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THE FIRST ISOMORPHISM THEOREM FOR (CO-ORDERED) Γ-SEMIGROUPS WITH APARTNESS

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Abstract. The notion of Γ-semigroups has been introduced in 1984 by Sen. This author introduced the concept of Γ-semigroups with apartness and analyzed their properties within the Bishop's constructive orientation. Many classical notions and processes of semigroups and Γ-semigroups have been extended to Γ-semigroups with apartness. Co-ordered Γ-semigroups with apartness have been studied by the author also. In this paper, as a continuation of previous research, the author investigates the specificity of two forms of the first isomorphism theorem for (co-ordered) Γ-semigroups with apartness which one of them has no a counterpart in the Classical theory. In addition, specific techniques used in proofs within algebraic Bishop's constructive orientation are exposed.

1. Introduction

Let (S, w) be a grupoid, where the set S is the carrier of this structure and $w : S \times S \longrightarrow S$ is a total function. If w is associative, then the structure is (S, w) a semigroup. Let us suppose that the logical environment in which we analyze this algebraic structure is Intuitionistic logic **IL** ([29]). This assumption implies that the axiom 'Principe TND' (tertium non datur - the logical principle of 'the exclusion of the third') is not valid in this setting. Just as equality '=' is one of the fundamental concepts in Classical Logic, in this logic, apart from equality, the diversity relation '≠' is a fundamental concept equal and independent to the equality and it is strongly connected with the concept of equality. Commitments under which we will obey in this paper is the Bishop's principled-philosophical orientation **Bish** (see, for example: [1, 2, 3, 13, 14, 21]). For the philosophical aspect of Bishop's constructive orientation, a reader can see the text [6] written by Laura Crosilla.

Now, we look at the carrier S as a relational system $(S, =, \neq)$, where $'='$ is the standard equality, and $'\neq'$ is a diversity relation on S. The symbol $'\neq'$ should

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be seen as a special relation and not as a negation of the relation of equality. This last relation is determined by the following axioms:

$(\forall x, y \in S)(x \neq y \implies \neg(x = y))$ (consistency),

$(\forall x, y \in S)(x \neq y \implies y \neq x)$ (symmetry).

This relation is extensive with respect to the equality in the standard way

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq$$

where $' \circ '$ is the standard composition of relations. A diversity relation $\neq$ is called apartness if it is co-transitive in the following sense

$(\forall x, y, z \in S)(x \neq z \implies (x \neq y \vee y \neq z))$ (co -transitivity).

In addition, any relation R on S, any functions f between such sets and any operation w in S appearing in this article are strongly extensional relative to the apartness (see, for example: [21]). For a strongly extensional mapping we will hereafter briefly write 'se-mapping'. Because of the specificity of **IL**, for some subsets of the S, their strictly extensive doubles will appears. For example, a (strongly extensional) subset K of a semigroup S with apartness is a *co-ideal* of S if holds

$$(\forall x, y \in S)(xy \in K \implies x \in K \vee y \in K).$$

It is not difficult to show that a strong complement

$$K^{\lhd} = \{x \in S : (\forall s \in K)(x \neq s)\}$$

of co-ideal K is an ideal in S. Conversely, in the general case, it does not have to be valid. Thus, the observed structure $S = ((S, =, \neq), w)$ is a *semigroup with apartness*. In this case, the internal binary operation w agrees with the relations of equality and apartness in the following sense

$$(\forall x, y, u, v \in S)((x = u \wedge y = v) \implies w(x, y) = w(u, v)),$$

$$(\forall x, y, u, v \in S)(w(x, y) \neq w(u.v) \implies (x \neq u \vee y \neq v)).$$

In the last 40 years, this author is alone, or in collaboration with other authors, has investigated structures of various types of semigroups with apartnesses [7, 8, 16, 17, 18, 19]. Independent, in the article [4], Cherubini and Frigeri introduce the concept of 'inverse semigroups with apartness'. A critical survey [15], written by M. Mitrović, M. N. Hounkonnou and M. A. Baroni, significantly deepens interest in the theory of semigroups with apartness. One of the main problems in these researches was "How to find and describe the duals of classical algebraic concepts?" in well-known algebraic structures.

In this text we interested in Γ-semigroups with apartness as a continuation of our research [20, 22, 23, 24]. The concept of the Γ-semigroup with apartness was introduced in the article [20]. Semilattice congruences in Γ-semigroup with apartness were the focus of the paper [22] while the article [23] analyzes some substructures of co-ordered Γ-semigroup with apartnss. A brief historical overview of the theory of Γ-semigroups with apartness within the Bishop-s constructive orientation can be found in the article [24].

Also, we will find and analyze some doubles of substructures of these semigroups. Our investigation the concept of Γ-semigroups with appartness consists of the observation of specificities that arise by placing the classically defined algebraic structure of Γ-semigroups ([5, 9, 10, 11, 12, 25, 26, 27, 28]) into a different logical environment and using specific Bishop's constructive algebra tools. Important for this analysis were the articles [5, 10] in which the isomorphism theorems in such semigroups are treated while in the papers [11, 12, 28] the properties of ordered Γ-semigroups are observed.

The rest of the paper is organized as follows: In Section 2, the concept of Γ-semigroup with apartness (Definition 2.1) and the concept of Γ-cosubsemigroup with apartness (Definition 2.2) are given. In subsection 2.2, the notion of ordered Γ-semigroup with apartess ordered under a co-order relation is given (Definition 2.4). While in subsection 2.3 we recall the concept of Γ-cocongruence, in subsection 2.4 we state the notion of homomorphism between Γ-semigroups.

In Section 3, which contains the main part of this paper, two forms (Theorem 3.2 and Theorem 3.5) of the first theorem on isomorphism between Γ-semigroups with apartness are presented, one of which has its a counterpart in the Classical Γ-semigroup theory. In Section 4, which is also the novelty in this scientific report, two forms of the First Isomorphism Theorem applied to Γ-semigroups ordered under co-order relations are designed (Theorem 4.4 and Theorem 4.6). In this case as well as in the previous one, the last of these theorems has no counterpart in the Classical theory of Γ-semigroups.

2. Preliminaries

2.1. **Γ-semigroup with apartness.** To explain the notions and notations used in this article, which but not previously described, we instruct a reader to look at the articles [11, 12, 25, 26, 27, 28]. Here we will introduce some specific substructures of this semigroups that appear only in the **Bish** version.

Definition 2.1. ([20], Definition 2.1) Let $(S, =, \neq)$ and $(\Gamma, =, \neq)$ be two inhabited (non-empty) sets with apartness. Then S is called a Γ-*semigroup with apartness* if there exist a strongly extensional total mapping $w_S : S \times \Gamma \times S \ni (x, a, y) \longmapsto xay \in S$ satisfying the condition

$$(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz = xa(ybz)).$$

We recognize immediately that the following implications

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq ubv \implies (x \neq u \vee a \neq b \vee y \neq v)),$$

and

$$(\forall x, y \in S)(\forall a, b \in \Gamma)(xay \neq xby \implies a \neq b)$$

are valid, because w_S is a strongly extensional function.

Definition 2.2. ([20], Definition 2.2) Let S be a Γ-semigroup with apartness. A subset T of S is said to be a Γ-*cosubsemigroup* of S if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)(xay \in T \implies (x \in T \vee y \in T)).$$

We will assume that the empty set $\emptyset$ is a Γ-cosubsemigroup of a Γ-semigroup S by definition.

Our first proposition in this section is the following:

Proposition 2.3 ([20], Proposition 2.1)**.** *If T is a Γ-cosubsemigroup of a Γ - semigroup with apartness S, then the set $T^{\lhd}$ is a Γ - subsemigroup of S.*

2.2. **Co-ordered Γ-semigroup with apartness.** The relation α is said to be a co-order on the set X with apartness if it is consistent $\alpha \subseteq \neq$, co-transitive $\alpha \subseteq \alpha * \alpha$, i.e.

$$(\forall x, y, z \in X)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha))$$

and linear in the following sense: $\neq \subseteq \alpha \cup \alpha^{-1}$. α is said to be a co-quasiorder on X if it is a consistent and co-transitive relation on X. An interested reader can find more information about this type of relations in sets (in algebraic structures also) in the following author's articles: [16, 17, 18, 19, 21]. A brief recapitulation of a number of algebraic structures ordered by co-quasiorder relation is presented in the review paper [21].

In the following definition we introduce the concept of co-order relations in Γ-semigroup with apartness.

Definition 2.4. ([22], Definition 3.1) Let S be a Γ-semigroup with apartness. A co-order relation $\nleqslant$ on S is *compatible* with the semigroup operations in S if the following holds

$$(\forall x, y, z \in S)(\forall a \in \Gamma)((xaz \nleqslant yaz \vee zax \nleqslant zay) \implies x \nleqslant y).$$

In this case it is said that S is an *ordered* Γ-semigroup *under co-order* $\nleqslant$ or it is *co-ordered* Γ-semigroup.

If we speak the language of classical algebra, then the relation $\nleqslant$ is compatible with the operation in S if this operation is cancellative with respect to the co-order.

2.3. **Γ-cocongruences.** The notion of the co-equality relation in sets with apartness introduced and analyzed by this author (see, for example, [16, 18]). The relation q is a co-equality on a set $(S, =, \neq)$ if it is a consistent, symmetric and co-transitive relation on S. A co-congruence on some algebraic structure $((S, =, \neq), \cdot)$ is a coequality relation on S which is compatible in one very specific sense with the internal operation in S. A reader can look at these specific features in the author previously published papers [17, 18, 19]. A co-equality relation q on a semigroup with apartness S is a co-congruence in S if the following holds

$$(\forall x, y, u, v \in S)((xu, yv) \in q \implies ((x, y) \in q \vee (u, v)).$$

The concept of Γ-cocongruence on Γ-semigroups with apartness was introduced in [20] by the following definition

Definition 2.5. ([20], Definition 2.6) Let S be a Γ-semigroup with apartness. A co-equality relation $q \subseteq S \times S$ is called a Γ-*cocongruence* on S if the following holds

(1) $((xau, ybv) \in q \implies ((x, y) \in q \vee a \neq b \vee (u, v) \in q))$

for any $x, y, u, v \in S$ and all $a, b \in \Gamma$.

The following lemma allows to a reader to better understand the previous implication.

Lemma 2.6 ([22], Lemma 3.2). *The condition* (1) *in Definition 2.5 is equivalent to the following two conditions*

(2) $(\forall x, y, z \in S)(\forall a, b \in \Gamma)((zax, zby) \in q, \implies ((x, y) \in q \vee a \neq b))$ *and*

(3) $(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xaz, ybz) \in q \implies ((x, y) \in q \vee a \neq b))$.

2.4. **Homomorphism between Γ-semigroups.** In this subsection, the determination of the concept of Γ-homomorphism between Γ-semigroups with apartness, taken from article [20], is considered.

Definition 2.7. ([20], Definition 2.7) Let S is a Γ-semigroup and T a Λ-semigroups with apartness. A pair (h, φ) of strongly extensional functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ is called a *se-homomorphism* from Γ-semigroup S to Λ-semigroup T if the following holds

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \varphi)(xay) :=_T h(x)\varphi(a)h(y)).$$

As can be seen, the determination of the Γ-homomorphism between two Γ-semigroups with separateness is somewhat different from the determination of this notion in the classical case. In our case, it is a pair (h, φ) of two strictly extensional functions assuming that the set acting on both structures does not have to be the same.

The following can be verified without difficulty.

Lemma 2.8. *Let (h, φ) be a se-homomorphism from Γ-semigroup with apartness S to a Λ-semigroup with apartness R. Then holds*

$$(h, \varphi) \circ w_S = w_T \circ (h, \varphi, h)$$

where (h, φ, h) is understood as follows

$$(\forall x, y \in S)(\forall a \in \Gamma)((h, \varphi, h)(x, a, y) := (h(x), \varphi(a), h(y))$$

Of course, the specificity of this determination is in the requirement that the functions $h : S \longrightarrow T$ and $\varphi : \Gamma \longrightarrow \Lambda$ must be strongly extensional functions. It is easily verified that (h, φ) is a correctly determined strongly extensive function.

Also, it is easy to see that:

Proposition 2.9. *Let $(h, \varphi) : S \longrightarrow T$ be a se-homomorphism. Then:*
- The relation $\rho := Ker(h, \varphi) := \{(x, y) \in S \times S : (h, \varphi)(x) =_T (h, \varphi)(y)\}$ is a Γ-congruence on Γ-semigroup S.
- The relation $q := Coker(h, \varphi) := \{(x, y) : (h, \varphi)(x) \neq_T (h, \varphi)(y)\}$ is a Γ-cocongruence on S.
- The subset $(h, \varphi)(S)$ is a Λ-subsemigroup of Λ-semigroup T.

For more information on the statements made in the previous proposition, a reader can look at Theorem 4 in [16], for example.

The relations ρ and q are associated in the following sense

$$\rho \circ q \subseteq q \text{ and } q \circ \rho \subseteq q.$$

3. The First Isomorphism Theorem of Γ-semigroups

In this section, the author designs two forms (Theorem 3.2 and Theorem 3.5) of the first isomorphism theorem on Γ-semigroups with apartness.

The Proposition 2.9 it allows to us to construct the structure $(S/(\rho, q), =_1, \neq_1)$ with

$$x\rho =_1 y\rho \iff (x, y) \in \rho \quad \text{and} \quad x\rho \neq_1 y\rho \iff (x, y) \in q$$

where $x, y \in S$. If the function

$$w_{S/(\rho,q)} : S/(\rho, q) \times \Gamma \times S/(\rho, q) \longrightarrow S/(\rho, q)$$

we determine as follows

$$(\forall x\rho, y\rho \in S/(\rho,q))(\forall a \in \Gamma)(w_{S/(\rho,q)}(x\rho, a, y\rho) := (xay)\rho),$$

the following theorem can be verified.

Theorem 3.1. *The structure $(S/(\rho,q), =_1, \neq_1)$ is a Γ-semigroup with apartness and there exists a unique se-epimorphism $(\pi, i) : S \ni x \longmapsto x\rho \in S/(\rho,q)$ such that the following holds*

$$w_{S/(\rho,q)} \circ (\pi, i, \pi) = (\pi, i) \circ w_S$$

where $i : \Gamma \longrightarrow \Gamma$ is the identical mapping.

Proof. It is clear that $w_{S/(\rho,q)}$ is a well-defined total function. Let us prove that $w_{S/(\rho,q)}$ is strongly extensional. Let $x\rho, y\rho, u\rho, v\rho \in S/(\rho,q)$ and $a \in \Gamma$ be arbitrary elements such that

$$w_{S/(\rho,q)}(x\rho, a, y\rho) \neq_1 w_{S/(\rho,q)}(u\rho, b, v\rho).$$

Then $(xay)\rho \neq_1 (ubv)\rho$. The latter means $(xay, ubv) \in q$. Thus $xay \neq ubv$ by consistency of q. Hence, $w_{S/(\rho,q)}$ is a strongly extensional total function.

Let us now show that $w_{S/(\rho,q)}$ satisfies the associativity condition in Definition 2.1. For $x\rho, y\rho, z\rho \in S/(\rho,q)$ and $a, b \in \Gamma$, we have

$$\begin{aligned} w_{S/(\rho,q)}(x\rho a y\rho, b, z\rho) &=_1 w_{S/(\rho,q)}((xay)\rho, b, z\rho) \\ &=_1 ((xay)bz)\rho =_1 (xa(ybz))\rho =_1 w_{S/(\rho,q)}(x\rho, a, (ybz)\rho). \end{aligned}$$

If we define the function $(\pi, i) : S \longrightarrow S/(\rho,q)$ in the following way $(\forall x \in S)((\pi, i)(x) := x\rho \in S/(\rho,q)$ then it can be checked without major difficulties that it is correctly defined a unique strongly extensional total surjective homomorphism.

Finally, let $x, y \in S$ and $a \in \Gamma$ be arbitrary elements. We have

$$\begin{aligned} &((\pi, i) \circ w_S)(x, a, y) =_1 (\pi, i)(w(x, a, y)) =_1 (\pi, i)(xay) =_1 (xay)\rho \\ &=_1 w_{S/(\rho,q)}(x\rho, a, y\rho) =_1 w_{S/(\rho,q)}(\pi(x), i(a), \pi(y)) =_1 w_{S/(\rho,q)}((\pi, i, \pi)(x, a, y)) \\ &=_1 (w_{S/\rho,q)} \circ (\pi, i, \pi))(x, a, y). \end{aligned}$$

□

We can look at the following theorem as the first theorem of isomorphism about Γ-semigroup with apartness.

Theorem 3.2. *Let $(h, \varphi) : S \longrightarrow T$ be a se-homomorphism from Γ-semigroup with apartness S into Λ-semigroup with apartness T. Then there exists a unique injective, embedding and surjective se-homomorphism (g, φ) from $S/(\rho,q)$ onto Λ-subsemigroup $((h, \varphi)(S), =_T, \neq_T)$ such that*

$$(h, \varphi) = (g, \varphi) \circ (\pi, i, \pi) \text{ and } w_T = (\pi, i) \circ w_S.$$

Proof. Let us define the mapping $(g, \varphi) : S/(\rho,q) \longrightarrow (h, \varphi)(S)$ in the following way

$$(\forall x \in S)((g, \varphi)(x\rho) := (h, \varphi)(x))$$

and check that this function thus determined is a well-defined se-homomorphism that has the required properties. Since it is known that (g, φ) is well-defined an injective function, let us check that (g, φ) is a se-mapping. Let $(h, \varphi)(x), (h, \varphi)(y) \in (h, \varphi)(S)$ be such that $(g, \varphi)(x\rho) =_T (h, \varphi)(x) \neq_T (h, \varphi)(y) =_T (g, \varphi)(y\rho)$. Then $(x, y) \in Coker(h, \varphi) = q$. Thus $x\rho \neq_1 y\rho$. So (g, φ) is a se-mapping.

The reverse is also valid. If $x\rho \neq_1 y\rho$ holds for some $x\rho, y\rho \in S/(\rho,q)$, then $(x, y) \in q = Coker(h, \varphi)$. This means $(h, \varphi)(x) \neq_T (h, \varphi)(y)$. Therefore $(g, \varphi)(x) \neq_T (g, \varphi)(y)$. This shows that (g, φ) is an embedding.

Since the equations $(h,\varphi)=(g,\varphi)\circ(\pi,i,\pi)$ and $w_T=(\pi,i)\circ w_S$ are obvious, this completes the proof of the theorem. $\square$

On the other hand, we can design the set $[S:q]:=\{xq : x\in S\}$ where the equality '$=_2$' and the apartness '$\neq_2$' are determined in the following way

$$(\forall\, xq, yq\in[S:q])(xq =_2 yq \iff (x,y)\lhd q \text{ and } xq\neq_2 yq \iff (x,y)\in q).$$

The operation $w_{[S:q]}$ on $[S:q]\times\Gamma\times[S:q]$ is defined as

$$(\forall xq, yq,\in[S:q])(\forall a\in\Gamma)(w_{[S:q]}(xq,a,yq):=(xay)q).$$

The following theorem proves the existence of another Γ-semigroup generated by the homomorphism (h,φ) which has no its counterpart in the Classical Γ-semigroup theory.

Theorem 3.3. *The structure* $([S:q],=_2,\neq_2)$ *is a* Γ*-semigroup and there is a unique se-morphism* $(\vartheta,i):S\longrightarrow[S:q]$ *such that the following holds*

$$w_{[S:q]}\circ(\vartheta,i,\vartheta)=(\vartheta,i)\circ w_S.$$

Proof. For arbitrary elements $xq,yq,uq,vq\in[S:q]$ and $a,b\in\Gamma$ such that it is $(xq,a,yq)=_{[S:q]\times\Gamma\times[S:q]}(uq,b,vq)$ we have $xq=_2 uq$, $a=b$ and $yq=_2 vq$. This means $(x,u)\lhd q$ and $(y,v)\lhd q$. Let $s,t\in S$ be such that $(s,t)\in q$. Then $(s,xay)\in q$ or $(xay,ubv)\in q$ or $(ubv,t)\in q$. As the second option gives $(x,u)\in q\vee a=b\vee(y,v)\in q$ which contradicts the adopted assumption, we conclude that the following holds $sq\neq_2(xay)q$ or $(ubv)q\neq_2 tq$. Since $s,t\in S$ were arbitrary, we conclude that $(xay,ubv)\lhd q$ is valid. Hence

$$w_{[S:q]}(xq,a,yq)=_2(xay)q=_2(ubv)q=_2 w_{[S:q]}(uq,b,vq).$$

This shows that $w_{[S:q]}$ is a well-defined function.

Let $xq,yq,uq,vq\in[S:q]$ and $a,b\in\Gamma$ be such that

$$w_{[S:q]}(xq,a,yq)=_2(xay)q\neq_2(ubv)q=_2 w_{[S:q]}(uq,b,vq).$$

It follows

$$(x,u)\in q\ \vee\ a\neq b\ \vee\ (y,v)\in q$$

from here. So, $(xq,a,yq)\neq_{[S:q]\times\Gamma\times[S:q]}(uq,b,vq)$ thus proving that $w_{[S:q]}$ is a se-mapping.

For $xq,yq,zq\in[S:q]$ and $a,b\in\Gamma$ we have

$$w_{[S:q]}(xq,a,(yz)a)=_2(xa(ybz))q=_2((xay)bz)q=_2 w_{[S:q]}(xay,b,z).$$

Therefore, $([S:q],=_2,\neq_2)$ is a Γ-semigroup.

Finally, if $x,y\in S$ and $a\in\Gamma$ arbitrary elements, then we have

$$\begin{aligned}(w_{[S:q]}\circ(\vartheta,i,\vartheta))(x,a,y)&=_2 w_{[S:q]}(xq,a,yq)=_2(xay)q=_2(\vartheta,i)(xay)\\&=_2((\vartheta,i)\circ w_S)(x,a,t).\end{aligned}$$

$\square$

Although Γ-semigroups $S/(\rho,q)$ and $[S:q]$ are different, there is a strong connection between them. It is known that $q^{\lhd}$ is a congruence on S compatible with q and that $\rho\subseteq q^{\lhd}$ holds (see, for example [16], Theorem 1). The structure $(S/(q^{\lhd},q),=_3,\neq_1)$ is a Γ-semigroup, where

$$(\forall x,y\in S)(xq^{\lhd}=_3 yq^{\lhd}\iff(x,y)\lhd q).$$

Proposition 3.4. *Let $(h,\varphi) : S \longrightarrow T$ be a se-homomorphism from Γ-semigroup with apartness S into Λ-semigroup with apartness T. There is a unique surjective and embedding se-homomorphism $\alpha : S/(\rho,q) \longrightarrow [S:q]$ such that $w_{[S:q]} = \alpha \circ w_{S/(\rho,q)}$. In particular, $\alpha : S/(q^{\lhd},q) \longrightarrow [S:q]$ is an se-isomorphism.*

Proof. Let us define the correspondence $\alpha : S/(\rho,q) \longrightarrow [S:q]$ as follows:

$$(\forall x\rho \in S/(\rho,q))(\alpha(x) := xq)$$

and show the properties of such a determined function.

First, to show that α is a well-defined function, we choose $x,y,s,t \in S$ such that $x\rho =_1 y\rho$ and $(s,t) \in q$. Then $(x,y) \in \rho$ and $(x,y) \notin q$. from $(s,t) \in q$ it follows

$$\begin{aligned}(s,t) \in q &\Longrightarrow (s,x) \in q \vee \vee (x,y) \in q \vee (y,t) \in q \\ &\Longrightarrow s \neq_S x \vee y \neq_S t \\ &\Longrightarrow (x,y) \neq_{S\times S} (s,t) \in q.\end{aligned}$$

This means $(x,y) \lhd q$, ie, $xq =_2 yq$. So α is a correctly defined function.

On the other hand, since

$$(\forall x,y \in S)(xq \neq_2 yq \iff x\rho \neq_1 y\rho),$$

holds, we conclude that α is an embedding se-mapping.

Finally, let us show that in the mentioned special case α is an injective mapping. Let $x,y \in S$ be arbitrary elements such that $xq =_2 yq$. Then $(x,y) \lhd q$. this means $xq^{\lhd} =_3 yq^{\lhd}$. □

The following theorem is another form of the first isomorphism theorem on Γ-semigroups with apartness which has no counterpart in the Classical Γ-semigroup theory.

Theorem 3.5. *Let $(h,\varphi) : S \longrightarrow T$ be a se-homomorphism from Γ-semigroup with apartness S into Λ-semigroup with apartness T. Then there exists a unique injective, embedding and surjective se-homomorphism (f,φ) from $[S:q]$ onto Λ-subsemigroup $((h,\varphi)(S), \neq_T^{\lhd}, \neq_T)$ such that*

$$(h,\varphi) = (f,\varphi) \circ (\vartheta, i, \vartheta) \text{ and } w_T = (\vartheta, i) \circ w_S.$$

Proof. Let us define $(f,\varphi) : [S:q] \longrightarrow S$ as follows

$$(\forall xq \in [S:q])((f,\varphi)(xq) := (h,\varphi)(x)).$$

Let $x,y,u,v,s,t \in S$ and $a,b \in \Gamma$ be arbitrary elements such that $(xay)q =_2 (ubv)q$ and $(s,t) \in q$. Then $(xay, ubv) \lhd q$. Now, we have

$$\begin{aligned}(s,t) \in q &\Longrightarrow (s,xay) \in q \vee (xay,ubv) \in q \vee (ubv,t) \in q \\ &\Longrightarrow sq \neq_2 (xay)q \vee (ubv)q \neq_2 tq \\ &\iff (s,xay) \in q \vee (ubv,t) \in q.\end{aligned}$$

Previously we recognize as

$$((f,\varphi)(x,a,y),(f,\varphi)(u,b,v)) \neq ((h,\varphi)(s),(h,\varphi)(t)) \in \neq_T .$$

So, the following holds

$$(f,\varphi)((xay)q) \neq_T^{\lhd} (f,\varphi)((ubv)q).$$

This shows that (f,φ) is a well-defined function.

Let $(xay)q, (ubv)q \in [S : q]$ be arbitrary elements such that

$$(f,\varphi)((xay)q) \neq_T (f,\varphi)((ubv)q).$$

Then $(h,\varphi)(xay) \neq_T (h,\varphi)(ubv)$ and $(xau, ubv) \in Coker(h,\varphi) = q$. This means $(xay)q \neq_2 (ubv)q$. Thus, it is shown that (f,φ) is a se-mapping.

To show the injectivity of the se-function (f,φ) let us take $x, y, u, v, s, t \in S$ and $a, b \in \Gamma$ such that $(s,t) \in q$ and

$$(f,\varphi)((xay)q) \neq_T^{\triangleleft} (f,\varphi)((ubv)q),$$

that is, such that it is

$$(h,\varphi)(xay) \neq_T^{\triangleleft} (h,\varphi)(ubv).$$

Then, we have

$$\begin{aligned}(s,t) \in q &\Longrightarrow (s, xay) \in q \vee (xay, ubv) \in q \vee (ubv, t) \in q\\ &\Longrightarrow (s, xay) \in q \vee (ubv, t) \in q\\ &\Longrightarrow (xay, ubv) \neq (s,t) \in q.\end{aligned}$$

Thus, we have $(xay)q =_2 (ubv)q$, proving that (f,φ) is an se-monomorphism.

Let $x, y, u, v \in S$ and $a, b \in \Gamma$ be arbitrary elements such that $(xay)q \neq_2 (ubv)q$. We have

$$\begin{aligned}(xay)q \neq_2 (ubv)q &\Longleftrightarrow (xay, ubv) \in q = Coker(h,\varphi)\\ &\Longleftrightarrow (h,\varphi)(xay) \neq_T (h,\varphi)(ubv)\\ &\Longleftrightarrow (f,\varphi)((xay)q) \neq_T (f,\varphi)((ubv)q).\end{aligned}$$

Since the equations $(h,\varphi) = (f,\varphi) \circ (\vartheta, i)$ and $w_T = (\vartheta, i) \circ w_S$ are obvious, this completes the proof of the theorem. □

4. The First Isomorphism Theorem of Co-ordered Γ-semigroups

Let us first present a brief analysis of homomorphisms between co-ordered Γ-semigroups:

Let $((S, =_S, \neq_S), \nleqslant_S)$ be a co-ordered Γ-semigroup and $((T, =_T, \neq_T), \nleqslant_T)$ be co-ordered Λ-semigroup. Note that the relation $\nleqslant_S$ (i.e., the relation $\nleqslant_T$) is compatible with the operation on S (respectively, on T) in the following sense

$$(\forall x, y, z \in S)(\forall a \in \Gamma)((xay \nleqslant_S yaz \vee zax \nleqslant_S zay) \Longrightarrow x \nleqslant_S y).$$

Let $(h,\varphi) : S \longrightarrow T$ be homomorphism. For (h,φ) is said to be:

- *isotone* if the following holds

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \nleqslant_S ubv \Longrightarrow (h,\varphi)(xay) \nleqslant_T (h,\varphi)(ubv));$$

- *reverse isotone* if the following holds

$$(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)((h,\varphi)(xay) \nleqslant_T (h,\varphi)(ubv) \Longrightarrow xay \nleqslant_S ubv).$$

In what follow, it will express in the following two lemmas several processes with homomorphisms that will be used.

Lemma 4.1. *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone homomorphism. Then $(h,\varphi)^{-1}(\nleqslant_T)$ is a co-quasiorder relation on S compatible with the operation on S such that $(h,\varphi)^{-1}(\nleqslant_T) \subseteq \nleqslant_S$.*

Proof. Let $x, t, z \in S$ be arbitrary elements.

$$\begin{aligned}
(x,y) \in (h,\varphi)^{-1}(\nleq_T) &\iff (h,\varphi)(x) \nleq_T (h,\varphi)(y) \\
&\Longrightarrow x \nleq_S y \\
&\Longrightarrow x \neq_S y. \\
(x,z) \in (h,\varphi)^{-1}(\nleq_T) &\iff (h,\varphi)(x) \nleq_T (h,\varphi)(z) \\
&\Longrightarrow (h,\varphi)(x) \nleq_T (h,\varphi)(y) \vee (h,\varphi)(y) \nleq_T (h,\varphi)(z) \\
&\iff (x,y) \in (h,\varphi)^{-1}(\nleq_T) \vee (y,z) \in (h,\varphi)^{-1}(\nleq_T). \\
(x,y) \in (h,\varphi)^{-1}(\nleq_T) &\iff (h,\varphi)(x) \nleq_T (h,\varphi)(y) \\
&\Longrightarrow x \nleq_S y. \\
(xaz, yaz) \in (h,\varphi)^{-1}(\nleq_T) &\iff (h,\varphi)(xaz) \nleq_T (h,\varphi)(yaz) \\
&\Longrightarrow xaz \nleq_S yaz \\
&\Longrightarrow x \nleq_S y.
\end{aligned}$$
□

Lemma 4.2. *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone homomorphism. Then holds*

$$q = Coker(h,\varphi) = ((h,\varphi)^{-1}(\nleq_T))^{-1} \cup (h,\varphi)^{-1}(\nleq_T).$$

Proof. The proof of this Lemma is analogous to the proof of Lemma 1 u [17], so we will omit it. □

For ease of writing, we will write $\not\prec := (h,\varphi)^{-1}(\nleq_T)$. So,

$$xay \not\prec ubv \iff (h,\varphi)(xay) \nleq_T (h,\varphi)(ubv),$$

$q = \sigma \cup \sigma^{-1}$ and $\not\prec \subseteq \nleq_S$ holds.

Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleq_S)$ into co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, \nleq_T)$. In the following theorem we describe the properties of the Γ-semigroup with apartness $S/(\rho, q)$ and the se-epimorphism (π, i) under new circumstances.

Theorem 4.3. *The structure $(S/(\rho,q), =_1, \neq_1, \npreceq)$ is an ordered Γ-semigroup with apartness under co-order '$\npreceq$', defined by the flowing way*

$$(\forall x, y \in S)(x\rho \npreceq y\rho \iff x \not\prec y),$$

and (π, i) is a reverse isotone se-epimorphism.

Proof. It suffices to prove that $\npreceq$ is a co-order on $(S/(\rho,q), =_1, \neq_1)$ and that (π, i) is a reverse isotone se-epimorphism.

Let $x, y \in S$ be arbitrary elements such that $x\rho \npreceq y\rho$. Then $x \not\prec y$. This means $(h,\varphi)(x) \nleq_T (h,\varphi)(y)$. Thus $x \nleq_S y$ because (h,φ) is a reverse isotone mapping. Thus, $\npreceq$ is a consistent relation on $S/(\rho, q)$.

Let $x, y, z \in S$ be arbitrary elements such that $x\rho \npreceq z\rho$. Then $x \not\prec z$. Thus $x \not\prec y \vee y \not\prec z$ by co-transitivity of $\not\prec$. Hence, $x\rho \npreceq y\rho \vee y\rho \npreceq z\rho$. So, the relation $\npreceq$ is a co-transitive relation on $S/(\rho, q)$.

Let $x, y \in S$ be arbitrary elements such that $x\rho \neq_1 y\rho$. Then $(x, y) \in q$. This means $(h,\varphi)(x) \neq_T (h,\varphi)(y)$. From here, we have

$$(h,\varphi)(x) \nleq_T (h,\varphi)(y) \ \vee \ (h,\varphi)(y) \nleq_T (h,\varphi)(x)$$

since $\nleq_T$ is a co-order relation. Therefore, $x\rho \npreceq y\rho \vee y\rho \npreceq x\rho$ which is shown that $\npreceq$ satisfies the condition of linearity.

Let $x, y, z \in S$ and $a \in \Gamma$ be arbitrary elements such that $(xaz)\rho \not\preceq (ybz)\rho$. Then $(xaz) \not\prec (yaz)$. This means $(h,\varphi)(xaz) \nleqslant_T (h,\varphi)(yaz)$, i.e. $h(x)\varphi(a)h(z) \nleqslant_T h(y)\varphi(a)h(z)$. Thus $h(x) \nleqslant_T h(y)$. So, $x\rho \not\preceq y\rho$ by definition of $\not\preceq$.

Let $x, y \in S$ be arbitrary elements such that $(\pi, i)(x) = x\rho \not\preceq y\rho = (\pi, i)(y)$. Then $x \not\prec y$. This means $(h,\varphi)(x) \nleqslant_T (h,\varphi)(y)$. Thus $x \nleqslant_S y$ since (h,φ) is a reverse isotone mapping. We have proved that (π, i) is a reverse isotone mapping also. □

Theorem 4.4. *Let $(h,\varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ into co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, \nleqslant_T)$. Then there exists a unique injective, embedding and surjective se-homomorphism (g,φ) from co-ordered Γ-semigroup with apartness $(S/(\rho,q), =_1, \neq_1, \not\preceq)$ onto co-ordered Λ-subsemigroup with apartness $((h,\varphi)(S), =_T, \neq_T, \nleqslant_T)$ such that $(h,\varphi) = (g,\varphi) \circ (\pi, i, \pi)$.*

Proof. According to Lemma 4.1, $(\pi, i) : S \longrightarrow S/(\rho,q)$ is a reverse isotone homomorphism. It suffices to prove that (g,φ) is a simultaneous isotone and reverse isotone mapping. This follows from the validity of the following equivalences

$$\begin{aligned} x\rho \not\preceq y\rho &\Longleftrightarrow x \not\prec y \\ &\Longleftrightarrow (h,\varphi)(x) \nleqslant_T (h,\varphi)(y) \\ &\Longleftrightarrow (g,\varphi)(x) \nleqslant_T (g,\varphi)(y). \end{aligned}$$

for arbitrary $x, y \in S$. □

Let us show that Γ-semigroup with apartness $[S, q]$ is ordered under the co-order relation $\not\preceq := \{(xq, yq) \in [S : q] \times [S; q] : x \not\prec y\}$ and that (ϑ, u) is a reverse isotone surjective mapping with respect to $\not\preceq$.

Theorem 4.5. *The structure $([S : q], =_2, \neq_2, \not\preceq)$ is a co-ordered Γ-semigroup and $(\vartheta, i) : S \longrightarrow [S : q]$ is a reverse isotone se-epimorphism.*

Proof. It has already been shown in Theorem 3.3 that the structure $[S, q]$ is a Γ-semigroup and $(\vartheta, i) : S \longrightarrow [S : q]$ is se-epimorphism.

We will show that $\not\preceq$ is a co-order relation on $[S : q]$, that is, we will show that $([S, q], \doteq_2, \neq_2, \not\preceq)$ is co-ordered Γ-semigroup.

Let $x, y \in S$ be such that $xq \not\preceq yq$. Then $x \not\prec y$, i.e. $(h,\varphi)(x) \nleqslant_T (h,\varphi)(y)$. Thus $(h,\varphi)(x) \neq_T (h,\varphi)(y)$ by consistency of $\nleqslant_T$. This means $(x, y) \in q$ and hence $xq \neq_2 yq$.

Let $x, t, z \in S$ such that $xq \not\preceq zq$. Then $x \not\prec z$. This means $(h,\varphi)(x) \nleqslant_T (h,\varphi)(z)$. Thus

$$(h,\varphi)(x) \nleqslant_T (h,\varphi)(y) \vee (h,\varphi)(y) \nleqslant_T (h,\varphi)(z)$$

by co-transitivity of $\nleqslant_T$. So, we have

$$xq \not\preceq yq \vee yq \not\preceq zq.$$

Let $x, y \in S$ be such that $xq \neq_2 yq$. Then $(h,\varphi)(x) \neq_T (h,\varphi)(y)$. Thus

$$(h,\varphi)(x) \nleqslant_T (h,\varphi)(y) \ \vee (h,\varphi)(y) \nleqslant_T (h,\varphi)(x)$$

by linearity of $\nleqslant_T$.

If $x, y, z \in S$ and $a \in \Gamma$ are such that $(xaz)q \not\preceq (yaz)q$, then we have

$$(h,\varphi)(xaz) \nleqslant_T (h,\varphi)(yaz),$$

i.e. we have

$$h(x)\varphi(a)h(z) \nleqslant_T h(y)\varphi(a)h(z).$$

Thus $h(x) \nleqslant_T h(y)$. This means $xq \npreceq yq$.

It remains to show that (ϑ, i) is a reverse isotone se-epimorphism. Let $xq, yq \in [S : q]$ be such that $(\vartheta, i)(x) \npreceq (\vartheta, i)(y)$. Then $xq \npreceq yq$. Thus $x \nprec y$ by definition of $\npreceq$. Hence $x \nleqslant_T y$ by Lemma 4.1. □

At the end of this paper, we show another form of the first theorem on isomorphisms between co-ordered Γ-semigroups which has no counterpart in the classical theory of ordered Γ-semigroups.

Theorem 4.6. *Let $(h, \varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ into co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, \nleqslant_T)$. Then there exists a unique injective, embedding and surjective se-homomorphism (f, φ) from co-ordered Γ-semigroup with apartness $([S : q], =_2, \neq_2, \npreceq)$ onto co-ordered Λ-subsemigroup with apartness $((h, \varphi)(S), \neq_T^{\triangleleft}, \neq_T, \nleqslant_T)$ of the co-ordered Λ-semigroup with apartness T such that*

$$(h, \varphi) = (f, \varphi) \circ (\vartheta, i, \vartheta) \text{ and } w_T = (\vartheta, i) \circ w_S.$$

Proof. The existence and required properties of the se-homomorphism (f, φ) are described in Theorem 3.5. It remains to prove that (f, φ) is isotone and reverse isotone homomorphism between co-ordered semigroups with apartness. Let $xq, yq \in [S : q]$ be arbitrary elements such that This statement is proved by the validity by the following sequent of equivalences. Let $x, y \in S$ be arbitrary elements. Then holds

$$\begin{aligned} xq \npreceq yq &\iff x \nprec y \\ &\iff (h, \varphi)(x) \nleqslant_T (h, \varphi)(y) \\ &\iff (f, \varphi)(x) \nleqslant_T (f, \varphi)(y). \end{aligned}$$ □

Of course, even in the case of co-ordered semigroups with apartness, a statement analogous to Proposition 3.4 can be proved.

Proposition 4.7. *Let $(h, \varphi) : S \longrightarrow T$ be a reverse isotone se-homomorphism from co-ordered Γ-semigroup with apartness $(S, =_S, \neq_S, \nleqslant_S)$ into co-ordered Λ-semigroup with apartness $(T, =_T, \neq_T, \nleqslant_T)$. There is a unique surjective and embedding reverse isotone se-homomorphism $\alpha : S/(\rho, q) \longrightarrow [S : q]$ from co-ordered Γ-semigroup with apartness $(S/(\rho, q) =_1, \neq_1, \npreceq)$ into co-ordered Γ-semigroup with apartness $([S : q], =_2, \neq_2, \npreceq)$ such that $w_{[S:q]} = \alpha \circ w_{S/(\rho,q)}$. In particular, $\alpha : S/(q^{\triangleleft}, q) \longrightarrow [S : q]$ is an se-isomorphism.*

Proof. The existence, uniqueness, and properties of the se-homomorphism α are described in Proposition 3.4. Let us show that α is a reverse isotone mapping. Let $x, y \in S$ be arbitrary elements such that $\alpha(x\rho) \npreceq \alpha(y\rho)$. Then $xq \npreceq yq$ and $x \nprec y$. Thus $(h, \varphi)(x) \nleqslant_T (h, \varphi)(y)$ by definition of $\nprec$. Hence $x\rho \npreceq y\rho$.

It remains to be shown that in a special case $\alpha : S/(q^{\triangleleft}, q) \longrightarrow [S, q]$, the mapping α is isotone and injective. Let $x, y \in S$ be such $\alpha(xq^{\triangleleft}) =_2 \alpha(yq^{\triangleleft})$ holds. Then $xq =_2 yq$. Thus $(x, y) \triangleleft q$. Tis means $xq^{\triangleleft} =_1 yq^{\triangleleft}$. On the other hand, let $x, y \in S$ be such that $xq^{\triangleleft} \npreceq yq^{\triangleleft}$. Then $x \nprec y$. Thus $(h, \varphi)(x) \nleqslant_T (h, \varphi)(y)$. Hence $xq \npreceq yq$. So, $\alpha(xq^{\triangleleft}) \npreceq \alpha(yq^{\triangleleft})$. □

5. Conclusion and Final Comments

Intuitionist logic, as a principled-philosophical and logical environment, enables the design of algebraic structures with apartness within the Bishop's constructive orientation, which is not just a different view of Algebra. A significant part of the contextual complexity of algebraic concepts and processes with them arises from the internal structure of the carriers of these structures. The carriers of algebraic structures with apartness relation are relational systems $(S, =, \neq)$, where '$\neq$' is a diversity relation on S self-existent and logically independent in relation to equality '$=$' but still firmly connected with it in the following sense:

$$= \circ \neq \subseteq \neq \text{ and } \neq \circ = \subseteq \neq .$$

The presence of this relation allows the analysis of the condition

$$(\forall u, v \in S)(P(u) \implies (u \neq v \vee P(v)))$$

for any predicate P existing over the elements of $(S, =, \neq)$. The same, the presence of this relation allows the analysis of the conditions

$$(\forall x, y, u, v \in S)((x, y) \in R \implies ((x, y) \neq (u, v) \vee (u, v) \in R))$$

for any relation R designed over $(S, =, \neq)$. In Classical logic, these conditions are equivalent to the extensionality of the predicate P and the relation R to equality (respectively), while in Intuitionist logic they are independent.

In this paper, se-homomorphisms between Γ -semigroups with apartness and se-homomorphisms between co-ordered Γ-semigroups with apartness are observed. In both cases, two forms of the First Theorem on Isomorphisms were presented: Theorem 3.2 and Theorem 3.5 in the first case, and Theorem 4.4 and Theorem 4.6 relating to co-ordered Γ-semigroups with apartness. Let us repeat once again that Theorem 3.5 and Theorem 4.6 are specifics of this principled-philosophical and logical orientation and they do not have their counterparts in the Classical Theory of Γ-semigroups.

It is estimated that the material presented in this paper may be a good basis for designing other isomorphism theorems between Γ-semigroups with apartness.

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