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Two Numerical Schemes for the Solution of the Generalized Rosenau Equation with the help of Operator Splitting Techniques

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Abstract
In the present manuscript, numerical solution of generalized Rosenau equation are applied quintic B-spline collocation and cubic B-spline lumped-Galerkin finite element methods (FEMs) together with both Strang splitting technique and the Ext4 and Ext6 techniques based on Strang splitting and derived from extrapolation. In the first instance, the problem is divided into two sub-equations as linear $U_t = \hat{A}(U)$ and nonlinear $U_t = \hat{B}(U)$ in the time term. Later, these sub-equations is implemented collocation and lumped-Galerkin (FEMs) using quintic and cubic B-spline functions respectively, with Strang ($S\Delta t = \hat{A} - \hat{B} - \hat{A}$), Ext4 and Ext6 splitting techniques. The numerical solutions of the system of ordinary differential equations obtained in this way are solved with help fourth order Runge-Kutta method. The aim of this study is to obtain superior results. For this, a test problem is selected to show the accuracy and efficiency of the method and the error norm results produced by these techniques have been compared among themselves and with the current study in the literature. İt can be clearly stated that it is concluded that the approximate results obtained with the proposed method are better than the study in the literature. So that one can see that the study has achieved its purpose.

Keywords: *Generalized Rosenau equation, quintic and cubic B-splines, collocation and Galerkin methods, Splitting techniques.*

AMS Subject Classification (2020): *35Q51; 74J35; 33F10.*

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1. Introduction

The Rosenau equation has been an important research and application topic in the fields of mathematics and physics by Philip Rosenau [1] since the 80s. Because the Korteweg-de Vries (KdV) equation which is one of the most

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important partial differential equations doesn't introduce wave to wave interaction and wave to wall interaction Chung [2], Rosenau [3] has exposed equation

$$Ut + Uxxxxt + UUx + Ux = 0, (x, t) \in \Omega x(0, T] \tag{1.1}$$

with the condition given at initial time

$$U(x, 0) = g_0(x), \tag{1.2}$$

and the conditions given at the boundaries

$$\begin{aligned} U(0, t) &= U(1, t) = 0, \\ U_{xx}(0, t) &= U_{xx}(1, t) = 0 \quad t > 0. \end{aligned} \tag{1.3}$$

Existence and uniqueness of equation with initial and boundary condition given in above have been studied by Park [4] and she has shown that the solution $U \in H^4(\Omega)$ is the only one solution for $U_0 \in H^4(\Omega)$. There are many studies in the literature about the Rosenau equation. These can be summarized as follows: Chung and Ha [5] proposed finite element Galerkin approximate solutions for a KDV-like Rosenau equation that models the dynamics of dense discrete systems to show existence and uniqueness of exact solutions and discussed the error estimates of the continuous time Galerkin solutions. Manickam *et al*.[6] performed a KDV-like Rosenau equation in one space variable using a second-order splitting method. Hence they employed an orthogonal cubic spline collocation procedure to approximate the resulting system. Chung [2] indicates existence and uniqueness of numerical solutions for the KDV-like Rosenau equation describing the dynamics of dense discrete systems. Lee [7] used the discrete Galerkin type approximations for solutions of the Rosenau equation. Sportisse [8] analyzed that the evolution equations to be simulated are stiff. Chung and Pani [9] showed a continuous in time finite element Galerkin method for a KDV-like Rosenau equation in several space variables and suggested several fully discrete schemes and build up connected convergence results.The Rosenau equation in several space variables was split into two second order equations and submited a lumped mass finite element method for piece wise linear elements by Chung and Pani [10]. Barreto *et al* [11] showed the existence of solutions of the Rosenau and Benjamin-Bona-Mahony equations known as hyperbolic equation. Choo *et al*. [12] achieved a posteriori error estimates of Rosenau equation using a discontinuous Galerkin method. Omrani *et al* [13] presented a conservative difference scheme for the KDV-like Rosenau equation appeared the unique solvability of numerical solutions. Hu and Zheng [14] suggested numerical solutions of generalized Rosenau equation and considered two energy conservative finite difference schemes. Wang *et al*.[15] considered the generalized Rosenau equation with a finite difference scheme and researched existence and uniqueness of numerical solution of equation. Mittal and Jain [16] solved some Rosenau type non-linear higher order evolution equations with Dirichlet's boundary conditions with the help of quintic B-splines collocation method. Atouani and Omrani [17] applied high-order conservative difference scheme for Rosenau equation. Abazari and Abazari [18] devoted numerical solution of KDV-like Rosenau equation using the quintic B-spline collocation scheme. Cai *et al*.[19] employed the variational discretizations for the generalized Rosenau-type equations. Ramos and Garcia-Lopez [20] studied numerically a generalized viscous Rosenau equation with the help of an implicit second-order accurate method in time. Safdari-Vaighani *et al*. [21] implemented radial basis function method (RBF) approximations methods for numerical solution of Rosenau equation, where they extented the fictitious point method and the resampling method to study by means of an RBF collocation. References to [22–25] and [26–35] can also be looked at for different methods applied for this type of partial differential equations. Atouani *et al*. [36] performed mixed finite element methods for Rosenau equation via splitting technique. Also, one can have a look at the references [37–43] on the splitting technique. In this study, we will consider generalized Rosenau equation given with form

$$2Ut + Uxxxxt + 3Ux - 60U^2U_x + 120U^4U_x = 0 \tag{1.4}$$

with an initial condition

$$U(x, 0) = sech(x) = g_0(x), x \in [x_L, x_R], \tag{1.5}$$

and with boundary condition

$$\begin{aligned} U(x_L, t) &= sech(x_L - t) = g_1(x), \\ U(x_R, t) &= sech(x_R - t) = g_2(x), \end{aligned} \tag{1.6}$$

$$\begin{aligned} U_x(x_L, t) &= -sech(x_L - t)tanh(x_L - t) = g_3(x), \\ U_x(x_R, t) &= sech(x_R - t)tanh(x_R - t) = g_4(x). \end{aligned} \tag{1.7}$$

The present work is summarized as follow: In section 2, about operator splitting method is informed. In section 3-4, generalized Rosenau equation is split into two sub-equations and each equation is implemented collocation and lumped-Galerkin (FEMs) with quintic and cubic B-splines respectively, and they are coverted of system order-differential equations and solved with operator splitting techniques using fourth order Runge-Kutta technque (RK-4). In Section 5, a test problem given with initial and boundary conditions is considered. Error norms L_2 and L_∞ obtained by operator splitting techniques are among themselves and with study available in the literature are compared. In Section 6, to emphasize the importance of the present method, a brief conclusion is given.

By using fourth order Runge-Kutta method via quintic B-spline collocation and cubic B-spline lumped-Galerkin finite element methods (FEMs) together with operator time splitting techniques, numerical solutions of the main problem in the study can be easily obtained. Thus, one can see that it has been produced quite good results with the method proposed in the study.

2. Splitting techniques

The splitting technique achieved with the half-time step Δt is sometimes known as Marhuk [42] and is the second-order symmetric technique proposed by Strang [44]. The mentioned technique can be defined as follows by changing the locations of the operators

$$S_{\Delta t} = e^{\frac{\Delta t}{2}\hat{A}} e^{\Delta t \hat{B}} e^{\frac{\Delta t}{2}\hat{A}} \text{ or } S^*_{\Delta t} = e^{\frac{\Delta t}{2}\hat{B}} e^{\Delta t \hat{A}} e^{\frac{\Delta t}{2}\hat{B}}. \tag{2.1}$$

This technique has the local turncation error called as splitting error which is in form

$$\begin{aligned} Te &= \frac{(e^{\Delta t(\hat{A}+\hat{B})} - e^{\frac{\Delta t}{2}\hat{A}} e^{\Delta t \hat{B}} e^{\frac{\Delta t}{2}\hat{A}})U(t_n)}{\Delta t} \\ &= \frac{\Delta t^2}{24}(2\left[\hat{B},\left[\hat{B},\hat{A}\right]\right] - \left[\hat{A},\left[\hat{A},\hat{B}\right]\right])U(t_n) + O(\Delta t^3) \end{aligned}$$

and this shows the fact that the proposed technique is of the second-order. The procedure for Strang splitting scheme can be presented as

$$\begin{aligned} \frac{dU^*(t)}{dt} &= \hat{A}U^*(t), \;\; U^*(t_n) = U_n^0, && t \in \left[t_n, t_{n+\frac{1}{2}}\right] \\ \frac{dU^{**}(t)}{dt} &= \hat{B}U^{**}(t), \;\; U^{**}(t_n) = U^*(t_{n+\frac{1}{2}}), && t \in [t_n, t_{n+1}] \\ \frac{dU^{***}(t)}{dt} &= \hat{A}U^{***}(t), \;\; U^{***}(t_{n+\frac{1}{2}}) = U^{**}(t_{n+1}), && t \in \left[t_{n+\frac{1}{2}}, t_{n+1}\right] \end{aligned} \tag{2.2}$$

in which $t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2}$. Here the desired solutions are easily obtained through the equation of $U(t_{n+1}) = U^{***}(t_{n+1})$. When this scheme is defined as $\hat{A} - \hat{B} - \hat{A}$, a new scheme can be obtained as $\hat{B} - \hat{A} - \hat{B}$. In the present study, extrapolation techniques [45] presented below are used to further improve convergence

$$\frac{4}{3}\phi\frac{\Delta t}{2} * \phi\frac{\Delta t}{2} - \frac{1}{3}\phi\Delta t,$$

and

$$\frac{81}{40}\phi\frac{\Delta t}{3} * \phi\frac{\Delta t}{3} * \frac{\Delta t}{3} * -\frac{16}{15}\phi\frac{\Delta t}{2} * \phi\frac{\Delta t}{2} * +\frac{1}{24}\phi\Delta t.$$

In addition to the Strang splitting technique, the fourth and sixth order convergence techniques presented below are used, respectively.

$$Ext4 = \frac{4}{3}\phi^{\hat{B}}_{\frac{\Delta t}{4}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}} * \phi^{\hat{B}}_{\frac{\Delta t}{2}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}} \phi^{\hat{B}}_{\frac{\Delta t}{2}} - \frac{1}{3}\phi^{\hat{B}}_{\frac{\Delta t}{2}} \phi^{\hat{A}} \Delta t \phi^{\hat{B}}_{\frac{\Delta t}{2}}$$

$$\begin{aligned} Ext6 = \frac{81}{40}\phi^{\hat{B}}_{\frac{\Delta t}{6}} * \phi^{\hat{A}}_{\frac{\Delta t}{3}} * \phi^{\hat{B}}_{\frac{\Delta t}{3}} \phi^{\hat{A}}_{\frac{\Delta t}{3}} * \phi^{\hat{B}}_{\frac{\Delta t}{3}} * \phi^{\hat{A}}_{\frac{\Delta t}{3}} \phi^{\hat{B}}_{\frac{\Delta t}{6}} - \frac{16}{15}\phi^{\hat{B}}_{\frac{\Delta t}{4}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}} * \phi^{\hat{B}}_{\frac{\Delta t}{2}} * \phi^{\hat{A}}_{\frac{\Delta t}{2}} \phi^{\hat{B}}_{\frac{\Delta t}{4}} \\ + \frac{1}{24}\phi^{\hat{B}}_{\frac{\Delta t}{2}} \phi^{\hat{A}} \Delta t \phi^{\hat{B}}_{\frac{\Delta t}{2}}. \end{aligned}$$

3. Scheme 1: Operator time-splitting solution by quintic B-spline collocation method of generalized Rosenau equation

Let be given as $x_L = x_0 \leq x_1 \leq ... \leq x_N = x_R$ at the knots x_m, $(m = 0(1)N-1)$ a uniform partition of closed interval $x_L \leq x \leq x_R$ about to be the solution domain $x_L \leq x \leq x_R$ in which $h = \frac{x_R - x_L}{N}$. Quintic B-spline functions $\varphi_m(x)$ are defined such as in the following on the domain $x_L \leq x \leq x_R$ with knot points x_m, $(m = -2(1)N+2)$

$$\varphi_m(x) = \frac{1}{h^5}\begin{cases}(x-x_{m-3})^5, & x\in[x_{m-3},x_{m-2}]\\ (x-x_{m-3})^5-6(x-x_{m-2})^5, & x\in[x_{m-2},x_{m-1}]\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5, & x\in[x_{m-1},x_m]\\ (x_{m+3}-x)^5-6(x_{m+2}-x)^5+15(x_{m+1}-x)^5, & x\in[x_m,x_{m+1}]\\ (x_{m+3}-x)^5-6(x_{m+2}-x)^5, & x\in[x_{m+1},x_{m+2}]\\ (x_{m+3}-x)^5, & x\in[x_{m+2},x_{m+3}]\\ 0, & otherwise.\end{cases} \tag{3.1}$$

[46]. All the quintic B-spline functions outside of $\varphi_{m-2}(x), \varphi_{m-1}(x), \varphi_m(x), \varphi_{m+1}(x), \varphi_{m+2}(x), \varphi_{m+3}(x)$, are zero except that the elements on $[x_m, x_{m+1}]$. Global approximation $U_N(x,t)$ corresponding exact solution $U(x,t)$ of eq.(1.4) can be written with the following formula on $[x_m, x_{m+1}]$ in terms of quintic B-spline functions and quantities $\delta_j(t)$ that need to be found

$$U_N(x,t) = \sum_{j=m-2}^{m+3} \varphi_j(x)\delta_j(t). \tag{3.2}$$

Using knot values in eqs.(3.1) and (3.2), derivatives up to fourth order of U and U with respect to variable x in terms of time dependent parameters are obtained as follows:

$$\begin{aligned} U_N^e(x_m,t) = (U_N^e)_m &= (\delta_{m-2}+26\delta_{m-1}+66\delta_m+26\delta_{m+1}+\delta_{m+2}),\\ (U_N^e)'_m = U'_m &= \frac{5}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2}),\\ (U_N^e)''_m = U''_m &= \frac{20}{h^2}(\delta_{m-2}+2\delta_{m-1}-6\delta_m+2\delta_{m+1}+\delta_{m+2}),\\ (U_N^e)'''_m = U'''_m &= \frac{60}{h^3}(-\delta_{m-2}+2\delta_{m-1}-6\delta_m-2\delta_{m+1}+\delta_{m+2}),\\ (U_N^e)^{(4)}_m = U^{(4)}_m &= \frac{120}{h^4}(\delta_{m-2}-4\delta_{m-1}+6\delta_m-4\delta_{m+1}+\delta_{m+2}). \end{aligned} \tag{3.3}$$

Generalized Rosenau equation (1.4) is split into as follows:

$$2Ut + Uxxxxt + 3Ux = 0 \tag{3.4}$$

$$2Ut + Uxxxxt - 60U^2U_x + 120U^4U_x = 0. \tag{3.5}$$

If the values of U, U', U'', U''' and $U^{(4)}$ in (3.3) are replaced in equations (3.4) and (3.5), the following system of first ordinary differential equations (3.6) and (3.7) to be consisted of $(N+1)$ equations and $(N+5)$ unknowns are found as follows:

$$\begin{aligned} 2\dot{\delta}_{m-2}+52\dot{\delta}_{m-1}+132\dot{\delta}_m+52\dot{\delta}_{m+1}+2\dot{\delta}_{m+2}+\frac{120}{h^4}(\dot{\delta}_{m-2}-4\dot{\delta}_{m-1}+6\dot{\delta}_m-4\dot{\delta}_{m+1}+\dot{\delta}_{m+2})\\ +\frac{15}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2}) \end{aligned} \tag{3.6}$$

$$\begin{aligned} 2\dot{\delta}_{m-2}+52\dot{\delta}_{m-1}+132\dot{\delta}_m+52\dot{\delta}_{m+1}+2\dot{\delta}_{m+2}-\frac{120}{h^4}(\dot{\delta}_{m-2}-4\dot{\delta}_{m-1}+6\dot{\delta}_m-4\dot{\delta}_{m+1}+\dot{\delta}_{m+2})\\ \frac{-300z_m}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2})+\frac{600g_m}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2})=0 \end{aligned} \tag{3.7}$$

where the symbol ″.″ denotes derivation with respect to time t and $z_m = U^2, g_m = U^4$ are considered as linearization process respectively in the following form

$$z_m = (\delta_{m-2}+26\delta_{m-1}+66\delta_m+26\delta_{m+1}+\delta_{m+2})^2,$$

$$g_m = (\delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2})^4.$$

Eliminating the parameters δ_{-2}, δ_{-1} and $\delta_{N+2}, \delta_{N+1}$ which are outside of the solution region from systems (3.6) and (3.7) using the boundary conditions $U(x_L, t) = sech(x_L - t), U(x_R, t) = sech(x_R - t)$ and $U_x(x_L, t) = sech(x_L - t)tanh(x_L - t), U_x(x_R, t) = sech(x_R - t)tanh(x_R - t)$ given by equation (1.4) , we obtain matrix systems $(N+1)$ x $(N+1)$ for systems (3.6) and (3.7) with form

$$\dot{\delta}^{n+1} = A_1^{-1} B_1 \delta^n$$

$$\dot{\delta}^{n+1} = A_1^{-1} B_2 \delta^n$$

such that $\delta^n = (\delta_0, ..., \delta_0 \delta_N)^T$ where A_1, B_1 and B_2 are matrices of dimensional $(N+1)$ x $(N+1)$ acquired as

$$A_1 = [a_{ij}] = \begin{cases} a_{11} = \frac{7650}{h^4}, a_{12} = \frac{4500}{h^4}, a_{13} = \frac{450}{h^4}, \\ a_{21} = \frac{175}{4} - \frac{975}{h^4}, a_{22} = \frac{255}{2} - \frac{990}{h^4}, a_{23} = \frac{207}{4} - \frac{495}{h^4}, a_{23} = a_{24} = 2 + \frac{120}{h^4} \\ a_{i,i-2} = a_{24}, a_{i,i-1} = 52 - \frac{480}{h^4}, a_{ii} = 132 + \frac{720}{h^4} \\ a_{i,i+1} = a_{i,i-1}, a_{i,i+2} = a_{24}; i = 3(1)N-1, \\ a_{N,N-2} = a_{24}, a_{N,N-1} = a_{23}, a_{N,N} = a_{22}, a_{N,N+1} = a_{21} \\ a_{N+1,N+1} = a_{11} \end{cases} \tag{3.8}$$

$$B_1 = [b_{ij}] = \begin{cases} b_{11} = 0, \\ b_{21} = \frac{705}{8h}, b_{22} = -\frac{135}{4h}, b_{23} = -\frac{1215}{8h}, b_{24} = -\frac{15}{h} \\ b_{i,i-2} = -b_{24}, a_i, a_{i-1} = \frac{150}{h}, b_{ii} = 0, \\ b_{i,i+1} = -\frac{150}{h}, b_{i,i+2} = b_{24}, i = 3(1)N-1, \\ b_{N,N-2} = -b_{24}, b_{N,N-1} = -b_{23}, b_{N,N} = -b_{22}, b_{N,N+1} = -b_{21} \\ a_{N+1,N+1} = 0. \end{cases} \tag{3.9}$$

$$B_2 = [b_{ij}] = \begin{cases} b_{11} = 0, \\ b_{21} = -\frac{3525}{2h} z_1 + \frac{3525}{h} g_1, b_{22} = \frac{675}{h} z_1 - \frac{1350}{h} g_1, \\ b_{23} = \frac{6075}{2h} z_1 - \frac{6075}{h} g_1, b_{24} = \frac{300}{h} z_1 - \frac{600}{h} g_1 \\ b_{i,i-2} = -\frac{300}{2h} z_m + \frac{600}{h} g_m, a_i, a_{i-1} = -\frac{3000}{2h} z_m + \frac{6000}{h} g_m, b_{ii} = 0, \\ b_{i,i+1} = -b_{i,i-1}, b_{i,i+2} = -b_{i,i-2}; i = 3(1)N-1, \\ b_{N,N-2} = -\frac{300}{h} z_N + \frac{600}{h} g_N, b_{N,N-1} = -\frac{6075}{2h} z_N + \frac{6075}{h} g_N, b_{N,N} = -\frac{675}{h} z_N + \frac{1350}{h} g_N, \\ b_{N,N+1} = \frac{3525}{2h} z_N - \frac{3525}{h} g_N, \\ b_{N+1,N+1} = 0. \end{cases} \tag{3.10}$$

As a solution method, after splitting the main equation, the FEMs presented in the article are applied to each equation. Then, the obtained system of ordinary differential equations is solved using the Runge Kutta method (RK4) with the help of Strang, Ext4 and Ext6 splitting algorithms. Furthermore, in order to obtain better results at each time step, systems (3.6) and (3.7) are applied three-five times an inner iteration given by

$$(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1}).$$

Now, for solution of the systems (3.6) and (3.7), we need to obtain the initial vector δ_m^0 with the help of initial condition $U(x, 0) = g_0(x)$ and boundary conditions

$$\begin{aligned} U_x(x_L, t) = g_3(x), U_x(x_R, t) = g_4(x), \\ U_{xx}(x_L, t) = g_5(x), U_{xx}(x_R, t) = g_6(x). \end{aligned} \tag{3.11}$$

Finally, the matrix equation for the initial vector δ_m^0 is acquired as

$$\begin{bmatrix} 54 & 60 & 6 & & & & & \\ 25.25 & 67.5 & 26.25 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & & & \ddots & & & & \\ & & 1 & 26 & 66 & 26 & 1 & \\ & & & 1 & 26.25 & 67.5 & 25.25 & \\ & & & & 6 & 60 & 54 & \end{bmatrix} \begin{bmatrix} {\delta^0}_0 \\ {\delta^0}_1 \\ {\delta^0}_2 \\ . \\ . \\ . \\ {\delta^0}_{N-2} \\ {\delta^0}_{N-1} \\ {\delta^0}_N \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ . \\ . \\ . \\ U_{N-2} \\ U_{N-1} \\ U_N \end{bmatrix}.$$

These matrices are easy to obtain with a symbolic programming language. In this study, Matlab 2019b with a memory 20GB and 64 bit has been used.

4. Scheme 2: Operator time-splitting solution by cubic B-spline Lumped Galerkin method of generalized Rosenau equation

In this section, we will handle with cubic B-spline lumped Galerkin method to numerical solution of (1.4) equation with the initial-boundary conditions (1.5) and (1.6)-(1.7). Here, the solution region of the problem is taken as in section 3. Cubic B-spline functions $\varphi_m(x)$, $(m = -1(1)N+1)$ at knots point x_m on the solution domain $[x_L, x_R]$ are described by Prenter [46] as follows:

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}) \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x \in [x_{m-1}, x_m) \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x \in [x_m, x_{m+1}) \\ (x_{m+2} - x)^3, & x \in [x_{m+1}, x_{m+2}] \\ 0, & otherwise. \end{cases} \tag{4.1}$$

Approximate solution $U_N(x,t)$ corresponding exact solution $U(x,t)$ of eq.(1.4) can be given in the following form in terms of cubic B-splines on $[x_m, x_{m+1}]$

$$U_N(x,t) = \sum_{j=-1}^{N+1} \varphi_j(x)\delta_j(t) \tag{4.2}$$

where $\varphi_j(x)$ are element shape functions and $\delta_j(t)$ are unknown time-dependent element parameters obtained with boundary conditions and weighted residual conditions. Using the local coordinate transformation given by $\xi = x - x_m$ such that $0 \le \xi \le h$ on the finite element $[x_m, x_{m+1}]$, cubic B-spline shape functions in terms of ξ on the region [0,h] can be submitted as follows:

$$\begin{aligned} \varphi_{m-1} &= \frac{1}{h^3}(h-\xi)^3, \\ \varphi_m &= \frac{1}{h^3}(4h^3 - 6h\xi^2 + 3\xi^3), \\ \varphi_{m+1} &= \frac{1}{h^3}(h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3), \\ \varphi_{m+2} &= \frac{1}{h^3}(\xi^3). \end{aligned} \tag{4.3}$$

Thus, the approximate solution $U_N(x,t)$ can be given as

$$U_N(\xi,t) = \sum_{j=m-1}^{m+2} \delta_j\varphi_j. \tag{4.4}$$

whole splines except $\varphi_{m-1}(x), \varphi_m(x), \varphi_{m+1}(x), \varphi_{m+2}(x)$ are zero over the domain $[x_m, x_{m+1}]$. Using cubic B-spline functions (4.1) and trial functions (4.2), u_N and its first and second derivatives at knots x_m according to x in terms of the element parameters δ_j are presented by

$$\begin{aligned} U_m = Ux_m &= \delta_{m+1} + 4\delta_m + \delta_{m-1} \\ U_m^{'} = U^{'}(x_m) &= \frac{3}{h}(\delta_{m+1} - \delta_{m-1}) \\ U_m^{''} = U^{''}(x_m) &= \frac{6}{h^2}(\delta_{m+1} - 2\delta_m + \delta_{m-1}). \end{aligned} \tag{4.5}$$

Generalized Rosenau equation (1.4) is split into two sub-equations as follows:

$$2U_t + U_{xxxxt} + 3U_x = 0, \tag{4.6}$$

$$2U_t + U_{xxxxt} - 60U^2U_x + 120U^4U_x = 0. \tag{4.7}$$

When applying the Galerkin method to (4.6) and (4.7) equations, respectively, the weak form of (4.6) and (4.7) equations is obtained as follows:

$$\int_{x_L}^{x_R} W\Big[2U_t + U_{xxxxt} + 3U_x\Big]dx = 0 \tag{4.8}$$

$$\int_{x_L}^{x_R} W\Big[2U_t + U_{xxxxt} - 60U^2U_x + 120U^4U_x\Big]dx = 0. \tag{4.9}$$

Here, due to the use of the Galerkin method, the weight function is chosen the same as the approximate functions and the approximate functions are B-splines and at the same time the smoothness of the weight function is guaranteed. If is used transformation $\xi = x - x_m$, we can get the following equations

$$\int_0^h W\Big[2U_t + U_{\xi\xi\xi\xi t} + 3U_\xi\Big]d\xi = 0 \tag{4.10}$$

$$\int_0^h W\Big[2U_t + U_{\xi\xi\xi\xi t} - 60U^2U_\xi + 120U^4U_\xi\Big]d\xi = 0. \tag{4.11}$$

In which U^2 and U^4 are considerd to be a constant such that z_m and g_m respectively. Applying partial integration to (4.10) and (4.11) equations lead to

$$\int_0^h \Big[2WU_t + W_{\xi\xi}U_{\xi\xi t} + 3WU_\xi\Big]d\xi = [-WU_{\xi\xi\xi t} + W_\xi U_{\xi\xi t}]\ |_0^h \tag{4.12}$$

$$\int_0^h \Big[2WU_t + W_{\xi\xi}U_{\xi\xi t} - 60z_mWU_\xi + 120Wg_mU_\xi\Big]d\xi = [-WU_{\xi\xi\xi t} + W_\xi U_{\xi\xi t}]\ |_0^h\ . \tag{4.13}$$

If it is taken the weight function as cubic B-spline base functions presented by equation (4.3) and replacing approximation (4.4) in integral equations (4.12) and (4.13) with some manipulation, the element contributions are given in the form

$$\sum_{j=m-1}^{m+2} \Big[2\Big(\int_0^h (\varphi_i\varphi_j + \varphi_i^{''}\varphi_j^{''})d\xi\Big)\Big]\dot{\delta}_j + 3\Big(\int_0^h \varphi_i\varphi_j^{'}d\xi\Big)\delta_j = -\Big[\Big((\varphi_i\varphi_j^{'''}) + (\varphi_i^{'}\varphi_j^{''})\Big)\ |_0^h\ \Big]\dot{\delta}_j \tag{4.14}$$

$$\sum_{j=m-1}^{m+2} \Big[2\Big(\int_0^h (\varphi_i\varphi_j + \varphi_i^{''}\varphi_j^{''})d\xi\Big)\Big]\dot{\delta}_j + \Big(-60z_m(\int_0^h \varphi_i\varphi_j^{'}d\xi) + 120g_m(\int_0^h \varphi_i\varphi_j^{'}d\xi)\Big)\delta_j = -\Big[\Big((\varphi_i\varphi_j^{'''}) + (\varphi_i^{'}\varphi_j^{''})\Big)\ |_0^h\ \Big]\dot{\delta}_j \tag{4.15}$$

In matrix form, (4.14) and (4.15) equations can be written as follows:

$$(2A^e + 3B^e - C^e + D^e)\dot{\delta}^e + 3C^e\delta^e = 0 \tag{4.16}$$

$$(2A^e + 3B^e - C^e + D^e)\dot{\delta}^e - (60C_1^e + 120C_2^e)\delta^e = 0 \tag{4.17}$$

respectively, where

$$A^e = \int_0^h \varphi_i\varphi_j d\xi = \frac{h}{140}\begin{bmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 160 & 933 & 1188 & 129 \\ 1 & 60 & 1129 & 20 \end{bmatrix}$$

$$B^e = \int_0^h \varphi_i^{''}\varphi_j^{''} d\xi = \frac{6}{h^3}\begin{bmatrix} 2 & -3 & 0 & 1 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 1 & 0 & -3 & 2 \end{bmatrix}$$

$$C^e = \int_0^h \varphi_i \varphi_j^{'} d\xi = \frac{1}{20}\begin{bmatrix} -10 & -9 & 18 & 1 \\ -71 & -150 & 183 & 38 \\ -38 & -183 & 150 & 71 \\ -1 & -18 & 9 & 10 \end{bmatrix}$$

$$D^e = \varphi_i \varphi_j^{'''} \mid_h^0 = \frac{6}{h^3}\begin{bmatrix} 1 & -3 & 3 & -1 \\ 3 & -9 & 9 & -3 \\ -3 & 9 & -9 & 3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$E^e = \varphi_i^{'}\varphi_j^{''} \mid_h^0 = \frac{18}{h^3}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

with sub-indexes $i, j = m-1, m, m+1, m+2$. Since U^2 and U^4 are considered constants like z_m and g_m, respectively, their lumped values are taken as $z_m = (\frac{\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})}{2})^2$ and $g_m = (\frac{\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2}}{2})^4$ respectively. Combining together contributions from all elements, we have the following matrix equations

$$\dot{\delta} = (2A + 3B - C + D)^{-1}(3C)\delta \tag{4.18}$$

$$\dot{\delta} = (2A + 3B - C + D)^{-1}(60C_1 + 120C_2)\delta^e \tag{4.19}$$

respectively, where C_1 and C_2 are matrices Z_mC and g_mC, $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$ is a vector and "$\cdot$" shows the derivative according to time. Here $\delta = (\delta_{-1}, \delta_0, ..., \delta_N, \delta_{N+1})^T$ is global element parameters. The A,B,C,D and z_mC and g_mC, are septa-diagonal matrices and their m.th rows are

$$A = \frac{1}{140}(1, 120, 1191, 2416, 1191, 120, 1),$$

$$B = \frac{6}{h^3}(1, 0, -9, 16, -9, 0, 1),$$

$$C = \frac{1}{20}(-1, -56, -245, 0, 245, 56, 1),$$

$$D = \frac{6}{h^3}(-1, 0, 9, -16, 9, 0, -1),$$

$$E = (0, 0, 0, 0, 0, 0, 0),$$

$$Z_mD = \frac{1}{20}(-Z_1, -18Z_1 - 38Z_2, 9Z_1 - 183Z_2 - 71Z_3, 10Z_1 + 150Z_2 - 150Z_3 - 10Z_4, 71Z_2 + 183Z_3 - 9Z_4, 38Z_3 + 18Z_4, Z_4)$$

in which

$$Z_1 = \frac{1}{4}(\delta_{m-2} + 5\delta_{m-1} + 5\delta_m + \delta_{m+1})^2$$

$$Z_2 = \frac{1}{4}(\delta_{m-1} + 5\delta_m + 5\delta_{m+1} + \delta_{m+2})^2$$

$$Z_3 = \frac{1}{4}(\delta_m + 5\delta_{m+1} + 5\delta_{m+2} + \delta_{m+3})^2$$

$$Z_4 = \frac{1}{4}(\delta_{m+1} + 5\delta_{m+2} + 5\delta_{m+3} + \delta_{m+4})^2.$$

Similar operations are written for g_m to the fourth power. Equation systems (4.18) and (4.19) comprise $(N+3)$ unknowns and $(N+3)$ equations. Using boundary conditions (1.6) and (1.7) given in (1.4) and the values $U_N(x,t)$ at knot points for $m=0$ and $m=N$, we can get the following equations

$$\delta_{-1}(t) + 4\delta_0(t) + 4\delta_1(t)$$

$$\delta_{N-1}(t) + 4\delta_N(t) + 4\delta_{N+1}(t).$$

If the δ_{-1} and $\delta_{N+1}(t)$ parameters from equations system (4.18) and (4.19) are eliminated using the above equations, we obtain system of matrices $(N+1)$ x $(N+1)$ for systems (4.18) and (4.19) This system is solved by means of the Thomas algorithm. In order to mimimize the nonlinearity , we need to two - five time inner iterations $(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ via fourth-order Runge-Kutta technique. In order to start the Runge-Kutta technique, the initial values of the parameters are needed. These values are obtained from $U(x,0) = f(x)$ initial conditions and approximate solutions $U_N(x_m,0) = \sum_{j=-1}^{N+1} \varphi_j(x_m)\delta_j^0(t)$ at $t=0$. Thus, equation systems consisted from $(N+1)$ equation and $(N+3)$ unknown for equations (4.18) and (4.19) are obtained in the following form

$$U(x_0,0) = \delta_{-1}(t) + 4\delta_0(t) + 4\delta_1(t)$$

$$U(x_1,0) = \delta_0(t) + 4\delta_1(t) + 4\delta_2(t)$$

...

$$U(x_{N-1},0) = \delta_{N-2}(t) + 4\delta_{N-1}(t) + 4\delta_N(t).$$

$$U(x_N,0) = \delta_{N-1}(t) + 4\delta_N(t) + 4\delta_{N+1}(t).$$

To solve this systems, we need to two auxiliary equations. These assistant equations are obtained utilizing the second derivative boundary conditions submited by (1.7) at $t=0$.

$$U_m^{''}(x_0,0) = \frac{6}{h^2}(\delta_{-1} - 2\delta_0 + \delta_1)$$

$$U_m^{''}(x_N,0) = \frac{6}{h^2}(\delta_{N-1} - 2\delta_N + \delta_{N+1}).$$

As a result, the systems (4.18) and (4.19) is $(N+3)$ x $(N+3)$-dimensional and we can be easily calculated the initial vector δ^0 from the following matrix equations

$$\begin{bmatrix} 1 & -2 & 1 & & & & & \\ 1 & 4 & 1 & & & & & \\ & & & . & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & 1 & 4 & 1 \\ & & & & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} {\delta^0}_{-1} \\ {\delta^0}_0 \\ . \\ . \\ . \\ {\delta^0}_N \\ {\delta^0}_{N+1} \end{bmatrix} = \begin{bmatrix} U_0'' \\ U_0 \\ . \\ . \\ U_N \\ U_N'' \end{bmatrix}.$$

5. Numerical examples and results

In this section, we have calculated with one example existing in the literature the difference between numerical solution with exact solution to demonstrate the accuracy and performance of the presented method. For this purpose, we have utilized error norms L_2 and L_∞ presented in the following form with the Matlab 2019b computer program which has a memory 20GB and 64 bit

$$L_2 = ||U - U_N||_2 = \sqrt{h\sum_{j=0}^{N}(U - U_N)^2}$$

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|.$$

Table 1. The error norm values for different values of h at $t = 0.2$ of Scheme I.

	SΔt		Ext 4		Ext 6	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$1.57E-3$	$1.48E-3$	$1.51E-3$	$1.45E-3$	$1.48E-3$	$1.36E-3$
100	$1.06E-3$	$9.91E-4$	$0.98E-3$	$9.33E-4$	$0.95E-3$	$8.34E-4$
120	$8.02E-4$	$6.91E-4$	$7.09E-4$	$6.43E-4$	$6.92E-4$	$5.44E-4$
140	$6.72E-4$	$5.06E-4$	$5.69E-4$	$4.53E-4$	$5.62E-4$	$4.17E-4$

Table 2. The error norm values for different values of h at $t = 0.2$ of Scheme II.

	SΔt		Ext 4		Ext 6	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$2.69E-3$	$2.23E-3$	$1.40E-3$	$1.11E-3$	$0.81E-3$	$0.74E-3$
100	$2.65E-3$	$22.07E-4$	$0.98E-3$	$7.35E-4$	$0.54E-3$	$5.30E-4$
120	$26.93eE-4$	$22.09E-4$	$7.88E-4$	$6.22E-4$	$5.23E-4$	$4.22E-4$
140	$27.37eE-4$	$22.16E-4$	$6.92E-4$	$5.60E-4$	$5.61E-4$	$3.96E-4$

Example 5.1. In this study, the solution region of the problem given with (1.4)-(1.7) is taken as $[-10, 10]$. Initial condition and the exact solution of the problem are submitted as follows:

$$U(x, 0) = sech(x), \quad x \in [-10, 10],$$

and

$$U(x, t) = sech(x - t), \quad x \in [-10, 10].$$

For different space values $h = 1/4, 1/5, 1/6, 1/7$ at $t = 0.2$, firstly, Tables 1 and 2 present comparison of the error norm values L_2 and L_∞ produced for both scheme I and scheme II with the help of operator splitting techniques (SΔt, Ext4,Ext6) and fourth order Runge-Kutta technque (RK-4) using quintic B-spline collocation and cubic B-spline lumped Galerkin methods respectively, for generalized Rosenau equation. As can be seen from these tables, the techniques Ext4 and Ext6 have lower error norm results produced for decreasing h values. In addition, the Strang splitting technique with the quintic B-spline collocation method applied in scheme I produces better results than the Strang splitting technique with the cubic B-spline lumped Galerkin method applied in scheme II. However, the results obtained with Ext4 and Ext6 techniques in scheme II are better than those in scheme I. After, Table 3-4 presents comparison with those in study [16] of the error norm values obtained for schemes I-II. It is clear from this table that the error norms L_2 and L_∞ acquired with the techniques Ext4 and Ext6 for both scheme I and scheme II are better from those of the values given in [16]. In the Table 5, for descending values of Δt and different space step length at time $t = 0.2$, we have calculated the error norms L_2 and L_∞ to show effectiveness of SΔt technique in Scheme II. We have seen that the error norms are significantly reduced, when the time step Δt become smaller and also we have observed that scheme I produces the same results as scheme II for the same parameter values.

Table 3. A comparison of the error norms with those given in [16] for different values of h at $t = 0.2$ of Scheme I.

	SΔt		Ext 4		Ext 6		[16]	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$1.57E-3$	$1.48E-3$	$1.51E-3$	$1.45E-3$	$1.48E-3$	$1.36E-3$	$1.53eE-3$	$1.49eE-3$
100	$1.06E-3$	$9.91E-4$	$0.98E-3$	$9.33E-4$	$0.95E-3$	$8.34E-4$	$1.01E-3$	$9.84eE-4$
120	$8.02E-4$	$6.91E-4$	$7.09E-4$	$6.43E-4$	$6.92E-4$	$5.44E-4$	$7.38eE-4$	$6.91eE-4$
140	$6.72E-4$	$5.06E-4$	$5.69E-4$	$4.53E-4$	$5.62E-4$	$4.17E-4$	$6.04E-4$	$5.00eE-4$

Table 4. A comparison of the error norms with those given in [16] for different values of h at $t = 0.2$ of Scheme II.

	SΔt		Ext 4		Ext 6		[16]	
N	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
80	$2.69E-3$	$2.23E-3$	$1.40E-3$	$1.11E-3$	$0.81E-3$	$0.74E-3$	$1.53E-3$	$1.49eE-3$
100	$2.65E-3$	$22.07E-4$	$0.98E-3$	$7.35E-4$	$0.54E-3$	$5.30E-4$	$1.01E-3$	$9.84E-4$
120	$26.93eE-4$	$22.09E-4$	$7.88E-4$	$6.22E-4$	$5.23E-4$	$4.22E-4$	$7.38E-4$	$6.91E-4$
140	$27.37eE-4$	$22.16E-4$	$6.92E-4$	$5.60E-4$	$5.61E-4$	$3.96E-4$	$6.04E-4$	$5.00E-4$

Table 5. The computed of the error norms for different values of h and Δt at $t = 0.2$ of Scheme II.

		L_2	L_∞			L_2	L_∞
h=1/4	$\Delta t = 0.02$	$1.66E-3$	$1.35E-3$	$h = 1/5$	$\Delta t = 0.02$	$1.33E-3$	$11.46E-4$
	$\Delta t = 0.01$	$1.40E-3$	$1.08E-3$		$\Delta t = 0.01$	$0.98E-3$	$8.13E-4$
	$\Delta t = 0.005$	$1.30E-3$	$1.03E-3$		$\Delta t = 0.005$	$0.83E-3$	$6.43E-4$
	$\Delta t = 0.0025$	$1.26E-3$	$1.04E-3$		$\Delta t = 0.0025$	$0.78E-3$	$6.43E-4$
h=1/6	$\Delta t = 0.02$	$12.17E-4$	$10.49E-4$	$h = 1/7$	$\Delta t = 0.02$	$11.70E-4$	$9.96E-4$
	$\Delta t = 0.01$	$7.94E-4$	$6.83E-4$		$\Delta t = 0.01$	$7.03E-4$	$6.17E-4$
	$\Delta t = 0.005$	$6.15E-4$	$5.08E-4$		$\Delta t = 0.005$	$4.97E-4$	$4.29E-4.$
	$\Delta t = 0.0025$	$5.42E-4$	$4.20E-4$		$\Delta t = 0.0025$	$4.12E-4$	$3.36E-4$

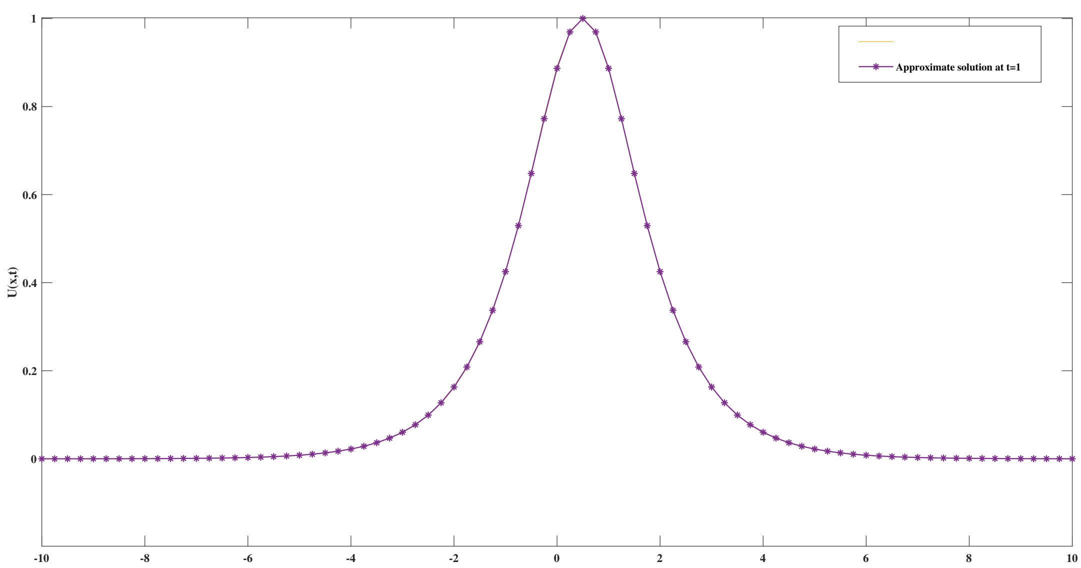

Figure 1. The overlapping of the approximate and the exact solution at $t = 1$ for Scheme I with $h = 0.25, \Delta t = 0.1$.

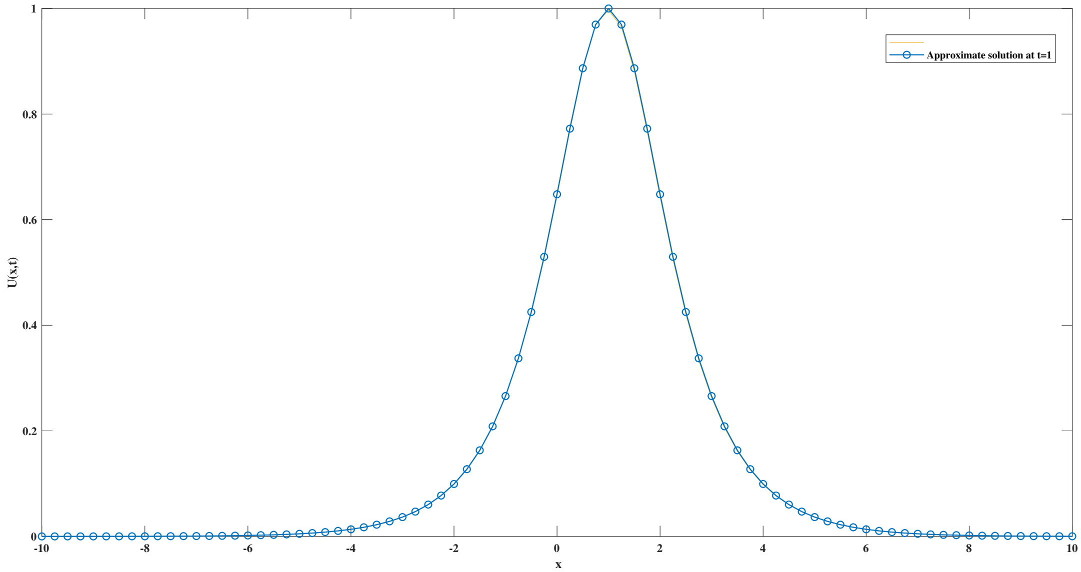

Figure 2. The overlapping of the approximate and the exact solution at $t = 1$ for Scheme II with $h = 0.25, \Delta t = 0.1$.

6. Conclusion

In this study, the numerical solution of generalized Rosenau equation with the initial and boundary conditions are computed by applying the fourth order Runge -Kutta method to systems obtained using collocation and lumped Galerkin methods (FEMs) with quintic and cubic B-splines with help operator time splitting techniques (Strang($S\Delta t$), Ext4 and ext6). It is selected a test problem available in the literature to measure the effectiveness of the method. Results obtained by the application of the method have been compared among themselves and with the present study in the literature. As result of comparisons, it is understood that the Ex6 technique are better than the Ext4 and the Ext4 technique than Strang splitting technique. Here, it is clear that the best among operator splitting techniques is the Ext6 technique. As a conclusion, one can observe that performence of the present method applied for generalized Rosenau equation is very well. Furthermore, operator time splitting techniques can be easily applied to partial differential equations used in different types of science.

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