^2\right]$ by $\left[\frac{\omega_c^2}{4}, \frac{\omega_c^2}{2}\right]$ and continue.

v) If nonexistence of a root on some box $\left[\frac{\omega_c^2}{2^{k+1}}, \frac{\omega_c^2}{2^k}\right] \times \dots \times \left[\frac{\omega_c^2}{2^{k+1}}, \frac{\omega_c^2}{2^k}\right] \times Q$ can not be verified then stop. The number $\widetilde{\omega}_c = \sqrt{\frac{\omega_c^2}{2^k}} = \omega_c \cdot 2^{-\frac{k}{2}}$ is a new cutoff frequency.

Example 1 [1, p. 183] Consider the model of an experimental oblique wing aircraft. The aircraft transfer function is

$$P(s) = \frac{64s + 128}{s^4 + 3.7s^3 + 65.6s^2 + 32s}.$$

We replace $P(s)$ by the following interval plant family $\mathcal{P}$ described by

$$P(s, q, r) = \frac{q_1 s + q_0}{s^4 + r_3 s^3 + r_2 s^2 + r_1 s + r_0} \tag{10}$$

with uncertainty bounds $90 \le q_0 \le 166$, $54 \le q_1 \le 74$, $-0.1 \le r_0 \le 0.1$, $30.1 \le r_1 \le 33.9$, $50.4 \le r_2 \le 80.8$, $2.8 \le r_3 \le 4.6$.

For this interval plant family $\mathcal{P}$, consider the problem of existence of a robustly stabilizing PI controller

$$C(s) = K_1 + \frac{K_2}{s}.$$

Assume that $0.8 \leq K_1 \leq 1.24$, $0.01 \leq K_2 \leq 0.84$. The closed loop characteristic polynomial is

$$p(s,q,r,K_1,K_2) = s^5 + r_3s^4 + r_2s^3 + (r_1 + q_1K_1)s^2 + (r_0 + q_0K_1 + q_1K_2)s + q_0K_2. \tag{11}$$

This polynomial family is multilinear. Calculation ω_c by (5) gives $\omega_c = 269.1$. Now apply Algortihm 1 to reduce ω_c. After 4 steps we arrive at the value $\omega_c = 16.81875$. The above intervals for the parameters K_1 and K_2 are taken from [1] (see [1], p. 186, Fig. 11.5.2) where by using Routh tables the set of robust stabilizing parameters are obtained graphically. In the next section, we will prove that the set of all (K_1, K_2) satisfying $0.8 \leq K_1 \leq 1.24$, $0.01 \leq K_2 \leq 0.84$ are stabilizing by using Sixteen Plant Theorem and the value $\omega_c = 16.81875$.

3. CONSTANT REGULAR INERTIA PROBLEM FOR A MULTILINEAR FAMILY

Let polynomial (1) with $a_n \neq 0$ be given. Define the ordered triple (n_1, n_2, n_3) where $n_1 + n_2 + n_3 = n$ and n_1 is the number of roots in the open left half plane, n_2 is the number of roots on the imaginary axis and n_3 is the number of roots in the open right-half plane. The ordered triple (n_1, n_2, n_3) is called the inertia of polynomial (1). If polynomial (1) is Hurwitz stable then its inertia is $(n, 0,0)$. In the case of $n_2 = 0$ the inertia is called regular.

Theorem 4 Let multilinear family (2) be given, assume that the point $s = 0$ is not a root of $\mathcal{P}$ and $\mathcal{P}$ has at least one polynomial with regular inertia $(\alpha, 0, \beta)$. Then $\mathcal{P}$ has constant regular inertia $(\alpha, 0, \beta)$ if and only if the system (8) has no solution on $[0, \omega_c^2] \times Q \subset \mathbb{R}^{l+1}$.

After finding a suitable ω_c by using the results from the previous section, we get the following algorithm for checking the inertia of a multilinear family.

Algorithm 2

i) Let multilinear family (2) be given and the least one polynomial has regular inertia $(\alpha, 0, \beta)$. Using Theorem 3, check the nonexistence of the root $s = 0$. Otherwise $\mathcal{P}$ has no constant regular inertia $(\alpha, 0, \beta)$.

ii) Obtain the equations $f_1(t, q) = 0$, $f_2(t, q) = 0$.

iii) Multilinearize this system by replacing $t = t_1$ and introducing new variables t_1, t_2, ..., t_k and new equation

$$\begin{gathered} f_1(t_1, t_2, \dots, t_k, q) = 0, \\ f_1(t_1, t_2, \dots, t_k, q) = 0, \\ t_2 - t_1 = 0, \\ t_3 - t_1 = 0, \\ \vdots \\ t_k - t_1 = 0 \end{gathered} \tag{12}$$

where $(t_1, t_2, \dots, t_k, q) \in [0, \omega_c^2] \times \cdots \times [0, \omega_c^2] \times Q \subset \mathbb{R}^{k+l}$.

iv) Using Theorem 3, calculate the ranges of all functions (12). If at least one range does not contain zero, then stop. The family $\mathcal{P}$ has regular inertia $(\alpha, 0, \beta)$. Otherwise apply the next step.

Example 2 Consider again the characteristic polynomial (11). According to the Sixteen Plant Theorem, the family (11) is robust stable if and only if all 16 Kharitonov polynomials are stable [1, p. 182]. All Kharitonov polynomials are affine with respect to (K_1, K_2) and have members with inertia (5,0,0). We have applied Algorithm 2 to all Kharitonov polynomials. As a result, all 16 families have regular inertia (5,0,0) and each (K_1, K_2) from the rectangle $0.8 \leq K_1 \leq 1.24$, $0.01 \leq K_2 \leq 0.84$ is stabilizing for the plant (10) (see Figure 1 and Table 1). Numbers of required steps for each Kharitonov polynomial are given in Table 1.

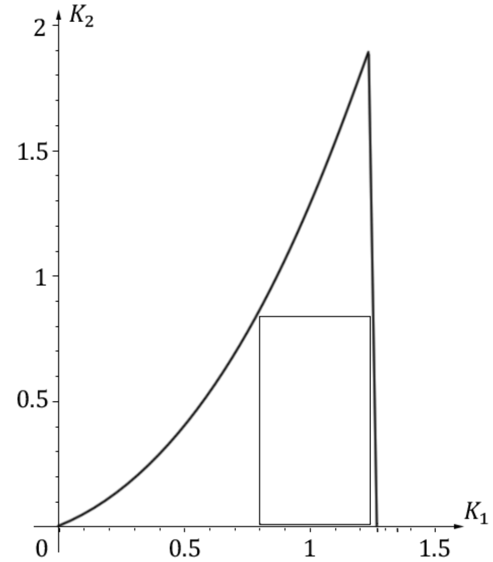

Figure 1: The set of robust stabilizing parameters and the rectangle $0.8 \leq K_1 \leq 1.24$, $0.01 \leq K_2 \leq 0.84$

Table 1. For Example 2, the numbers of steps are given in the second column

Kharitonov polinomial families correspond to the polynomial family (11)	steps
$p_{1,1}(s,K_1,K_2) = s^5 + 4.6s^4 + 80.8s^3 + (54K_1 + 30.1)s^2 + (90K_1 + 54K_2 - 0.1)s + 90K_2$	62
$p_{1,2}(s,K_1,K_2) = s^5 + 2.8s^4 + 50.4s^3 + (54K_1 + 33.9)s^2 + (90K_1 + 54K_2 + 0.1)s + 90K_2$	48
$p_{1,3}(s,K_1,K_2) = s^5 + 4.6s^4 + 50.4s^3 + (54K_1 + 30.1)s^2 + (90K_1 + 54K_2 + 0.1)s + 90K_2$	44
$p_{1,4}(s,K_1,K_2) = s^5 + 2.8s^4 + 80.8s^3 + (54K_1 + 33.9)s^2 + (90K_1 + 54K_2 - 0.1)s + 90K_2$	42
$p_{2,1}(s,K_1,K_2) = s^5 + 4.6s^4 + 80.8s^3 + (74K_1 + 30.1)s^2 + (166K_1 + 74K_2 - 0.1)s + 166K_2$	44
$p_{2,2}(s,K_1,K_2) = s^5 + 2.8s^4 + 50.4s^3 + (74K_1 + 33.9)s^2 + (166K_1 + 74K_2 + 0.1)s + 166K_2$	116
$p_{2,3}(s,K_1,K_2) = s^5 + 4.6s^4 + 50.4s^3 + (74K_1 + 30.1)s^2 + (166K_1 + 74K_2 + 0.1)s + 166K_2$	38
$p_{2,4}(s,K_1,K_2) = s^5 + 2.8s^4 + 80.8s^3 + (74K_1 + 33.9)s^2 + (166K_1 + 74K_2 - 0.1)s + 166K_2$	38
$p_{3,1}(s,K_1,K_2) = s^5 + 4.6s^4 + 80.8s^3 + (54K_1 + 30.1)s^2 + (166K_1 + 54K_2 - 0.1)s + 166K_2$	110
$p_{3,2}(s,K_1,K_2) = s^5 + 2.8s^4 + 50.4s^3 + (54K_1 + 33.9)s^2 + (166K_1 + 54K_2 + 0.1)s + 166K_2$	50
$p_{3,3}(s,K_1,K_2) = s^5 + 4.6s^4 + 50.4s^3 + (54K_1 + 30.1)s^2 + (166K_1 + 54K_2 + 0.1)s + 166K_2$	40
$p_{3,4}(s,K_1,K_2) = s^5 + 2.8s^4 + 80.8s^3 + (54K_1 + 33.9)s^2 + (166K_1 + 54K_2 - 0.1)s + 166K_2$	52
$p_{4,1}(s,K_1,K_2) = s^5 + 4.6s^4 + 80.8s^3 + (74K_1 + 30.1)s^2 + (90K_1 + 74K_2 - 0.1)s + 90K_2$	42
$p_{4,2}(s,K_1,K_2) = s^5 + 2.8s^4 + 50.4s^3 + (74K_1 + 33.9)s^2 + (90K_1 + 74K_2 + 0.1)s + 90K_2$	60
$p_{4,3}(s,K_1,K_2) = s^5 + 4.6s^4 + 50.4s^3 + (74K_1 + 30.1)s^2 + (90K_1 + 74K_2 + 0.1)s + 90K_2$	36
$p_{4,4}(s,K_1,K_2) = s^5 + 2.8s^4 + 80.8s^3 + (74K_1 + 33.9)s^2 + (90K_1 + 74K_2 - 0.1)s + 90K_2$	42

4. CONCLUSIONS

In this work, two procedures are given for strict estimation of cutoff frequency and the constant inertia problem of a polynomial family. The cutoff frequency for a multilinear polynomial family obtained by the classical calculations can be minimized by Algorithm 1. After finding a suitable ω_c, the inertia of this family can be checked by Algorithm 2. An application of these algorithms to an interval plant family is also provided.

ACKNOWLEDGMENTS
The authors would like to thank the Editor and referees for the valuable comments.

REFERENCES

[1] Barmish BR. New Tools for Robustness of Linear Systems. Macmillan, New York, 1994.

[2] Bhattacharyya SP. Chapellat H and Keel LH. Robust Control: The Parametric Approach. Prentice Hall PTR, 1995.

[3] Petersen IR, Tempo R. Robust Control of Uncertain Systems: Classical Results and Recent Developments. Automatica, 50, pp 1315-1335, 2014.

[4] Marden M. Geometry of Polynomials. American Mathematical Society, Providence, R.I., 1966.

[5] Akyar H, Büyükköroğlu T and Dzhafarov V. On stability of parametrized families of polynomials and matrices, Abstract and Applied Analysis, vol. 2010, Article ID 687951, 16 pages, 2010.