INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 1 PAGE 171–180 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1195178

An Investigation of Timelike Aminov Surface with respect to its Gauss Map in Minkowski Space-time

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

In this work, we handle timelike Aminov surfaces in $\mathbb{E}_1^4$ with respect to having pointwise one type Gauss map. First, we get the Laplacian of the Gauss map of this type of surface. Then, we obtain that there is no timelike Aminov surface having harmonic Gauss map and also pointwise one type Gauss map of the first kind in Minkowski $4-$ space. Further, we yield the conditions of having pointwise one type Gauss map of the second kind.

Keywords: *Aminov surface, Gauss map, Minkowski* $4-$*space.*
AMS Subject Classification (2020): *Primary: 53A05 ; Secondary: 53A10.*

1. Introduction

The Minkowski Space was described by the German mathematician Herman Minkowski (1864-1909) in 1907. In mathematical physics, the most suitable mathematical model for the relativity theory introduced by Einstein in 1905 is the Minkowski space-time. The $4-$dimensional Minkowski space (Minkowski space-time) model can be thought of as a $4-$dimensional manifold obtained by combining the general $3-$dimensions of space and $1-$dimension of time. Minkowski $4-$space is defined by Lorentzian metric as

$$g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

where $x = (x_1, .., x_4)$, $y = (y_1, .., y_4)$. In Minkowski space-time, a surface $M : F(s,t) : (s,t) \in U \ (U \subset \mathbb{E}^2)$ is known as a timelike surface if the metric g on M (induced metric) has index 1. Thus, four-dimensional Minkowski space can be decomposed into the tangent space and normal space of timelike surface M, at each point p as

$$\mathbb{E}_1^4 = T_pM \oplus T_p^{\perp}M.$$

Let F_1, F_2 be the orthonormal tangent vectors and N be the normal vector of M. By indicating the Levi-Civita connections with $\widetilde{\nabla}$ and ∇, we give the Gauss and Weingarten formulas as

$$\begin{aligned} \widetilde{\nabla}_{F_1}F_2 &= \nabla_{F_1}F_2 + h(F_1,F_2), \\ \widetilde{\nabla}_{F_1}N &= -A_NF_1 + D_{F_1}N, \end{aligned} \tag{1.1}$$

where D is the normal connection, A_N is the shape operator and h is the second fundamental tensor [7, 12].

If the considered surface is a timelike surface, then one of the first fundamental form coefficients (e or g) is negative definite, where the first fundamental form coefficients are $e = g(F_s, F_s)$, $f = g(F_s, F_t)$, $g = g(F_t, F_t)$. Hence, the normal frame field $\{N_1, N_2\}$ satisfies $g(N_1, N_1) = 1$, $g(N_2, N_2) = 1$ and $g(N_1, N_2) = 0$.

Received : *26–10–2022,* ***Accepted :*** *14–03–2023*
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Provided that the vectors F_s and F_t are orthogonal, the orthonormal tangent vectors can be written as

$$\begin{aligned} F_1 &= \frac{F_s}{\sqrt{|e|}}, \\ F_2 &= \frac{F_t}{\sqrt{g}}. \end{aligned} \tag{1.2}$$

Therefore, we write the second fundamental form with respect to F_1 and F_2 as;

$$\begin{aligned} h(F_1, F_1) &= h_{11}^1 N_1 + h_{11}^2 N_2, \\ h(F_1, F_2) &= h_{12}^1 N_1 + h_{12}^2 N_2, \\ h(F_2, F_2) &= h_{22}^1 N_1 + h_{22}^2 N_2, \end{aligned} \tag{1.3}$$

(see, [12]).

Whether a surface has pointwise one type Gauss map is important in categorizing surfaces. In many recent studies, especially in Euclidean spaces, the Laplacian of the Gauss map has been obtained and some analyzes have been made on surfaces [3, 4, 5, 13]. Regarding the Gauss map (G) of a surface, if the condition

$$\Delta G = \alpha \left(G + \overrightarrow{C} \right), \tag{1.4}$$

is hold, then the surface is said to have a pointwise one type Gauss map. Here, α is a nonzero differentiable function and $\overrightarrow{C}$ is a constant vector. In addition, if $\overrightarrow{C} = 0$ (or $\overrightarrow{C} \neq 0$), then the surface is said to have pointwise one type Gauss map of the first kind (or second kind) [10, 11].

In Euclidean spaces, the Laplacian of any differentiable function ψ defined on the surface M is known as

$$\Delta \psi = -\left(\widetilde{\nabla}_{F_i} \widetilde{\nabla}_{F_i} \psi - \widetilde{\nabla}_{\nabla_{F_i} F_i} \psi \right). \tag{1.5}$$

Also, let $\{e_1, e_2\}$ be tangent vector fields on the surface, and $\{e_3, .., e_n\}$ be normal vector fields, where $\{e_1, e_2, .., e_n\}$ is orthonormal frame. Then, Gauss map of M is defined by

$$G(p) = (e_1 \wedge e_2)(p).$$

In Minkowski spaces, with respect to these definitions, the Laplacian of the Gauss map is written by

$$\Delta G = -\varepsilon_i \left(\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} G - \widetilde{\nabla}_{\nabla_{e_i} e_i} G \right), \tag{1.6}$$

(see, [1, 6]). Here, $\varepsilon_i = g(e_i, e_i) = \pm 1$, where $\{e_1, ..e_i, ..e_n\}$ is the orthonormal frame in $\mathbb{E}_1^n$. In the study [6], the authors investigate Meridian surfaces in $\mathbb{E}_1^4$, which are also spacelike surfaces.

A special surface, named Aminov surface, is given by a Monge patch [2, 8]. Recently, the authors have studied this type of surface in $\mathbb{E}_1^4$ with hyperbolic angle. They characterized the flat, minimal and semiumbilical Aminov surfaces of hyperbolic type [9].

In this present study, we concentrate on timelike Aminov surface in $\mathbb{E}_1^4$ with its Gauss map. We obtain the Laplacian of the Gauss map of this type of surface. Then, we investigate the conditions of having harmonic and pointwise one type Gauss map of the first kind (second kind).

2. Timelike Aminov Surfaces in Minkowski Space-Time

Definition 2.1. Let $M : F(s,t)$ be a timelike surface in four-dimensional Minkowski space $\mathbb{E}_1^4$. If the parametrization of M is given by the Monge patch

$$F(s,t) = (r(s)\cosh t, r(s)\sinh t, s, t), \tag{2.1}$$

where $r(s)$ is a differentiable function, then this surface is congruent to a timelike Aminov surface with hyperbolic angle in E_1^4 [9].

Let M be a timelike Aminov surface in E_1^4. Then, tangent vectors are

$$\begin{aligned} F_s &= \left(r'(s)\cosh t, r'(s)\sinh t, 1, 0\right), \\ F_t &= \left(r(s)\sinh t, r(s)\coth t, 0, 1\right). \end{aligned}$$

Hence, the coefficients of the first fundamental form are

$$\begin{aligned} e &= 1-\left(r'\right)^2, \\ f &= 0, \\ g &= 1+r^2. \end{aligned} \tag{2.2}$$

Since our chosen surface is a timelike surface in $\mathbb{E}_1^4$, then $e = 1-\left(r'\right)^2$ is negative definite. Thus, we set for the later use; $W = \sqrt{f^2-eg} = \sqrt{(-e)\,g} = \sqrt{\left(\left(r'\right)^2-1\right)(1+r^2)}$.

The tangent vectors are orthogonal ($f=0$). Therefore, the orthonormal tangent vectors are

$$\begin{aligned} F_1 &= \frac{1}{\sqrt{(-e)}}F_s = \frac{1}{\sqrt{\left(r'\right)^2-1}}\left(r'(s)\cosh t, r'(s)\sinh t, 1, 0\right), \\ F_2 &= \frac{1}{\sqrt{g}}F_t = \frac{1}{\sqrt{1+r^2}}\left(r(s)\sinh t, r(s)\coth t, 0, 1\right). \end{aligned} \tag{2.3}$$

The normal vector fields yield as

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{A}}\left(1, 0, r'\cosh t, r\sinh t\right), \\ N_2 &= \frac{1}{\sqrt{A\,(-e)\,g}}\left(-B, A, r'g\sinh t, re\cosh t\right), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} A &= -1+r'^2\cosh^2 t+r^2\sinh^2 t, \\ B &= -\cosh t\sinh t\left(r'^2+r^2\right), \\ C &= 1+r'^2\sinh^2 t+r^2\cosh^2 t, \\ D &= AC-B^2 = W^2 = (-e)\,g. \end{aligned}$$

Moreover, with the help of Gauss and Weingarten formulas, we have

$$\begin{aligned} \widetilde{\nabla}_{F_1}F_1 &= -\lambda_1 N_1-\lambda_2 N_2, \\ \widetilde{\nabla}_{F_2}F_2 &= \lambda_3 F_1-\lambda_4 N_1-\lambda_5 N_2, \\ \widetilde{\nabla}_{F_2}F_1 &= \lambda_3 F_2-\lambda_6 N_1+\lambda_7 N_2, \\ \widetilde{\nabla}_{F_1}F_2 &= -\lambda_6 N_1+\lambda_7 N_2, \\ \widetilde{\nabla}_{F_1}N_1 &= -\lambda_1 F_1+\lambda_6 F_2+\lambda_8 N_2, \\ \widetilde{\nabla}_{F_1}N_2 &= -\lambda_2 F_1-\lambda_7 F_2-\lambda_8 N_1, \\ \widetilde{\nabla}_{F_2}N_1 &= -\lambda_6 F_1+\lambda_4 F_2+\lambda_9 N_2, \\ \widetilde{\nabla}_{F_2}N_2 &= \lambda_7 F_1+\lambda_5 F_2-\lambda_9 N_1. \end{aligned} \tag{2.5}$$

where the smooth functions λ_i $(i=1,..,9)$ are

$$\lambda_1 = \frac{r''\cosh t}{(-e)\sqrt{A}}, \quad \lambda_2 = \frac{r''g\sinh t}{(-e)\sqrt{A(-e)g}}, \quad \lambda_3 = \frac{r'r}{\sqrt{-eg}},$$

$$\lambda_4 = \frac{r\cosh t}{g\sqrt{A}}, \quad \lambda_5 = \frac{r\sinh t}{\sqrt{A(-e)g}}, \quad \lambda_6 = \frac{r'\sinh t}{\sqrt{A(-e)g}}, \tag{2.6}$$

$$\lambda_7=\frac{r'\cosh t}{g\sqrt{A}},\quad \lambda_8=\frac{r'\cosh t\sinh t\left(r''g+re\right)}{A(-e)\sqrt{g}},\quad \lambda_9=\frac{eg+A}{Ag\sqrt{-e}}.$$

Therefore, the tangent and normal components of the equation (2.5) are written as

$$\begin{array}{ll}\nabla_{F_1}F_1=0, & A_{N_1}F_1=\lambda_1F_1-\lambda_6F_2,\\ \nabla_{F_2}F_2=\lambda_3F_1, & A_{N_1}F_2=\lambda_6F_1-\lambda_4F_2,\\ \widetilde{\nabla}_{F_1}F_2=0, & A_{N_2}F_1=\lambda_2F_1+\lambda_7F_2,\\ \widetilde{\nabla}_{F_2}F_1=\lambda_3F_2, & A_{N_2}F_2=-\lambda_7F_1-\lambda_5F_2,\end{array}\tag{2.7}$$

and

$$\begin{array}{ll}h\left(F_1,F_1\right)=-\lambda_1N_1-\lambda_2N_2, & D_{F_1}N_1=\lambda_8N_2,\\ h\left(F_1,F_2\right)=-\lambda_6N_1+\lambda_7N_2, & D_{F_1}N_2=-\lambda_8N_1,\\ h\left(F_1,F_2\right)=-\lambda_4N_1-\lambda_5N_1, & D_{F_2}N_1=\lambda_9N_2,\\ & D_{F_2}N_2=-\lambda_9N_1.\end{array}\tag{2.8}$$

In addition, by the use of (2.8) and (2.6), we yield the coefficients h_{ij}^k;

$$\begin{array}{ll}h_{11}^1=\frac{-r''\cosh t}{(-e)\sqrt{A}}, & h_{11}^2=\frac{-r''g\sinh t}{(-e)\sqrt{A(-e)g}},\\ h_{12}^1=\frac{-r'\sinh t}{\sqrt{A(-e)g}}, & h_{12}^2=\frac{r'\cosh t}{g\sqrt{A}},\\ h_{22}^1=\frac{-r\cosh t}{g\sqrt{A}}, & h_{22}^2=\frac{-r\sinh t}{\sqrt{A(-e)g}}.\end{array}\tag{2.9}$$

So, we have the following lemma:

Lemma 2.1. *Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ given by (2.1). Then, the shape operator matrices are given by*

$$A_{N_1}=\begin{pmatrix}\frac{-r''\cosh t}{(-e)\sqrt{A}} & \frac{-r'\sinh t}{\sqrt{A(-e)g}}\\ \frac{-r'\sinh t}{\sqrt{A(-e)g}} & \frac{-r\cosh t}{g\sqrt{A}}\end{pmatrix},\tag{2.10}$$

$$A_{N_2}=\begin{pmatrix}\frac{-r''g\sinh t}{(-e)\sqrt{A(-e)g}} & \frac{r'\cosh t}{g\sqrt{A}}\\ \frac{r'\cosh t}{g\sqrt{A}} & \frac{-r\sinh t}{\sqrt{A(-e)g}}\end{pmatrix}.$$

Theorem 2.1. *[9]Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ given by the parametrization (2.1). Then, Gaussian curvature and the mean curvature vector are*

$$K=\frac{r''rg+\left(r'\right)^2e}{W^4}\tag{2.11}$$

and

$$H=\frac{\cosh t\left(r''g+re\right)}{2\sqrt{A}W^2}N_1+\frac{\sinh t\left(r''g+re\right)}{2\left(-e\right)\sqrt{A}W}N_2,\tag{2.12}$$

respectively.

3. An Investigation of Timelike Aminov Surface relating to its Gauss map in $\mathbb{E}_1^4$

Let $M:F\left(s,t\right)$ be a timelike surface in four-dimensional Minkowski space and F_1, F_2 are orthonormal tangent vectors, and N_1, N_2 are normal vector fields. Then, the Gauss map of the surface is

$$G=F_1\wedge F_2.$$

Since M is timelike, then we rewrite the equation (1.6) as;

$$\Delta G=\left(\widetilde{\nabla}_{F_1}\widetilde{\nabla}_{F_1}G-\widetilde{\nabla}_{\nabla_{F_1}F_1}G\right)-\left(\widetilde{\nabla}_{F_2}\widetilde{\nabla}_{F_2}G-\widetilde{\nabla}_{\nabla_{F_2}F_2}G\right).\tag{3.1}$$

Theorem 3.1. *Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ given by (2.1). Then, the Laplacian of the Gauss map of M is given by*

$$\begin{aligned} \Delta G &= \left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_4)^2 + (\lambda_5)^2 - 2(\lambda_6)^2 - 2(\lambda_7)^2\right) F_1 \wedge F_2 \\ &\quad + (-F_1[\lambda_6] + F_2[\lambda_4] - \lambda_7\lambda_8 - \lambda_5\lambda_9 - 2\lambda_3\lambda_6) F_1 \wedge N_1 \\ &\quad + (F_1[\lambda_7] + F_2[\lambda_5] - \lambda_6\lambda_8 + \lambda_4\lambda_9 + 2\lambda_3\lambda_7) F_1 \wedge N_2 \\ &\quad + (F_1[\lambda_1] - F_2[\lambda_6] - \lambda_2\lambda_8 + \lambda_3\lambda_4 - \lambda_7\lambda_9 + \lambda_1\lambda_3) F_2 \wedge N_1 \\ &\quad + (F_1[\lambda_2] + F_2[\lambda_7] + \lambda_1\lambda_8 - \lambda_6\lambda_9 + \lambda_3\lambda_5 + \lambda_2\lambda_3) F_2 \wedge N_2 \\ &\quad - 2(\lambda_7(\lambda_1 + \lambda_4) + \lambda_6(\lambda_2 + \lambda_5)) N_1 \wedge N_2, \end{aligned} \tag{3.2}$$

where $F_i[\lambda_j]$ are congruent to directional derivatives according to F_i and the functions λ_j $(j = 1, .., 9)$ are given by (2.6).

Proof. Using the components of Weingarten and Gauss formulas, the derivatives $\widetilde{\nabla}_{F_i}\widetilde{\nabla}_{F_i}G$ and $\widetilde{\nabla}_{\nabla_{F_i}F_i}G$ $(i = 1, 2)$ are obtained as

$$\begin{aligned} \widetilde{\nabla}_{F_1}\widetilde{\nabla}_{F_1}G &= \left((\lambda_1)^2 + (\lambda_2)^2 - (\lambda_6)^2 - (\lambda_7)^2\right) F_1 \wedge F_2 \\ &\quad + (-F_1[\lambda_6] - \lambda_7\lambda_8) F_1 \wedge N_1 + (F_1[\lambda_7] - \lambda_6\lambda_8) F_1 \wedge N_2 \\ &\quad + (F_1[\lambda_1] - \lambda_2\lambda_8) F_2 \wedge N_1 + (F_1[\lambda_2] + \lambda_1\lambda_8) F_2 \wedge N_2 \\ &\quad - 2(\lambda_1\lambda_7 + \lambda_2\lambda_6) N_1 \wedge N_2, \end{aligned}$$

$$\begin{aligned} \widetilde{\nabla}_{F_2}\widetilde{\nabla}_{F_2}G &= \left((\lambda_6)^2 + (\lambda_7)^2 - (\lambda_4)^2 - (\lambda_5)^2\right) F_1 \wedge F_2 \\ &\quad + (-F_2[\lambda_4] + \lambda_3\lambda_6 + \lambda_5\lambda_9) F_1 \wedge N_1 \\ &\quad + (-F_2[\lambda_5] - \lambda_3\lambda_7 - \lambda_4\lambda_9) F_1 \wedge N_2 \\ &\quad + (F_2[\lambda_6] - \lambda_3\lambda_4 + \lambda_7\lambda_9) F_2 \wedge N_1 \\ &\quad + (-F_2[\lambda_7] - \lambda_3\lambda_5 + \lambda_6\lambda_9) F_2 \wedge N_2 \\ &\quad + 2(\lambda_4\lambda_7 + \lambda_5\lambda_6) N_1 \wedge N_2, \end{aligned}$$

$$\widetilde{\nabla}_{\nabla_{F_1}F_1}G = 0,$$

$$\widetilde{\nabla}_{\nabla_{F_2}F_2}G = (-\lambda_3\lambda_6) F_1 \wedge N_1 + (\lambda_3\lambda_7) F_1 \wedge N_2 + (\lambda_1\lambda_3) F_2 \wedge N_1 + (\lambda_2\lambda_3) F_2 \wedge N_2.$$

Then, put these derivatives into the equation (3.1), we get the desired result.

Theorem 3.2. *Timelike Aminov surfaces given by the parameterization (2.1) cannot have harmonic Gauss map in Minkowski space-time.*

□

Proof. Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ given by (2.1). We rewrite the Laplacian of the Gauss map as

$$\Delta G = a_1(F_1 \wedge F_2) + a_2(F_1 \wedge N_1) + a_3(F_1 \wedge N_2) + a_4(F_2 \wedge N_1) + a_5(F_2 \wedge N_2) + a_6(N_1 \wedge N_2).$$

where a_i (i=1,..6) are indicated in (3.2). Suppose M has harmonic Gauss map, then $\Delta G = 0$, i.e $a_i = 0$. By substituting the functions λ_i into

$$a_6 = -2(\lambda_7(\lambda_1 + \lambda_4) + \lambda_6(\lambda_2 + \lambda_5)) = 0,$$

and

$$a_1 = \left((\lambda_1)^2 + (\lambda_2)^2 + (\lambda_4)^2 + (\lambda_5)^2 - 2(\lambda_6)^2 - 2(\lambda_7)^2\right) = 0,$$

we get

$$\frac{r'(r''g - re)}{(-e)^2 g^2} = 0, \tag{3.3}$$

$$\frac{(r'')^2 g^2 + \left(r^2 - 2r'^2\right)(-e)^2}{(-e)^3 g^2} = 0, \tag{3.4}$$

respectively.
If $r'(s) = 0$ in (3.3), then equation (3.4) is not satisfied. If $r''g = re$, from (3.4) we obtain

$$r'^2 = r^2. \tag{3.5}$$

Then, in (3.3) and (3.4), taking $e = 1 - r'^2$, $g = 1 + r^2$ and $r'^2 = r^2$, we yield

$$r(s) = 0.$$

This contradicts the regularity of the surface and completes the proof. □

Theorem 3.3. *Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ given by the parametrization (2.1). Then, M has pointwise one type Gauss map of the first kind if and only if*

$$\begin{aligned}
\frac{(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_4)^2 + (\lambda_5)^2 - 2(\lambda_6)^2 - 2(\lambda_7)^2}{\alpha} - 1 &= 0, \\
\frac{-F_1[\lambda_6] + F_2[\lambda_4] - \lambda_7\lambda_8 - \lambda_5\lambda_9 - 2\lambda_3\lambda_6}{\alpha} &= 0, \\
\frac{F_1[\lambda_7] + F_2[\lambda_5] - \lambda_6\lambda_8 + \lambda_4\lambda_9 + 2\lambda_3\lambda_7}{\alpha} &= 0, \\
\frac{F_1[\lambda_1] - F_2[\lambda_6] - \lambda_2\lambda_8 + \lambda_3\lambda_4 - \lambda_7\lambda_9 + \lambda_1\lambda_3}{\alpha} &= 0, \\
\frac{F_1[\lambda_2] + F_2[\lambda_7] + \lambda_1\lambda_8 - \lambda_6\lambda_9 + \lambda_3\lambda_5 + \lambda_2\lambda_3}{\alpha} &= 0, \\
\frac{-2\lambda_7}{\alpha}(\lambda_1 + \lambda_4) + \frac{-2\lambda_6}{\alpha}(\lambda_2 + \lambda_5) &= 0,
\end{aligned} \tag{3.6}$$

where α is a non-zero differentiable function.

Proof. Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ which has pointwise one type Gauss map. By using (1.4) and (3.2), we can write

$$\begin{aligned}
\alpha + \alpha g\left(\overrightarrow{C}, F_1 \wedge F_2\right) &= (\lambda_1)^2 + (\lambda_2)^2 + (\lambda_4)^2 + (\lambda_5)^2 - 2(\lambda_6)^2 - 2(\lambda_7)^2, \\
\alpha g\left(\overrightarrow{C}, F_1 \wedge N_1\right) &= -F_1[\lambda_6] + F_2[\lambda_4] - \lambda_7\lambda_8 - \lambda_5\lambda_9 - 2\lambda_3\lambda_6, \\
\alpha g\left(\overrightarrow{C}, F_1 \wedge N_2\right) &= F_1[\lambda_7] + F_2[\lambda_5] - \lambda_6\lambda_8 + \lambda_4\lambda_9 + 2\lambda_3\lambda_7, \\
\alpha g\left(\overrightarrow{C}, F_2 \wedge N_1\right) &= F_1[\lambda_1] - F_2[\lambda_6] - \lambda_2\lambda_8 + \lambda_3\lambda_4 - \lambda_7\lambda_9 + \lambda_1\lambda_3, \\
\alpha g\left(\overrightarrow{C}, F_2 \wedge N_2\right) &= F_1[\lambda_2] + F_2[\lambda_7] + \lambda_1\lambda_8 - \lambda_6\lambda_9 + \lambda_3\lambda_5 + \lambda_2\lambda_3, \\
\alpha g\left(\overrightarrow{C}, N_1 \wedge N_2\right) &= -2\left(\lambda_7(\lambda_1 + \lambda_4) + \lambda_6(\lambda_2 + \lambda_5)\right),
\end{aligned}$$

where α is non-zero differentiable function. With the help of the equation (1.4), the vector $\overrightarrow{C}$ can be written by

$$\begin{aligned}
\overrightarrow{C} = {} & K_1 F_1 \wedge F_2 + K_2 F_1 \wedge N_1 + K_3 F_1 \wedge N_2 \\
& + K_4 F_2 \wedge N_1 + K_5 F_2 \wedge N_2 + K_6 N_1 \wedge N_2,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned} K_1(s,t) &= \frac{(\lambda_1)^2+(\lambda_2)^2+(\lambda_4)^2+(\lambda_5)^2-2(\lambda_6)^2-2(\lambda_7)^2}{\alpha}-1, \\ K_2(s,t) &= \frac{-F_1[\lambda_6]+F_2[\lambda_4]-\lambda_7\lambda_8-\lambda_5\lambda_9-2\lambda_3\lambda_6}{\alpha}, \\ K_3(s,t) &= \frac{F_1[\lambda_7]+F_2[\lambda_5]-\lambda_6\lambda_8+\lambda_4\lambda_9+2\lambda_3\lambda_7}{\alpha}, \\ K_4(s,t) &= \frac{F_1[\lambda_1]-F_2[\lambda_6]-\lambda_2\lambda_8+\lambda_3\lambda_4-\lambda_7\lambda_9+\lambda_1\lambda_3}{\alpha}, \\ K_5(s,t) &= \frac{F_1[\lambda_2]+F_2[\lambda_7]+\lambda_1\lambda_8-\lambda_6\lambda_9+\lambda_3\lambda_5+\lambda_2\lambda_3}{\alpha}, \\ K_6(s,t) &= \frac{-2\lambda_7}{\alpha}(\lambda_1+\lambda_4)+\frac{-2\lambda_6}{\alpha}(\lambda_2+\lambda_5) \end{aligned} \tag{3.8}$$

are smooth functions. If M has pointwise one type Gauss map of first kind, then $C=0$ in the equation (2.1). With the help of (3.7) and (3.8), we get the result. □

Corollary 3.1. *Timelike Aminov surfaces given by (2.1) cannot have pointwise one type Gauss map of the first kind in* $\mathbb{E}^4_1$.

Proof. Assume that an Aminov surface M has pointwise one type Gauss map of the first kind in $\mathbb{E}^4_1$. Then, the equation system (3.6) is hold. By the use of the last equation in (3.6), we have

$$\frac{-2r'(r''g-re)}{\alpha(-e)^2g^2}=0.$$

Therefore, two cases ensue: $r'(s)=0$ or $r''(s)g-r(s)e=0$. By the use of the first case or second case (approving $r''g=re$) with the second equality of (3.6), we get

$$\frac{-r\sinh t}{(-e)\,g\sqrt{Ag}}=0.$$

It means that $r(s)=0$, which contradicts the regularity of the parametrization (2.1). Hence, it completes the proof. □

Theorem 3.4. *Let M be a timelike Aminov surface in $\mathbb{E}^4_1$ given by the parametrization (2.1). Then, the necessary and sufficient conditions for M to have pointwise one type Gauss map of the second kind are*

$$\begin{aligned} F_1[K_1]+K_2\lambda_6-K_3\lambda_7+K_4\lambda_1+K_5\lambda_2 &= 0, \\ F_1[K_2]-K_1\lambda_6-K_3\lambda_8+K_6\lambda_2 &= 0, \\ F_1[K_3]+K_1\lambda_7+K_2\lambda_8-K_6\lambda_1 &= 0, \\ F_1[K_4]+K_1\lambda_1-K_5\lambda_8+K_6\lambda_7 &= 0, \\ F_1[K_5]+K_1\lambda_2+K_4\lambda_8+K_6\lambda_6 &= 0, \\ F_1[K_6]+K_2\lambda_2-K_3\lambda_1-K_4\lambda_7-K_5\lambda_6 &= 0, \end{aligned}$$

and

$$\begin{aligned} F_2[K_1]+K_2\lambda_4+K_3\lambda_5+K_4\lambda_6-K_5\lambda_7 &= 0, \\ F_2[K_2]-K_1\lambda_4-K_3\lambda_9+K_4\lambda_3-K_6\lambda_7 &= 0, \\ F_2[K_3]-K_1\lambda_5+K_2\lambda_9+K_5\lambda_3-K_6\lambda_6 &= 0, \\ F_2[K_4]+K_1\lambda_6+K_2\lambda_3-K_5\lambda_9-K_6\lambda_5 &= 0, \\ F_2[K_5]-K_1\lambda_7+K_3\lambda_3+K_4\lambda_9+K_6\lambda_4 &= 0, \\ F_2[K_6]-K_2\lambda_7-K_3\lambda_6+K_4\lambda_5-K_5\lambda_4 &= 0, \end{aligned}$$

where $F_i[K_j]$ are congruent to directional derivatives according to F_i.

Proof. Let M be a timelike Aminov surface in $\mathbb{E}_1^4$ which has pointwise $1-$type Gauss map of the second kind. It means that $\overrightarrow{C}$ is constant in (1.4). With the help of the equation (3.7), the derivatives of the vector $\overrightarrow{C}$ are obtained as

$$\begin{aligned}\widetilde{\nabla}_{F_1} C &= (F_1[K_1] + K_2\lambda_6 - K_3\lambda_7 + K_4\lambda_1 + K_5\lambda_2) F_1 \wedge F_2 \\ &+ (F_1[K_2] - K_1\lambda_6 - K_3\lambda_8 + K_6\lambda_2) F_1 \wedge N_1 \\ &+ (F_1[K_3] + K_1\lambda_7 + K_2\lambda_8 - K_6\lambda_1) F_1 \wedge N_2 \\ &+ (F_1[K_4] + K_1\lambda_1 - K_5\lambda_8 + K_6\lambda_7) F_2 \wedge N_1 \\ &+ (F_1[K_5] + K_1\lambda_2 + K_4\lambda_8 + K_6\lambda_6) F_2 \wedge N_2 \\ &+ (F_1[K_6] + K_2\lambda_2 - K_3\lambda_1 - K_4\lambda_7 - K_5\lambda_6) N_1 \wedge N_2\end{aligned}$$

and

$$\begin{aligned}\widetilde{\nabla}_{F_2} C &= (F_2[K_1] + K_2\lambda_4 + K_3\lambda_5 + K_4\lambda_6 - K_5\lambda_7) F_1 \wedge F_2 \\ &+ (F_2[K_2] - K_1\lambda_4 - K_3\lambda_9 + K_4\lambda_3 - K_6\lambda_7) F_1 \wedge N_1 \\ &+ (F_2[K_3] - K_1\lambda_5 + K_2\lambda_9 + K_5\lambda_3 - K_6\lambda_6) F_1 \wedge N_2 \\ &+ (F_2[K_4] + K_1\lambda_6 + K_2\lambda_3 - K_5\lambda_9 - K_6\lambda_5) F_2 \wedge N_1 \\ &+ (F_2[K_5] - K_1\lambda_7 + K_3\lambda_3 + K_4\lambda_9 + K_6\lambda_4) F_2 \wedge N_2 \\ &+ (F_2[K_6] - K_2\lambda_7 - K_3\lambda_6 + K_4\lambda_5 - K_5\lambda_4) N_1 \wedge N_2\end{aligned}$$

Since $\overrightarrow{C}$ is constant, the derivatives vanish. It completes the proof. □

Example 3.1. Suppose that a surface M is given by the parameterization

$$M : F(s,t) = (2s\cosh t, 2s\sinh t, s, t),$$

then it is congruent to a timelike Aminov surface in Minkowski space-time. Having pointwise one type Gauss map of second kind for this surface means that the condition $\Delta G = \alpha G + \alpha C$ is satisfied, and the Laplacian of the Gauss map of M is calculated by

$$\begin{aligned}\Delta G &= \left(\frac{4(s^2-2)}{Ag}\left(\frac{\cosh^2 t}{g} + \frac{\sinh^2 t}{3}\right)\right) F_1 \wedge F_2 \\ &+ \left(-\frac{2\sinh t}{3}\frac{\partial}{\partial s}\left(\frac{1}{\sqrt{Ag}}\right) + \frac{2s}{g\sqrt{g}}\frac{\partial}{\partial t}\left(\frac{\cosh t}{\sqrt{A}}\right) + \frac{8s\cosh^2 t\sinh t}{Ag\sqrt{Ag}} - \frac{8s\sinh t\left(\sinh^2 t\left(1+s^2\right) - 3s^2\right)}{3Ag\sqrt{Ag}} - \frac{16s\sinh t}{3g\sqrt{Ag}}\right) F_1 \wedge N_1 \\ &+ \left(\frac{2\cosh t}{\sqrt{3}}\frac{\partial}{\partial s}\left(\frac{1}{g\sqrt{A}}\right) + \frac{2s}{\sqrt{3}g}\frac{\partial}{\partial t}\left(\frac{\sinh t}{\sqrt{A}}\right) + \frac{8s\cosh t\sinh^2 t}{Ag\sqrt{3A}} + \frac{8s\cosh t\left(\sinh^2 t\left(1+s^2\right) - 3s^2\right)}{Ag^2\sqrt{3A}} + \frac{16s\cosh t}{\sqrt{3}g^2}\right) F_1 \wedge N_2 \\ &+ \left(-\frac{2}{\sqrt{3}g}\frac{\partial}{\partial t}\left(\frac{\sinh t}{\sqrt{A}}\right) + \frac{8s^2\cosh t}{\sqrt{3A}g^2} - \frac{8\cosh t\left(\sinh^2 t\left(1+s^2\right) - 3s^2\right)}{Ag^2\sqrt{3A}}\right) F_2 \wedge N_1 \\ &+ \left(\frac{2}{g\sqrt{g}}\frac{\partial}{\partial t}\left(\frac{\cosh t}{\sqrt{A}}\right) + \frac{8s^2\sinh t}{3g\sqrt{Ag}} - \frac{8\sinh t\left(\sinh^2 t\left(1+s^2\right) - 3s^2\right)}{3Ag\sqrt{Ag}}\right) F_2 \wedge N_2 \\ &- \left(\frac{8s}{3g^2}\right) N_1 \wedge N_2\end{aligned}$$

where $A = -1 + 4\cosh^2 t + 4s^2\sinh^2 t$ and $g = 1 + 4s^2$. In addition, we can plot the 3d-projection of this surface with Mapple command plot3d([2*s*cosht,2*s*sinht,s+t], s=-2*pi..2*pi, t=-2*pi..2*pi):

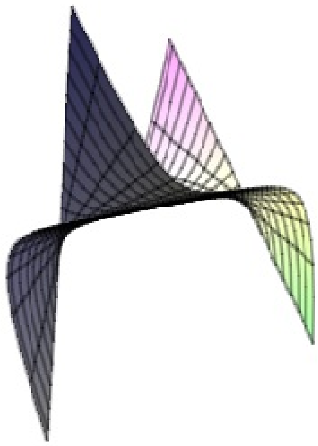

Figure 1. Timelike Aminov Surface

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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