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The Existence of Positive Solutions for the Caputo-Fabrizio Fractional Boundary Value Problems at Resonance

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Abstract. This paper deals with a class of nonlinear fractional boundary value problems at resonance with Caputo-Fabrizio fractional derivative. We establish some new necessary conditions for the existence of positive solutions for the fractional boundary value problems at resonance by using the Leggett-Williams norm-type theorem for coincidences due to O' Regan and Zima. Some examples are constructed to support our results.

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1. Introduction

This paper is concerned with positive solutions of the following two-point fractional boundary value problem (FBVP)

$$ {}^{CF}_{0}D^{\alpha}_{1}u(x) + f(x, u(x)) = 0, \quad \alpha \in (1, 2], \quad x \in [0, 1], \tag{1.1} $$

$$ u(0) = 0, \quad u'(0) = u'(1), \tag{1.2} $$

where ${}^{CF}_{0}D^{\alpha}_{1}$ is the Caputo-Fabrizio derivative of order α which has non-singular kernel defined below, $f \in C\Big([0, 1] \times [0, \infty), [0, \infty)\Big)$ is a L^1-Carathéodory function. FBVP (1.1) happens to be resonance because the following linear FBVP

$$ \begin{aligned} {}^{CF}_{0}D^{\alpha}_{1}u(x) + \lambda u(x) = 0,, \quad \alpha \in (1, 2], \quad x \in [0, 1], \\ u(0) = 0, \quad u'(0) = u'(1) \end{aligned} $$

has an eigenvalue $\lambda = 0$.

Over the recent decades, FBVPs have received much attention and have been extensively investigated due to their valuable role in the mathematical modelling of physical phenomena in the science and engineering [16, 20, 24] . There exist many papers devoted to the existence of solutions or positive solutions for nonresonant FVPs. (see, e.g, [2, 9, 10, 12, 13, 29, 31]). In [2], existence results for sequential fractional integro-differential equations and inclusions with non-local boundary conditions have been presented. Nonlinear fractional boundary value problems with integral boundary conditions have been considered in [9]. Positive solutions of nonlinear fractional differential equations have been studied in [29]. Uniqueness results for fractional boundary value problems have been established in [12]. New results on uniqueness of fractional differential equations have been also established in [13]. Positive solutions for a nonlocal fractional boundary value problems have been presented in [29]. Uniqueness results for higher order

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fractional differential equations have been presented in [31]. Unlike the nonresonant case, the study of the existence of positive solutions of FBVPs at resonance requires more efforts and there are limited works on this direction. The main tool in proving the existence of positive solution to FBVPs at resonance includes the coincidence degree theory and Leggett–Williams norm-type theorem. We cite some recent papers on FBVPs at resonance [17, 18, 21, 22, 28, 30]. The existence of positive solutions for multi-point boundary value problems at resonance using the Leggett–Williams norm-type theorem due to O'Regan and Zima has been studied in [17]. With the help of the coincidence degree theory due to Mawhin and constructing suitable operators, [18] has studied the existence of solutions to boundary value problems of fractional differential equations at resonance. In [21], solvability of three-point nonlinear FBVP at resonance using the Riemann-Liouville fractional derivative has been established by means of the coincidence degree theory. The authors in [22] have studied a resonant functional boundary value problem of fractional order using the coincidence degree theory of Mawhin. The existence of positive solution for a class of second-order m -point boundary value problems under different resonant conditions using the Leggett–Williams norm-type theorem due to O'Regan and Zima has been presented in [28]. The authors in [30] have considered three-point FBVP and have established sufficient conditions for the existence of positive solutions using the fixed point index theory and iterative technique.

To the best of the author's knowledge, the existing studies are mostly devoted to FBVPs at resonance using the Riemann-Liouville or Caputo fractional derivatives. There is no contribution on FBVPs at resonance with the Caputo-Fabrizio fractional derivative. The main goal of this paper is to fill this gap in the literature. The key ingredient in our analysis is the Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima [25]. The main feature of the Caputo-Fabrizio fractional derivative is that the kernel which consists of the convolution of the classical derivative with the exponential function is not singular [11]. Although the kernel of this new fractional operator is not singular, it has heterogeneous properties as the commonly used fractional derivative operator [6]. This new fractional derivative operator has been further investigated by some authors. Some linear fractional differential equations involving this new fractional derivative have been studied in [23]. Maximum principle of FBVPs using the Caputo-Fabrizio fractional derivative has been studied in [3, 5, 19]. The existence and uniqueness of the solutions for FBVPs involving this new fractional derivative have been extensively studied in the literature [1, 3, 6, 8, 26, 27]. Numerical solution of the space-time Caputo-Fabrizio fractional derivative with applications to groundwater pollution equation has been investigated in [7] and the mathematical modelling of wave movement on the surface of shallow water with the Caputo-Fabrizio fractional derivative has been studied in [4]. The Caputo-Fabrizio fractional derivative operator has been further used for modelling many physical phenomena including mass-spring-damper system [15, 32], fractional Fisher's equation [5], elasticity model [11], Lienard model [15] and KdV-Burgers equation with fractional order [4, 14].

In this study, a positive solution of FBVP (1.1)-(1.2) means that a nonnegative solution $u \in C[0,1]$ which has no zero on $[0,1]$ solves the problem (1.1)-(1.2) and ${}^{CF}_{\ 0}D^{\alpha}_{1} u \in L^1[0,1]$.

2. Preliminaries and Useful Lemmas

This section introduces some definitions and useful lemmas that will be needed in the following.

Definition 2.1. [11, 23] Let $f \in H^1(a,b)$, $\quad a < b$ and $\alpha \in (0,1]$. The fractional Caputo-Fabrizio derivative is defined as

$$^{CF}D^{\alpha}_{0^+} u(x) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^x \exp\Big(-\frac{\alpha}{1-\alpha}(x-t)\Big) u'(t)\, dt, \quad t \geq 0,$$

where $M(\alpha)$ is a normalization function with $M(0) = M(1) = 1$.

Definition 2.2. [23] The Caputo-Fabrizio fractional integral of order $\alpha \in (0,1)$ is defined as

$$^{CF}\mathcal{I}^{\alpha}_{0} u(x) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(x) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^x u(s)\, ds.$$

Imposing $\dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \dfrac{2\alpha}{(2-\alpha)M(\alpha)} = 1$, we can have an explicit expressing for $M(\alpha)$, $\quad \alpha \in (0,1]$ given as

$$M(\alpha) = \frac{2}{2-\alpha}.$$

The high order Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0, 1)$ and $n \in \mathbb{N}$ is defined as

$$^{CF}D^{\alpha+n}_{0^+}u(x) := {}^{CF}D^{\alpha}_{0^+}({}^{CF}D^{n}_{0^+}u(x)).$$

$AC[0, 1]$ denotes the space of absolutely continuous functions on the interval $[0, 1]$ and $AC_{loc}(0, 1]$ be the space consisting of functions that are absolutely continuous on every interval $[a, 1] \subset (0, 1]$.

Lemma 2.3. *[11] Assume $\alpha > 0$. For any $u \in L^1(0, 1)$, we have*

$$^{CF}\mathcal{I}^{\alpha CF}_0 D^{\alpha}_0 u(x) = u(x) - \sum_{k=0}^{n} \frac{u^{(k)}(0)}{k!}x^k, \quad x \in [0, 1], \quad u \in L^p[0, 1], \tag{2.1}$$

$$^{CF}D^{\alpha CF}_0 \mathcal{I}^{\alpha}_0 u(x) = u(x), \text{ where } n = [\alpha] + 1. \tag{2.2}$$

The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0, 1)$ and $n \in \mathbb{N}$ is given by [11]

$$\mathcal{L}\Big\{^{CF}D^{\sigma}_0 u(x)\Big\}(s) = \frac{s^{n+1}\mathcal{L}\{f(x)\}(s) - s^n f(0) - s^{n-1}f'(0)\cdots - f^{(n)}(0)}{s + \alpha(1 - s)}.$$

3. Main result

We need the following definition in proving the existence of positive solution.

Definition 3.1. A linear mapping $L : \operatorname{dom} L \subset X \to Y$ between two real normed spaces X and Y is called a Fredholm mapping provided that

(i) The dimension of $\operatorname{Ker} L$ is finite
(ii) $\operatorname{Im} L$ is closed and its codimension is finite.

The Fredholm index $\operatorname{Ind} L$ of a Fredholm mapping L is defined to be the integer given by $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$. In this paper, we consider Fredholm operator of index zero.

Consider the Banach space $X = C[0, 1]$ with the sup-norm $\|u\|_\infty = \max_{x\in[0,1]} |u(x)|$ and we set $Y = L^1[0, 1]$ with the usual norm denoted by $\|u\|_1 = \int_0^1 |u(t)|\, dt$.

Define the set $\operatorname{dom} L = \{u \in X : u \in AC^n[0, 1], {}^{CF}_{\ 0}D^{\alpha}_1 u(x) \in Y, \quad u(0) = 0, u'(0) = u'(1)\}$. Let the operator $L : \operatorname{dom} L \to Y$ be given by

$$Lu(x) = -{}^{CF}_{\ 0}D^{\alpha}_1 u(x)\},$$

where $1 < \alpha \leq 2$. Let $N : X \to Y$ be defined as $Nu(x) = f(x, u(x)), x \in [0, 1]$. Then, FBVP problem (1.1) is equivalent to the equation $Lu = Nu, \quad u \in domL$.

Throughout the paper, we assume that the operator L is a Fredholm operator of index zero, that is, ImL is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$. This assumption guarantees that there are two projections $P : X \to X, \quad Q : Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L,$$
$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that $L|_{\operatorname{dom} L\cap \operatorname{Ker} P} \to \operatorname{Im} L$ is invertible. We denote the inverse by K_P. The generalized inverse of L is denoted by $K_{P,Q} : Y \to \operatorname{dom} L \cap \operatorname{Ker} P$ and is defined by $K_{P,Q} = K_P(I - Q)$.

Definition 3.2. Let $L : \operatorname{dom} L \subset\to X \to Y$ be a Fredholm operator and Z be a metric space. A mapping $N : Z \to Y$ is called L-compact on Z provided that $QN : Z \to Y$ is continuous and bounded and $K_{P,Q}N : Z \to X$ is compact on Z.

Definition 3.3. The function f satisfies the Carathéodory conditions with respect to $L^1[0, 1]$ if the following conditions hold:

(f1) For each $u \in \mathbb{R}$, the mapping $x \mapsto f(x, u)$ is Lebesgue measurable on $[0, 1]$,
(f2) For a.e. $x \in [0, 1]$, the mapping $u \mapsto f(x, u)$ is continuous on $\mathbb{R}$,
(f3) For each $r > 0$, there exists a nonnegative $\phi_r(x) \in L^1[0, 1]$ such that, for a.e. $x \in [0, 1]$ and every u such that $|u| \leq r$, we have $|f(x, u)| \leq \phi_r(x)$.

In order to prove the existence of a positive solution of the equation $Lu = Nu$, we will need the following coincidence degree theorem due to O' Regan and Zima [25].

Lemma 3.4. *Let* $L : \operatorname{dom} L \subset X \to Y$ *be a Fredholm operator of index zero and* $\mathrm{N} : \mathrm{X} \to \mathrm{Y}$ *be L-compact on* $\bar{\Omega}$*. Assume that the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in ((\operatorname{dom} L \backslash \operatorname{Ker} L) \cap \partial\Omega) \times (0, 1)$,
(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$,
(3) $\deg\left(QN|_{\operatorname{Ker} L}, \Omega \bigcap \operatorname{Ker} L, 0\right) \neq 0$*, where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then, the equation $Lu = Nu$ *has at least one solution in* $\operatorname{dom} L \cap \bar{\Omega}$.

Lemma 3.5. *Let L be defined as above, then*

$$\operatorname{Ker} L = \{u \in X \mid u(x) = c_1 x, c_1 \in \mathbb{R}, \forall x \in [0, 1]\},$$
$$\operatorname{Im} L = \left\{ v \in Y \mid \int_0^1 \Big[(\alpha - 1)v(s) - (2 - \alpha)v'(s)\Big] ds = 0 \right\}.$$

Proof. From (2.2), the equation $Lu = 0$ has a solution $u(x) = c_0 + c_1 x$. Applying the boundary conditions (1.2) implies $u(x) = c_1 x$, thus we have the first assertion.

For $v \in \operatorname{Im} L$, there exists $u \in \operatorname{dom} L$ such that $v = Lu \in Y$. Using (2.1), we have

$$u(x) = \frac{2 - \alpha}{M(\alpha - 1)} \int_0^x v(s) ds + \frac{\alpha - 1}{M(\alpha - 1)} \int_0^x (x - s) v(s) ds + c_0 + c_1 x. \tag{3.1}$$

From the boundary condition $u(0) = 0$, we find $c_0 = 0$ and that

$$u'(x) = \frac{2 - \alpha}{M(\alpha - 1)} v(x) + \frac{\alpha - 1}{M(\alpha - 1)} \int_0^x v(s) ds + c_1.$$

Now the boundary condition $u'(0) = u'(1)$ implies that v satisfies $\int_0^1 \Big[(\alpha - 1)v(s) - (2 - \alpha)v'(s)\Big] ds = 0$. Thus, we get the second assertion. On the other hand, suppose $v \in Y$ and satisfies $\int_0^1 \Big[(\alpha - 1)v(s) - (2 - \alpha)v'(s)\Big] ds = 0$. Let $u(x) = {}^{CF}_{0}\mathcal{I}^{\alpha}_{1} v(x)$, then $u \in \operatorname{dom} L$ and ${}^{CF}_{0}D^{\alpha}_{1} u(x) = v(x)$. So that, $v \in \operatorname{Im} L$. The proof is completed. □

Lemma 3.6. *Let* $L : domL \subset X \to Y$ *be defined as above. Then the operator L is a Fredholm operator of index zero and the linear continuous projector operators* $P : X \to X$ *and* $Q : Y \to Y$ *can be defined as*

$$Pu(x) = u'(0)x, \quad \forall x \in [0, 1],$$
$$Qv(x) = \frac{1}{\alpha - 1} \int_0^1 \Big[(\alpha - 1)v(s) - (2 - \alpha)v'(s)\Big] ds,$$

and the linear operator $K_P : \mathrm{ImL} \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be expressed as*

$$K_P v = \frac{2 - \alpha}{M(\alpha - 1)} \int_0^x v(s) ds + \frac{\alpha - 1}{M(\alpha - 1)} \int_0^x (x - s) v(s) ds.$$

Proof. It is easy to see that

$$P^2 u = Pu, \quad ImP = \operatorname{Ker} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P. \tag{3.2}$$

Moreover, we observe that $\operatorname{Ker} Q = \mathrm{ImL}$. For $y \in Y$, we can easily get that $Q^2 y = Qy$, i.e. $Q : Y \to Y$ is a projection. Set $y = (y - Qy) + Qy$. Then, $y - Qy \in \operatorname{Ker} Q = \operatorname{Im} L$, $\quad Qy \in \operatorname{Im} Q$. It is easy to see that $\operatorname{Im} Q \cap \mathrm{ImL} = \{0\}$. Thus, we have $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. This, together with (3.2), means that L is a Fredholm operator of index zero.

With the help of (3.1), one can prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. Therefore, the proof is now completed. □

Lemma 3.7. *Assume* $\Omega \subset X$ *is an open bounded subset and* $\operatorname{dom} L \cap \bar{\Omega} \neq \emptyset$*, then the operator N is L-compact on* $\overline{\Omega}$.

Proof Since $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a L^1-Carathéodory function, we infer that $QN(\bar{\Omega})$ is continuous and bounded and $K_P(I - Q)N(\bar{\Omega})$ is continuous due to the Lebesgue dominated convergence theorem. By the Arzelá-Ascoli theorem, we shall prove $K_P(I - Q)N : X \to X$ is equicontinuous.

Since f a L^1-Carathéodory function, for each $r > 0$, there exists a constant r > 0 such that $|(I - Q)Nu| \leq M$ for $|u| \leq r$, a.e. $x \in [0, 1]$, where $M := \|\phi_r\|_{L^1(\Omega)}$.

For $0 \leq x_1 < x_2 \leq 1, \quad |u| \leq r$, we have

$$\begin{aligned}|K_P(I-Q)Nu\,(x_2) - K_P(I-Q)Nu\,(x_1)| &\leq \frac{1}{M(\alpha-1)}\Big[|\,(2-\alpha)\int_{x_1}^{x_2}(I-Q)Nu(s)ds \\ &\quad + (\alpha-1)\Big(\,|\int_0^{x_1}(x_2-x_1)\,ds + \int_{x_1}^{x_2}(x_2-s)(I-Q)Nu(s)ds\,|\,\Big) \\ &\leq \frac{K}{M(\alpha-1)}\Big[(x_2-x_1)+(x_2^2-x_1^2)\Big),\end{aligned}$$

where $K := \max\{M, 2-\alpha, \frac{1}{2}\}$. The right-hand side of the above inequality tends to zero as $x_2 \to x_1$, thus we have proved that $K_P(I-Q)N(\bar{\Omega})$ is equicontinuous. It follows from Ascoli-Arzela theorem that $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. The proof is completed.

We are now in a position to prove our main results. We need to impose the following conditions.

(H1) $|f(x,u)| \leq \psi(x) + g(x)|u|, \quad \psi, g \in C[0,1]$
with $M(\alpha-1) - 2\max\{2-\alpha, \alpha-1\}\|g\|_\infty > 0$.

(H2) There exists a $c_0 > 0$ such that $\forall u \in \mathbb{R}$ with $|u| > c_0$ and
$c_0\Big[\int_0^1\Big[(\alpha-1)Nu(s) - (2-\alpha)(Nu(s))'\Big]ds\Big] \neq 0$.

Lemma 3.8. *Suppose the conditions (H1) and (H2) hold true, then the set* $\Omega_1 = \{u \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lu = \lambda Nu, \lambda \in (0,1)\}$ *is bounded.*

Proof. For $u \in \Omega_1$, we have $u \in \operatorname{dom} L\setminus \operatorname{Ker} L$ and $Nu \in \operatorname{Im} L$. By Lemma 3.5, $\int_0^1\Big[(\alpha-1)Nu(s)-(2-\alpha)(Nu(s))'\Big]ds = 0$. Using the condition (H2), there exists x_0 such that $|u(x_0)| \leq c_0$. From the equation $Lu = \lambda Nu, \quad u(0) = 0$, we have the following equation with $L = \frac{\lambda}{M(\alpha-1)}$

$$u(x) = L\Big[(2-\alpha)\int_0^x Nu(s)\,ds + (\alpha-1)\int_0^x (x-s)Nu(s)ds\Big] + u'(0)x. \tag{3.3}$$

Hence, using the condition (H1) we obtain

$$\begin{aligned}\Big|u'(0)x_0\Big| &\leq c_0 + \frac{\lambda}{M(\alpha-1)}\Big[(2-\alpha)\int_0^{x_0}|Nu(s)|\,ds + (\alpha-1)\int_0^{x_0}(x_0-s)|Nu(s)|ds\Big] \\ &\leq c_0 + \frac{A}{M(\alpha-1)}\Big[\int_0^{x_0}|Nu(s)|\,ds + \int_0^{x_0}(x_0-s)|Nu(s)|ds\Big] \\ &\leq c_0 + \frac{A}{M(\alpha-1)}\left(\|\psi\|_\infty + \|g\|_\infty\|u\|_\infty\right),\end{aligned}$$

where $A = \max\{2-\alpha, \alpha-1\}$ Then, plugging this into the equation (3.3) gives $\|u\| \leq c_0 + \frac{2A}{M(\alpha-1)}\left(\|\psi\|_\infty + \|g\|_\infty\|u\|_\infty\right)$. Therefore, the condition $M(\alpha-1) - 2A\|g\|_\infty > 0$ implies that $\|u\| \leq \dfrac{c_0M(\alpha-1) + 2A\|\psi\|_\infty}{M(\alpha-1) - 2A\|g\|_\infty} := M_u$, So Ω_1 is bounded. The proof is now completed. □

Lemma 3.9. *Suppose (H2) holds, then the set*

$$\Omega_2 = \{u \mid u \in \operatorname{Ker} L, \quad Nu \in \operatorname{Im} L\} \textit{ is bounded.}$$

Proof. For $u \in \Omega_2$, for some constant c we have $u(x) = cx$ and $N(cx) \in \operatorname{Im} L = \operatorname{Ker} Q$. Thus, we have $Q(N(cx)) = 0$ which means that

$$\int_0^1\Big[(\alpha-1)N(cs)\,ds - (2-\alpha)(N(cs))'ds = 0.$$

Using the condition (H2), we obtain that $|u| \leq c_0$. This implies Ω_2 is bounded. The proof is now completed. □

Lemma 3.10. *Suppose* $(H2)$ *holds, then the set*

$$\Omega_3 = \{u \in \operatorname{Ker} L \mid \lambda u + (1-\lambda)QNu = 0, \lambda \in [0,1]\} \textit{ is bounded.}$$

Proof. For $u \in \Omega_3$ we have $u(x) = cx$ for some constant c and

$$\lambda cx + (1-\lambda)\frac{1}{1-\alpha}\int_0^1 \left[(\alpha-1)N(cs) - (2-\alpha)(N(cs))'\right] ds = 0. \tag{3.4}$$

If $\lambda = 1$, then $c = 0$ and we are done. If $\lambda \in [0,1)$ and $|u| > c_0$, then by $(H2)$ we infer that,

$$(1-\lambda)\frac{1}{1-\alpha}\int_0^1 \left[(\alpha-1)N(cs) - (2-\alpha)N'(cs)\right] ds > 0.$$

This together with (3.4) leads to $\lambda c_0 x < 0$. Since $x \in [0,1]$, $\lambda \in [0,1)$, one has to have $c_0 < 0$, a contradiction. As a result, Ω_3 is bounded. □

Theorem 3.11. *Suppose $(H1)-(H2)$ hold true and f is a L^1-Carathéodory function with the propriety that $f(x,u) \not\equiv 0$ for $x \in \Omega$. Then, the problem (1.1)-(1.2) has at least one solution in X.*

Proof. Let Ω be an open and bounded such that $\Omega = \cup_{n=1}^{3}\Omega_n$, where $\Omega_n, n = 1,2,3$ is given in above lemmas 3.8-3.10. It follows from Lemma 3.6 and Lemma 3.7 that L is a Fredholm operator of index zero and N is L-compact on $\bar{\Omega}$. In the light of Lemma 3.8 and Lemma 3.9, the conditions (1) and (2) of Lemma 3.4 are satisfied. Define

$$H(u,\lambda) = \pm\lambda u + (1-\lambda)QNu.$$

It follows from Lemma 3.10 that $H(u,\lambda) \neq 0$ for $u \in \operatorname{Ker} L \cap \partial\Omega$. Since the degree is invariant with respect to a homotopy, for $u \in \operatorname{Ker} L \cap \partial\Omega$ we have

$$\begin{aligned}
\deg\left(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\right) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0)\\
&= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0)\\
&= \deg(\pm I, \Omega \cap \operatorname{Ker} L, 0) \neq 0.
\end{aligned}$$

Thus, the condition (3) of Lemma 3.4 is satisfied. Consequently, by Lemma 3.4, there exists at least one solution to the equation $Lx = Nx$ in $\operatorname{dom} L \cap \bar{\Omega}$. Therefore, the BVP (1.1) has at least one solution that is not identically zero since $f(x,u) \not\equiv 0$ for $x \in \Omega$. Therefore, the proof is completed. □

4. Examples

In this section, we give numerical examples to illustrate our theoretical results.

Example 4.1. Consider the following FBVP at resonance

$$\begin{aligned}
{}^{CF}_{0}D_1^{\frac{3}{2}} u(x) &= f(x,u(x)), \quad 0 < x < 1,\\
u(0) &= 0, \quad u'(0) = u'(1),
\end{aligned} \tag{4.1}$$

where

$$f(x,u) = \frac{1}{4}(u-10) + \frac{1}{2}e^{-|u|}.$$

Define $\psi(x) = 4$, $g(x) = \frac{1}{4}$. Then, we have $f(x,u) \leq |\psi(x)| + g(x)|u|$ and $M\left(\frac{3}{2}-1\right) - 2(3/2-1)\frac{1}{4} = \frac{3}{8} > 0$. Thus, (H1) is fulfilled. Moreover, we take $c_0 = 11$ and (H2) is satisfied. Then, by Theorem 3.11, the FBVP (4.1) has at least one positive solution.

Example 4.2. Consider the FBVP (4.1) with

$$f(x,u) = \frac{5}{2} + \frac{1}{10}(x+1)(u^2(x) + u(x)).$$

Define $\psi(x) = \frac{5}{2}$, $g(x) = \frac{2}{5}$. Then, we have $f(x,u) \leq |\psi(x)| + g(x)|u|$ and $M\left(\frac{3}{2}-1\right) - 2(3/2-1)\frac{2}{5} = \frac{17}{20} > 0$. Thus, (H1) is fulfilled. Moreover, we take $c_0 = 1$ and (H2) is satisfied. Then, by Theorem 3.11, the FBVP (4.1) has at least one positive solution.

5. Conclusion

We present new results on the existence of positive solutions for the Caputo-Fabrizio fractional differential equation at resonance assuming that the growth conditions on the nonlinear function hold. The main tool in proving the existence of positive solution is the the Leggett-Williams norm-type theorem for coincidences due to O' Regan and Zima. Some numerical examples are given to illustrate the efficiency of the theoretical results.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author confirms sole responsibility for the following: study conception and design, analysis and interpretation of results, and manuscript preparation.

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