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CAYLEY SUBSPACE SUM GRAPH OF VECTOR SPACES

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Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $S(\mathbb{V})$ be the set of all subspaces of $\mathbb{V}$ and $\mathbb{A} \subseteq S^*(\mathbb{V}) = S(\mathbb{V})\backslash\{0\}$. In this paper, we define the Cayley subspace sum graph of $\mathbb{V}$, denoted by $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$, as the simple undirected graph with vertex set $S^*(\mathbb{V})$ and two distinct vertices X and Y are adjacent if $X+Z = Y$ or $Y+Z = X$ for some $Z \in \mathbb{A}$. Having defined the Cayley subspace sum graph, we study about the connectedness, diameter and girth of several classes of Cayley subspace sum graphs $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ for a finite dimensional vector space $\mathbb{V}$ and $\mathbb{A} \subseteq S^*(\mathbb{V}) = S(\mathbb{V})\backslash\{0\}$.

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1. Introduction

In recent years, lot of attention has been given for construction of graphs from algebraic structures. In particular, intersection graphs associated with subspaces of a vector space have been studied by many authors. The subspace inclusion graph of a vector space is introduced and studied by Das [9] and, further properties like Hamiltonian, Eulerian, planar, toroidal, independence number and domination number of the subspace inclusion graph have been studied in [11,13]. Also Das [11] posed four conjectures out of which two of them are solved by Wong [20] and remaining two are proved by Peter J. Cameron et al. [7]. Various other graphs associated with vector spaces like nonzero component union graph and nonzero component graph of finite dimensional vector spaces have been introduced and studied in [10,8,16,19]. The Cayley graph is a powerful tool to connect the algebra and graph theory and there are worthwhile applications for Cayley graphs like routing networks in parallel computing. The Cayley graph of finite groups and rings are well studied in the literature and one can see [1,2,4,12,14,17,15]. The both directed and undirected Cayley sum graph of ideals of commutative rings is defined in [3] and some basic properties such as connectivity, girth, clique number, planar and outer planar are studied. Later Tamizh Chelvam et al. [18] studied about connectedness, Eulerian, Hamiltonian and toroidal properties of Cayley sum graph of ideals of commutative rings. Any interested reader can refer the monograph [5] for complete literature on graphs from rings.

2. Preliminaries

Throughout this paper, $\mathbb{V}$ is a finite dimensional vector space of dimension n over the finite field $\mathbb{F}$ containing q elements and $\mathfrak{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of $\mathbb{V}$. In this regard, $\mathfrak{B}(\mathbb{W})$ denotes a basis of a subspace W of $\mathbb{V}$ in general $\mathfrak{B}(\mathbb{V})$ denotes a basis of $\mathbb{V}$. Let $S(\mathbb{V})$ be the set of all subspaces of $\mathbb{V}$ and let $\mathbb{A}$ be a subset of $S^*(\mathbb{V}) = S(\mathbb{V})\backslash\{0\}$. The *Cayley subspace sum graph* of $\mathbb{V}$ with respect to $\mathbb{A}$ is the simple undirected graph with vertex set $S^*(\mathbb{V})$ and two distinct vertices X and Y are adjacent if and only if $X + Z = Y$ or $Y + Z = X$ for some $Z \in \mathbb{A}$ and the same is denoted as $Cay(S^*(\mathbb{V}), \mathbb{A})$. Any $k(\leq n)$ dimensional subspace W of $\mathbb{V}$ spanned by $\{\beta_1, \dots, \beta_k\}$ is written as $\langle\beta_1, \dots, \beta_k\rangle$. When $\dim(V) = n$, the number of distinct subspaces of $\mathbb{V}$ with $k \geq 1$ dimension is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1)\dots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\dots(q - 1)}.$$

Thus V has $\sum\limits_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q$ distinct non-zero subspaces and so the Cayley subspace sum graph $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ contains $\sum\limits_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q$ vertices.

Now, we recall some definitions and notations on graphs. By a graph $G = (V, E)$, we mean a simple undirected graph with non-empty vertex set V and edge set E. The number of elements in V is called the order n of G and the number of elements in E is called the size m of G. A graph G is said to be complete if any two distinct vertices in G are adjacent and the complete graph of order n is denoted by K_n. A graph G is said to be bipartite if the vertex V can be partitioned into two disjoint subsets with no pair of vertices in one subset is adjacent. A star graph is a bipartite graph with any one of the subsets in the bipartite graph containing a single vertex and the same is called as the center of the star. A graph G is n-partite if the vertex V can be partitioned into n disjoint subsets with no pair of vertices in one subset is adjacent.

A walk in a graph G is a finite non-null sequence $W = v_0e_1v_1e_2\dots e_kv_k$, whose terms are alternatively vertices and edges, such that, for $1 \leq i \leq k$ and ends of

e_i are v_{i-1} and v_i. The walk W is said to be a trial if the edges $e_1, \ldots, e_k$ of the walk W are distinct. Further if vertices $v_0, v_1, \ldots, v_k$ are distinct, then W is called a path. A cycle is a path with starting and terminating vertex are same. A graph is said to be Hamiltonian if it contains a cycle containing all the vertices of G. A graph G is said to be connected if there exists a path between every pair of distinct vertices in G. The diameter of a connected graph is the supremum of the shortest distance between pairs of vertices in G and is denoted by $\mathrm{diam}(G)$. The girth of G is defined as length of the shortest cycle in G and is denoted by $\mathrm{gr}(G)$. We take $\mathrm{gr}(G) = \infty$ if G contains no cycles. A complete subgraph of a graph G is called a clique. The clique number of G, written as $\omega(G)$, is the maximum size of a clique in G. A subset D of V is called dominating set if any vertex in $V \setminus D$ is adjacent with at least one vertex in D. The minimum cardinality of D is called domination number and it is denoted by $\gamma(G)$. A planar graph is a graph that can be embedded in the plane and the genus of planar graphs is zero. For undefined terms in graph theory, we refer [6].

3. Cayley subspace sum graph

Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$, $S(\mathbb{V})$ be the set of all subspaces of $\mathbb{V}$ and $\mathbb{A} \subseteq S^*(\mathbb{V}) = S(\mathbb{V})\backslash\{0\}$. The Cayley subspace sum graph $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A}) = (V, E)$ is the simple undirected graph with vertex set $S^*(\mathbb{V})$ and two distinct vertices X and Y are adjacent $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ if $X + Z = Y$ or $Y + Z = X$ for some $Z \in \mathbb{A}$. In this section, we observe some properties of $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$.

Theorem 3.1. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{W_1, \ldots, W_k\} \subseteq S^*(\mathbb{V})$. Then $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected if and only if $\bigcup\limits_{i=1}^{k} \mathfrak{B}(W_i) = \mathfrak{B}(\mathbb{V})$ where $\mathfrak{B}(W_i)$ is a basis of the subspace W_i of $\mathbb{V}$.*

Proof. Let $Cay(S^*(\mathbb{V}), \mathbb{A})$ be connected. Without loss of generality one can assume that $\mathfrak{B}(W_i) \subseteq \mathfrak{B}(\mathbb{V})$. Suppose $\bigcup\limits_{i=1}^{k} \mathfrak{B}(W_i) \subset \mathfrak{B}(\mathbb{V})$. Then there exists at least one vector $\beta \in \mathbb{V}$ such that $\bigcup\limits_{i=1}^{k} \mathfrak{B}(W_i) \bigcup \{\beta\} \subseteq \mathfrak{B}(\mathbb{V})$. Let $V_1 = \{X \in S^*(\mathbb{V}) : \beta_i$ and β are linearly independent for all $\beta_i \in \mathfrak{B}(X)\}$ and $V_2 = S^*(\mathbb{V}) \setminus V_1$. For $X \in V_1$, we have $X + W_i = X' \in V_1$ for all $W_i \in A, X \in V_1$ and $Y + W_i = Y' \in V_2$ for all $W_i \in A, Y \in V_2$. This implies that two vertices in different partitions V_1 and V_2

of $S^*(\mathbb{V})$ are not connected by a path, which is a contradiction to the assumption that $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected. Hence $\bigcup_{i=1}^{k} \mathfrak{B}(W_i) = \mathfrak{B}(\mathbb{V})$.

Conversely, assume that $\bigcup_{i=1}^{k} \mathfrak{B}(W_i) = \mathfrak{B}(\mathbb{V})$ where $\mathbb{A} = \{W_1, \ldots, W_k\} \subseteq S^*(\mathbb{V})$. For $X \in S^*(\mathbb{V})$, there exists a path $P = X - (X + W_1) - (X + W_1 + W_2) - \cdots - (X + \sum_{i=1}^{\ell} W_i) - \cdots - (X + \sum_{i=1}^{k-1} W_i) - \mathbb{V}$ between X and $\mathbb{V}$. Hence, every vertex $X \in S^*(\mathbb{V})$ is connected with $\mathbb{V}$ so $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected. □

Theorem 3.2. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field of order q with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{W_1, \ldots, W_k\} \subseteq S^*(\mathbb{V})$. If $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected then, it is not a path or cycle.*

Proof. Let $Cay(S^*(\mathbb{V}), \mathbb{A})$ be connected and $\mathbb{V}_{n-1} \subset S^*(\mathbb{V})$ be the set of all $n-1$ dimensional subspaces of $\mathbb{V}$. Then

$$|\mathbb{V}_{n-1}| = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q = \frac{q^n - 1}{q - 1} \geq 3.$$

We claim that every vertex in $\mathbb{V}_{n-1}$ is adjacent to $\mathbb{V}$. If not, there exists $X \in V_{n-1}$ which is not adjacent to $\mathbb{V}$. This in turn implies that there exists $\beta \in \mathbb{V}$ such that $\mathfrak{B}(X) \cup \{\beta\} = \mathfrak{B}(\mathbb{V})$ and β is linearly independent with all the elements in $\mathfrak{B}(W_i)$ for all $W_i \in \mathbb{A}, 1 \leq i \leq k$. In this case $\bigcup_{i=1}^{k} \mathfrak{B}(W_i) \subset \mathfrak{B}(\mathbb{V})$, which is a contradiction to Theorem 3.1. Hence $\deg(\mathbb{V}) \geq |\mathbb{V}_{n-1}| = 3$ and so $Cay(S^*(\mathbb{V}), \mathbb{A})$ can never be a path or cycle. □

Lemma 3.3. *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$. Then $Cay(S^*(\mathbb{V}), \mathbb{V})$ is a star graph.*

Proof. Let $W \in S^*(\mathbb{V})$ be a non-zero subspace of $\mathbb{V}$. Then $W + \mathbb{V} = \mathbb{V}$, i.e, $\mathbb{V}$ is adjacent to all $W \in S^*(\mathbb{V})$. Hence Cay$(S^*(\mathbb{V}), \mathbb{V})$ is a star graph with $\mathbb{V}$ as the central vertex. □

Now, we observe certain instances where the Cayley subspace sum graph is connected and they are consequences of Theorem 3.1.

Corollary 3.4. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field. Then $Cay(S^*(\mathbb{V}), \mathbb{V})$ is connected and diam($Cay(S^*(\mathbb{V}), \mathbb{V}) = 2$.*

Corollary 3.5. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field. Then $Cay(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is connected and diam($Cay(S^*(\mathbb{V}), S^*(\mathbb{V})) = 2$.*

Corollary 3.6. *Let p be a prime number and $k \geq 1$ be an integer. Let $\mathbb{V}$ be a two dimensional vector space over a finite field F of order $q = p^k$ with basis $\mathfrak{B} = \{\alpha_1, \alpha_2\}$. Then Cay($S^*(\mathbb{V}), S^*(\mathbb{V})) = K_{1,q+1}$.*

Proof. Let $\mathfrak{B} = \{\alpha_1, \alpha_2\}$ be a basis for $\mathbb{V}$. The set of all non-zero one dimensional subspaces of $\mathbb{V}$ are $V_1 = \{\langle\alpha_1\rangle, \langle\alpha_2\rangle, \langle\alpha_1 + a\alpha_2\rangle\}$ for $0 \neq a \in \mathbb{F}$ where as $\mathbb{V}$ is the only two dimensional trivial subspace of $\mathbb{V}$. Note that $|V_1| = p^k + 1$ and $|V_2| = 1$ and $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{V}) = K_{1,q+1}$. □

Now, we find the girth of $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$.

Theorem 3.7. *Let $\mathbb{V}$ be an $n(\geq 3)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$. Then the girth gr(Cay($S^*(\mathbb{V}), S^*(\mathbb{V})$))=3.*

Proof. For an integer m, $1 \leq m \leq n-2$, let $X = \langle\alpha_1, \dots, \alpha_m\rangle, Y = \langle\alpha_1, \dots, \alpha_{m+1}\rangle$ and $Z = \langle\alpha_1, \dots, \alpha_{m+2}\rangle$ be $m, m+1$ and $m+2$ dimensional subspaces of $\mathbb{V}$ respectively. Let $X' = \langle\alpha_{m+1}\rangle, Y' = \langle\alpha_{m+2}\rangle$ and $Z' = \langle\{\alpha_{m+1}, \alpha_{m+2}\}\rangle \in \mathbb{A}$. Then $X + X' = Y$, $Y + Y' = Z$ and $X + Z' = Z$. Hence $X - Y - Z - X$ is a cycle of length 3 in $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$. □

Theorem 3.8. *Let $\mathbb{V}$ be a finite dimensional vector space of dimension $n(\geq 2)$ over a finite field $\mathbb{F}$ and $\mathbb{A} \subseteq S^*(\mathbb{V})$. Then Cay($S^*(\mathbb{V}), \mathbb{A}$) is an n-partite graph.*

Proof. Let S^*_m be the collection of all non-zero m-dimensional subspaces of $\mathbb{V}$. Then $\{S^*_m : 1 \leq m \leq n\}$ is a partition of $S^*(\mathbb{V})$. To conclude the proof, it is enough to prove that no two vertices in one partition S^*_m are adjacent in $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$. For, let $X, Y \in S^*_m$ for some m. Then $X = \langle\beta_1, \dots, \beta_m\rangle$ and $Y = \langle\beta'_1, \dots, \beta'_m\rangle$. Let $Z \in \mathbb{A}$ and $\dim(Z) = \ell$.

Case 1. If $Z \subseteq X$, then $X + Z = X$.

Case 2. If $Z \nsubseteq X$, then let $Y' = X + Z$. Note that $\dim(X \cap Z) < \ell$ and so $\dim(Y') = \dim(X) + \dim(Z) - \dim(X \cap Z) = m + \ell - \dim(X \cap Z) > m$. Hence $X + Z \neq Y$ for any $Z \in \mathbb{A}$ and so there exists no $Z \in \mathbb{A}$ such that $X + Z = Y$. □

Using Theorem 3.8, we obtain the clique number of $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ where $\mathbb{V}$ is a finite dimensional vector space.

Theorem 3.9. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$. Then ω(Cay($S^*(\mathbb{V}), S^*(\mathbb{V})$)) $= n$.*

Proof. Consider the set of subspaces $\{W_1, \dots, W_n\}$ where $W_i = \langle\alpha_1, \dots, \alpha_i\rangle$ is an i dimensional subspace of $\mathbb{V}$.

Given W_i, W_j $1 \leq i < j \leq n$, let $U_{ij} = \langle \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{i+j} \rangle \in S^*(\mathbb{V})$. Then $W_i + U_{ij} = W_j$ and so the subgraph induced by $\{W_i : 1 \leq i \leq n\}$ is complete and so $\omega(\text{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))) \geq n$. By Theorem 3.8, $\omega(\text{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))) \leq n$. Hence $\omega(\text{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))) = n$. $\square$

For a finite dimensional vector space $\mathbb{V}$ over a finite field $\mathbb{F}$, the subspace inclusion graph $In(\mathbb{V})$ of $\mathbb{V}$ was introduced and studied by Das [9]. The subspace inclusion graph $In(\mathbb{V})$ of $\mathbb{V}$ is the simple undirected graph with the set of all nontrivial subspaces of $\mathbb{V}$ as the vertex set and two vertices are adjacent if one is contained in other. If $\mathbb{V} \in \mathbb{A}$, then $\mathbb{V}$ is adjacent to all the vertices in $\text{Cay}(S^*(\mathbb{V}), \mathbb{A}))$. Hence it is necessary to study about the Cayley subspace sum graph by excluding by considering $\mathbb{V} \notin \mathbb{A}$. Let $S^{**}(\mathbb{V}) = S(\mathbb{V}) \backslash \{0, \mathbb{V}\}$ and $\mathbb{A} \subseteq S^{**}(\mathbb{V})$. Now we prove that the subspace inclusion graph $In(\mathbb{V})$ can be realized as a Cayley subspace sum graph with vertex set $S^{**}(\mathbb{V}) = S(\mathbb{V}) \backslash \{0, \mathbb{V}\}$.

Theorem 3.10. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field. Then Cay$(S^{**}(\mathbb{V}), S^{**}(\mathbb{V}))$ is isomorphic to $In(\mathbb{V})$.*

Proof. Note that the vertex sets of both $\text{Cay}(S^{**}(\mathbb{V}), S^{**}(\mathbb{V}))$ and $In(\mathbb{V})$ are nontrivial proper subspaces of $\mathbb{V}$. If X and Y are two adjacent vertices in the graph $\text{Cay}(S^{**}(\mathbb{V}), S^{**}(\mathbb{V}))$, by definition there exists some $Z \in S^{**}(\mathbb{V})$ such that $X + Z = Y$ or $Y + Z = X$. This gives that $X \subset Y$ or $Y \subset X$. Hence X and Y are adjacent in $In(\mathbb{V})$.

On the other hand, let X and Y be adjacent in $In(\mathbb{V})$. By definition either $X \subset Y$ or $Y \subset X$. Without loss of generality let us take $X \subset Y$. Then $X + W = Y$ where W is a subspace isomorphic to quotient space Y/X. From this X and Y are adjacent in $\text{Cay}(S^{**}(\mathbb{V}), S^{**}(\mathbb{V}))$. $\square$

Now we characterize all finite dimensional vector spaces $\mathbb{V}$ for which $\text{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is planar. We recall the following well known characterization for planar graphs.

Theorem 3.11. ([6, Kuratowski's theorem pp. 151]) *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

Theorem 3.12. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$. Then Cay$(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is planar if and only if $n = 2$.*

Proof. Assume that $\text{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is planar where $\mathbb{V}$ is an n-dimensional vector space. Suppose $n \geq 3$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathfrak{B}(\mathbb{V})$. Consider the subspaces

$W_1 = \langle\alpha_1\rangle$, $W_2 = \langle\alpha_2\rangle$, $W_3 = \langle\alpha_3\rangle$, $W_4 = \langle\alpha_1+\alpha_2\rangle$, $W_5 = \langle\alpha_1+\alpha_3\rangle$, $W_6 = \langle\alpha_2+\alpha_3\rangle$, $W_7 = \langle\alpha_1,\alpha_2\rangle$, $W_8 = \langle\alpha_1,\alpha_3\rangle$, $W_9 = \langle\alpha_2,\alpha_3\rangle$, $W_{10} = \langle\alpha_1,\alpha_2+\alpha_3\rangle$, $W_{11} = \langle\alpha_2,\alpha_1+\alpha_3\rangle$, $W_{12} = \langle\alpha_3,\alpha_1+\alpha_2\rangle$. The subgraph H of $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ induced by $\{W_i : 1 \leq i \leq 12\}$ is given in Fig. 1.

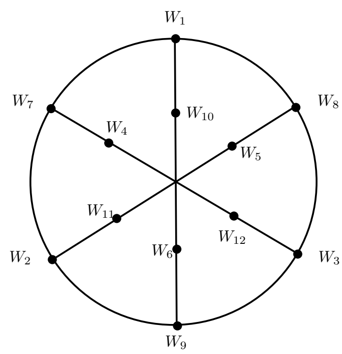

Fig. 1: The graph H

Note that the graph H is a subdivision graph of $K_{3,3}$ as given in Fig. 2.

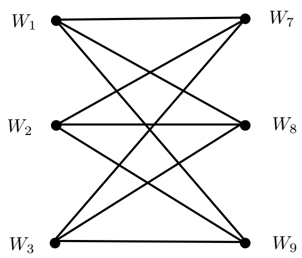

Fig. 2: $K_{3,3}$

From this $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ contains a subdivision of $K_{3,3}$ which is a contradiction to Theorem 3.11. Hence $n = 2$.

Conversely, assume that $n = 2$. By Corollary 3.6 $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is a star graph and so planar. □

4. Properties of Cay$(S^*(\mathbb{V}), \mathbb{A})$

In this section, we study $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ where $\mathbb{V}$ is an n-dimensional vector space over a finite field of order q with basis $\mathfrak{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \dots, \langle\alpha_n\rangle\}$. In view of Theorem 3.1, we have the following.

Lemma 4.1. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \dots, \langle\alpha_n\rangle\}$. Then Cay$(S^*(\mathbb{V}), \mathbb{A})$ is connected.*

Theorem 4.2. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. If X and Y are adjacent in Cay$(S^*(\mathbb{V}), \mathbb{A})$, then $|\dim(X) - \dim(Y)| = 1$.*

Proof. Let the vertices $X, Y \in S^*(\mathbb{V})$ be adjacent in $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$. By definition, there exists a subspace $\langle\alpha_i\rangle \in \mathbb{A}$ such that $X + \langle\alpha_i\rangle = Y$ or $Y + \langle\alpha_i\rangle = X$ for some $\alpha_i \in \mathfrak{B}$. Suppose $X + \langle\alpha_i\rangle = Y$ and $\dim(X) = k$. Then $\dim(Y) = \dim(X + \langle\alpha_i\rangle) = k + 1$. Hence $|\dim(X) - \dim(Y)| = |k - (k + 1)| = 1$. □

Note that the converse of Theorem 4.2 is not true. For, let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space with basis $\mathfrak{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \langle\alpha_2\rangle, \langle\alpha_3\rangle\}$. Let $X = \langle\alpha_1\rangle$ and $Y = \langle\alpha_1, \alpha_1 + \alpha_2\rangle$. Then $|\dim(X) - \dim(Y)| = 1$ but there exists no $Z \in \mathbb{A}$ such that $X + Z = Y$ or $Y + Z = X$.

From Theorem 4.2, we have the following corollary.

Corollary 4.3. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. Then no two non-zero subspaces of same dimension are adjacent in Cay$(S^*(\mathbb{V}), \mathbb{A})$.*

Theorem 4.4. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. Then Cay$(S^*(\mathbb{V}), \mathbb{A})$ is a bipartite graph.*

Proof. Consider the partition $V_1 = \{X \in S^*(\mathbb{V}) : \dim(X) \text{ is odd}\}$ and $V_2 = \{X \in S^*(\mathbb{V}) : \dim(X) \text{ is even}\}$ of $S^*(\mathbb{V})$. Let X and Y be two vertices in the same partition V_i for $i = 1, 2$. If X and Y are of same dimension, then by Corollary 4.3, X and Y are not adjacent. If X and Y are of different dimension, then $|\dim(X) - \dim(Y)| \geq 2$. By Theorem 4.2, X and Y cannot be adjacent. Hence no two vertices in the same partition V_i for $i = 1, 2$ are adjacent in $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$. □

Since a bipartite graph is bi-chromatic, we have the following corollary from Theorem 4.4.

Corollary 4.5. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. Then $\omega(Cay(S^*(\mathbb{V}), \mathbb{A})) = 2$.*

Also we have following corollary from Theorem 4.4.

Corollary 4.6. *Let $\mathbb{V}$ be a two dimensional vector space over a finite field of order q with basis $\mathfrak{B} = \{\alpha_1, \alpha_2\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \langle\alpha_2\rangle\}$. Then Cay$(S^*(\mathbb{V}), \mathbb{A})$ is the star graph $K_{1,q+1}$.*

Theorem 4.7. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. Then the girth of Cay($S^*(\mathbb{V}), \mathbb{A}$)) is given by*

$$gr(Cay(S^*(\mathbb{V}), \mathbb{A})) = \begin{cases} 4 & \text{if } n \geq 3; \\ \infty & \text{if } n = 2. \end{cases}$$

Proof. Case 1. Let $\mathbb{V}$ be an $n \geq 3$ dimensional vector space with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$. By Theorem 4.4, Cay($S^*(\mathbb{V}), \mathbb{A}$) is a bipartite graph and so it contains no cycle of length 3. Consider the subspaces $W_1 = \langle\alpha_1\rangle$, $W_2 = \langle\alpha_1, \alpha_2\rangle$, $W_3 = \langle\alpha_1, \alpha_2, \alpha_3\rangle$ and $W_4 = \langle\alpha_1, \alpha_3\rangle$. Note that $W_1 - W_2 - W_3 - W_4 - W_1$ is a cycle of length 4 in Cay($S^*(\mathbb{V}), \mathbb{A}$)) and so gr(Cay($S^*(\mathbb{V}), \mathbb{A}$)) = 4.

Case 2. Let $\mathbb{V}$ be a 2 dimensional vector space. By Theorem 4.6 Cay($S^*(\mathbb{V}), \mathbb{A}$) is a star graph and so in this case $gr(Cay(S^*(\mathbb{V}), \mathbb{A}))$ is ∞. □

Theorem 4.8. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. Then diam(Cay($S^*(\mathbb{V}), \mathbb{A}$)) = $2(n-1)$.*

Proof. Let $X \in S^*(\mathbb{V})$. Assume that $\dim(X) = m \geq 1$ and $X = \langle\beta_1, \ldots, \beta_m\rangle$. Without loss of generality one can assume that $\mathfrak{B}(X) \subseteq \mathfrak{B}(\mathbb{V})$ and $\mathfrak{B}(\mathbb{V})$ is obtained from $\mathfrak{B}(X)$ by adjoining $\gamma_1, \gamma_2, \ldots, \gamma_{n-m}$.

Consider the trial $P : X - \langle\gamma_1, \beta_1, \ldots, \beta_m\rangle - \langle\gamma_1, \gamma_2, \beta_1, \ldots, \beta_m\rangle - \cdots - \langle\gamma_1, \gamma_2, \ldots, \gamma_{n-m}, \beta_1, \ldots, \beta_m\rangle = \mathbb{V}$ from X to $\mathbb{V}$ is of length $n - m$ which contains a $(X, \mathbb{V})$ path. Similarly, there exists a path of length at most $n - m$ for any other vertex Y to $\mathbb{V}$. From this, one can visualize a path of length at most $2(n - m)$ between X and Y in Cay($S^*(\mathbb{V}), \mathbb{A}$)). Hence diam(Cay($S^*(\mathbb{V}), \mathbb{A}$)) $\leq 2(n-m) \leq 2(n-1)$.

Consider the two one dimensional subspaces $U = \langle\alpha_1\rangle$ and $W = \langle\alpha_1 + \alpha_2 + \ldots + \alpha_n\rangle$ of $\mathbb{V}$. Then $P : U - \langle\alpha_1, \alpha_2\rangle - \langle\alpha_1, \alpha_2, \alpha_3\rangle - \cdots - \langle\alpha_1, \ldots, \alpha_n\rangle$ is a path of length $n - 1$ between U and $\mathbb{V}$ and so $d(U, \mathbb{V}) = n - 1$. On the other hand $Q : W - \langle\alpha_1, W\rangle - \langle\alpha_1, \alpha_2, W\rangle - \cdots - \langle\alpha_1, \ldots, \alpha_{n-1}, W\rangle$ is path of length $n - 1$ between Y and $\mathbb{V}$ and so $d(W, \mathbb{V}) = n - 1$. Therefore $d(X, Y) = 2(n - 1)$ and so diam(Cay($S^*(\mathbb{V}), \mathbb{A}$)) = $2(n - 1)$. □

Now we characterize all finite dimensional vector spaces for which Cay($S^*(\mathbb{V}), \mathbb{A}$) is planar.

Theorem 4.9. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A} = \{\langle\alpha_1\rangle, \ldots, \langle\alpha_n\rangle\}$. Then Cay($S^*(\mathbb{V}), \mathbb{A}$)) is planar if and only if $n = 2$.*

Proof. Suppose $n = 2$. By Theorem 3.12, $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is planar. Thus $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})) \subseteq \mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is planar.

Conversely assume that $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ is planar. Suppose $n \geq 3$. Consider the subspaces $W_1 = \langle \alpha_1 \rangle$, $W_2 = \langle \alpha_2 \rangle$, $W_3 = \langle \alpha_3 \rangle$, $W_4 = \langle \alpha_1 + \alpha_2 \rangle$, $W_5 = \langle \alpha_1 + \alpha_3 \rangle$, $W_6 = \langle \alpha_2 + \alpha_3 \rangle$, $W_7 = \langle \alpha_1 + \alpha_2 + \alpha_3 \rangle$, $W_8 = \langle \alpha_1, \alpha_2 \rangle$, $W_9 = \langle \alpha_1, \alpha_3 \rangle$, $W_{10} = \langle \alpha_2, \alpha_3 \rangle$, $W_{11} = \langle \alpha_1, \alpha_2 + \alpha_3 \rangle$, $W_{12} = \langle \alpha_2, \alpha_1 + \alpha_3 \rangle$, $W_{13} = \langle \alpha_3, \alpha_1 + \alpha_2 \rangle$, $W_{14} = \langle \alpha_1 + \alpha_2, \alpha_1 + \alpha_3 \rangle$ and $W_{15} = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. The induced subgraph H induced by $\{W_i : 1 \leq i \leq 15\}$ is a subgraph of $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A}))$. The graph H is given in Fig. 3.

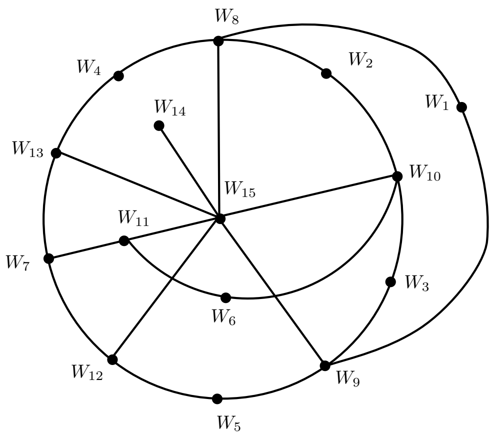

Fig. 3: Graph H

Now let us prove that the graph H cannot have a planar embedding. Note that the subgraph induced by $\{W_2, W_3, W_4, W_5, W_7, W_8, W_9, W_{10}, W_{12}, W_{13}\}$ is the cycle $C_1 = W_8 - W_2 - W_{10} - W_3 - W_9 - W_5 - W_{12} - W_7 - W_{13} - W_4 - W_8$.

Case 1. Let us place the vertex W_{15} in the interior face of C_1 as in Fig. 3. Now we get five cycles $C_2 = W_{13} - W_{15} - W_{12} - W_7 - W_{13}, C_3 = W_8 - W_{15} - W_{13} - W_4 - W_8, C_4 = W_{10} - W_{15} - W_8 - W_2 - W_{10}, C_5 = W_9 - W_{15} - W_{10} - W_3 - W_9$ and $C_6 = W_{12} - W_{15} - W_9 - W_5 - W_{12}$. Now one has to place the vertex W_{11} in an interior face of one of the cycles C_2, C_3, C_4, C_5 and C_6. Without loss of generality let us place W_{11} in the interior face of C_2 as in the Fig. 3. Similarly place the vertex W_6 in one of the interior faces and without loss of generality let us place W_6 in the interior face of C_6 as in Fig. 3. Note that the vertex W_6 is adjacent to W_{10} and W_{11}. It is clear from Fig. 3 that one cannot draw the edges W_6W_{10} and W_6W_{11} without crossing another edge. Hence H is not planar.

Case 2. Now let us consider the possibility that the vertex W_{15} is placed in the outer face of C_1. Note that the subgraph H' induced by $\{W_1, W_2, W_3, W_4, W_5, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}, W_{13}, W_{15}\}$ is given in Fig. 4.

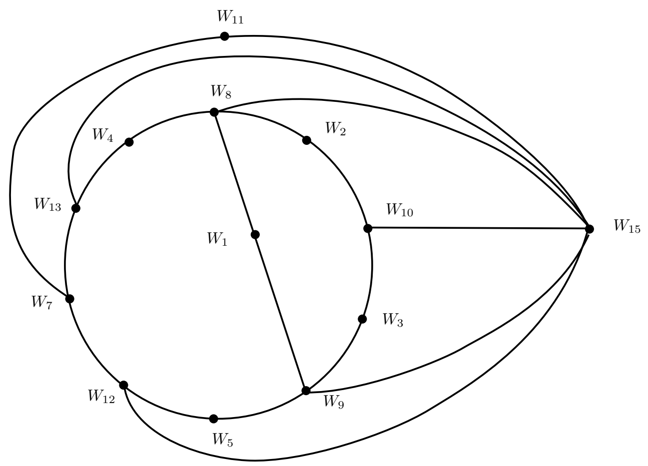

Fig. 4: H'

Consider the circle $C = W_1 - W_8 - W_{15} - W_9 - W_1$ in H'. The vertex W_{10} is inside C and W_{11} is outside the circle C. One cannot draw edges W_6W_{10} and W_6W_{11} in H' without crossings. Therefore the graph H is non-planar.

From the above $\mathrm{Cay}(S^*(\mathbb{V}), S^*(\mathbb{V}))$ is non-planar, which is a contradiction. Hence $n = 2$. □

5. Another class of Cay$(S^*(\mathbb{V}), \mathbb{A})$

In this section, we study $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ where $\mathbb{A}$ is set of all $m(1 \le m < n)$ dimensional nonzero subspaces of $\mathbb{V}$ for some fixed m.

Theorem 5.1. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ and $\mathbb{A}$ be the set of all $1 \le m < n$ dimensional nonzero subspaces of $\mathbb{V}$. If X and Y are adjacent in Cay$(S^*(\mathbb{V}), \mathbb{A})$, then $|\dim(X) - \dim(Y)| \le m$.*

Proof. Let $X, Y \in S^*(\mathbb{V})$ be adjacent in $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$. Then there exists some $Z \in \mathbb{A}$ such that $X + Z = Y$ or $Y + Z = X$. Without loss of generality, let us take $X + Z = Y$. Then $\dim(X + Z) = \dim(Y)$ and so

$$\begin{aligned}
|\dim(X) - \dim(Y)| &= |\dim(X) - \dim(X + Z)| \\
&= |\dim(X) - (\dim(X) + \dim(Z) - \dim(X \cap Z))| \\
&= |\dim(Z) - \dim(X \cap Z)| \le m.
\end{aligned}$$ □

Now, we have the following corollary for $n-1$ dimensional subspaces of $\mathbb{V}$.

Corollary 5.2. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A}$ be the set of all $n-1$ dimensional nonzero subspaces of $\mathbb{V}$. Then $\mathbb{V}$ is adjacent to all the vertices in $Cay(S^*(\mathbb{V}), \mathbb{A})$.*

Theorem 5.3. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A}$ be the set of all $m(\geq 1)$ dimensional nonzero subspaces of $\mathbb{V}$. Then $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected.*

Proof. Let $X \in S^*(\mathbb{V})$. Assume that $\dim(X) = k$ and $\{\beta_1, \ldots, \beta_k\}$ be a basis of X. By division algorithm, $n = mt + r$ where t and $r < m$ are integers. Then $P : X - \langle \beta_1, \ldots, \beta_k, \alpha_1, \ldots, \alpha_m \rangle - \langle \beta_1, \ldots, \beta_k, \alpha_1, \ldots, \alpha_{2m} \rangle - \cdots \langle \beta_1, \ldots, \beta_k, \alpha_1, \ldots, \alpha_{tm} \rangle - \mathbb{V}$ contains a path between the arbitrary vertex X and $\mathbb{V}$ and so $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected. □

Theorem 5.4. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A}$ is the set of all $m(\geq 1)$ dimensional nonzero subspaces of $\mathbb{V}$. Then the girth of $Cay(S^*(\mathbb{V}), \mathbb{A}))$ is 3.*

Proof. Let $X = \langle \beta_1, \ldots, \beta_{m-1} \rangle, Y = \langle \beta_1, \ldots, \beta_m \rangle$ and $Z = \langle \beta_1, \ldots, \beta_{m+1} \rangle$ be subspaces of $\mathbb{V}$ of dimension $m, m+1$ and $m+2$ respectively. Then the subspaces $X' = \langle \beta_1, \ldots, \beta_m \rangle$ and $Y' = \langle \beta_2, \ldots, \beta_{m+1} \rangle$ satisfy $X + X' = Y,\ Y + Y' = Z$ and $X + Y' = Z$. Hence $X - Y - Z - X$ is a cycle in $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A}))$ of length 3 and so the girth of $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A}))$ is 3. □

Theorem 5.5. *Let $\mathbb{V}$ be an $n(\geq 2)$ dimensional vector space over a finite field with basis $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathbb{A}$ is the set of all $m(\geq 1)$ dimensional nonzero subspaces of $\mathbb{V}$. Then diam(Cay($S^*(\mathbb{V}), \mathbb{A}$)) $= 2\lceil \frac{n}{m} \rceil$.*

Proof. By division algorithm $n = tm + r$ where t, r be integers with $r < m$. Consider the subspaces $Y_k = \langle \alpha_{(k-1)m+1}, \alpha_{(k-1)m+2}, \ldots, \alpha_{(k-1)m+m} \rangle$ for $1 \leq k \leq t$ of dimension m and $Y_{t+1} = \langle \alpha_{tm+1}, \alpha_{tm+2}, \ldots, \alpha_n \rangle$ of $\mathbb{V}$ of dimension r. For a nonzero subspace $X \in S^*(\mathbb{V})$, let $Z_0 = X$ and $Z_i = X + \sum_{j=1}^{i} Y_j$ for $i = 1, 2, \ldots, t+1$. Then the trail $W : Z_0 - Z_1 - Z_2 - \cdots Z_{t+1} = \mathbb{V}$ from X to $\mathbb{V}$ of length $t + 1 = \lceil \frac{n}{m} \rceil$ contains a $(X, \mathbb{V})$ path. Similarly a trial W' between another subspace $X' \in S^*(\mathbb{V})$ to $\mathbb{V}$ of length $t + 1 = \lceil \frac{n}{m} \rceil$ contains a $(X', \mathbb{V})$ path. Hence there exists a path of length at most $2\lceil \frac{n}{m} \rceil$ between two arbitrary subspaces $X, X' \in S^*(\mathbb{V})$ and so $Cay(S^*(\mathbb{V}), \mathbb{A})$ is connected and $diam(Cay(S^*(\mathbb{V}), \mathbb{A})) \leq 2\lceil \frac{n}{m} \rceil$.

Consider the one dimensional subspaces $U = \langle\alpha_1\rangle$ and $U' = \langle\alpha_1 + \alpha_2 + \ldots + \alpha_n\rangle$ of $\mathbb{V}$. Then $P : Z_0 - Z_1 - Z_2 - \ldots Z_{t+1} = \mathbb{V}$ is a path of length $\lceil \frac{n}{m} \rceil$ between U and $\mathbb{V}$. Similarly $P' : U' - Z_1 - Z_2 - \ldots Z_{t+1} = \mathbb{V}$ is a path of length $\lceil \frac{n}{m} \rceil$ between U' and $\mathbb{V}$. Therefore $d(U, U') = 2\lceil \frac{n}{m} \rceil$ and so $diam(Cay(S^*(\mathbb{V}), \mathbb{A})) = 2\lceil \frac{n}{m} \rceil$. □

6. Properties of Cay$(S^*(\mathbb{V}), \mathbb{A})$ where $\mathbb{V}$ is 3 dimensional

In this section, we discuss some special properties of Cayley subspace sum graphs of three dimensional vector spaces over finite field. First we obtain some adjacency relations of Cay$(S^*(\mathbb{V}), \mathbb{A})$ for the different possibilities of $\mathbb{A}$. Let $\mathbb{V}$ be the finite dimensional vector space with $\{\alpha_1, \alpha_2, \alpha_3\}$ as basis over the finite field $\mathbb{F}$ of order q. One can see that the following are the complete list of non-zero subspaces of $\mathbb{V}$.

One dimensional subspaces

(1) $\langle\alpha_i\rangle : i = 1, 2, 3;$
(2) $\langle\alpha_i + a\alpha_j\rangle : i, j = 1, 2, 3;\ i \neq j, a \in \mathbb{F} \setminus \{0\};$
(3) $\langle\alpha_1 + a\alpha_2 + b\alpha_3\rangle : a, b \in \mathbb{F} \setminus \{0\}.$

Two dimensional subspaces

(1) $\langle\alpha_i, \alpha_j\rangle : i, j = 1, 2, 3;\ i \neq j;$
(2) $\langle\alpha_i, \alpha_j + a\alpha_k\rangle : i, j, k = 1, 2, 3;\ i \neq j \neq k, a \in \mathbb{F} \setminus \{0\};$
(3) $\langle\alpha_1 + a\alpha_2, \alpha_1 + b\alpha_3\rangle : a, b \in \mathbb{F} \setminus \{0\}.$

Note that total number of nonzero subspaces of $\mathbb{V}$ is $2(q^2+q)+3$. Suppose $|V_i|$ is the number of i dimensional nonzero subspaces of $\mathbb{V}$, then $|V_1| = |V_2| = q^2 + q + 1$. Note that Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ is a subgraph of Cay$(S^*(\mathbb{V}), \mathbb{A})$ with vertex set $S^{**}(\mathbb{V}) = S(\mathbb{V}) \setminus \{0, \mathbb{V}\}$.

Theorem 6.1. *Let $\mathbb{V}$ be a three dimensional vector space over a finite field and $\mathbb{A}$ be the set of all one dimensional non-zero proper subspaces of $\mathbb{V}$. Any two vertices in Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ are adjacent if and only if one of them is properly contained in the other.*

Proof. Let X and Y be any two nonzero proper subspaces of $\mathbb{V}$ and assume that they are adjacent in Cay$(S^{**}(\mathbb{V}), \mathbb{A})$. This implies there exists $Z \in \mathbb{A}$ such that $X + Z = Y$ or $Y + Z = X$. In the first case $X \subset Y$ where as in the second case $Y \subset X$.

Conversely, let X and Y be two nonzero proper subspaces of $\mathbb{V}$ and $X \subset Y$. Without loss of generality dim(X)=1, dim(Y)=2 and so $X = \langle\beta\rangle$ and $Y = \langle\beta, \beta'\rangle$ for $\beta, \beta' \in \mathbb{V}^*$. Then $X + Z = Y$ where $Z = \langle\beta'\rangle \in \mathbb{A}$, i.e., X and Y are adjacent. □

Theorem 6.2. *Let $\mathbb{V}$ be a three dimensional vector space over a finite field and $\mathbb{A}$ be the set of all two dimensional proper subspaces of $\mathbb{V}$. Any two subspaces are adjacent in Cay$(S^*(\mathbb{V}), \mathbb{A})$ if and only if one is properly contained in the other.*

Proof. The proof of "only if" part is similar to that of Theorem 6.1.

Conversely, let X and Y in $S^*(\mathbb{V})$ and $X \subset Y$. Then there are three possibilities. Suppose dim(X)=1, dim(Y)=2, $X = \langle\beta_1\rangle$ and $Y = \langle\beta_1, \beta_2\rangle$. Then $X + Y = Y$, i.e., X and Y are adjacent. Similar proof follows in the cases of dim(X)=1, dim(Y)=3; and dim(X)=2, dim(Y)=3. Thus in all the cases X and Y are adjacent. □

Corollary 6.3. *Let $\mathbb{V}$ be a three dimensional vector space over a finite field and $\mathbb{A}$ be the set of all two dimensional proper subspaces of $\mathbb{V}$. Any two subspaces are adjacent in Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ if and only if one is properly contained in the other.*

In similar to the proof of Theorem 6.2, one can prove the following.

Theorem 6.4. *Let $\mathbb{V}$ be a four dimensional vector space over a finite field and $\mathbb{A}$ be the set of all two dimensional proper subspaces of $\mathbb{V}$. Any two subspaces are adjacent in Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ if and only if one is properly contained in the other.*

Remark 6.5. Let $\mathbb{V}$ be a three dimensional vector space and $\mathbb{A}$ be either the set of all one dimensional proper subspaces or the set of all two dimensional proper subspaces of $\mathbb{V}$. By Theorem 6.1 and Corollary 6.3, Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ as same as In$(\mathbb{V})$. Let $\mathbb{V}$ be a four dimensional vector space and $\mathbb{A}$ be the set of all two dimensional proper subspaces of $\mathbb{V}$. By Theorem 6.4, Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ as same as In$(\mathbb{V})$. This property is not true for four dimensional vector spaces with other choices for $\mathbb{A}$. For let $\mathbb{A}_1$ be set of all one dimensional proper subspaces and $\mathbb{A}_2$ be the set of all three dimensional proper subspaces of $\mathbb{V}$. Then the subspaces $\langle\alpha_1\rangle$ and $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ are not adjacent Cay$(S^{**}(\mathbb{V}), \mathbb{A}_1)$ even though $\langle\alpha_1\rangle \subset \langle\alpha_1, \alpha_2, \alpha_3\rangle$. Similarly $\langle\alpha_1\rangle$ and $\langle\alpha_1, \alpha_2\rangle$ are not adjacent in Cay$(S^{**}(\mathbb{V}), \mathbb{A}_2)$ even though $\langle\alpha_1\rangle \subset \langle\alpha_1, \alpha_2\rangle$.

Remark 6.6. By Theorem 6.4, Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ is same as In$(\mathbb{V})$. Also, by [11, Corollary 6.3] Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ is a $q + 1$-regular graph.

Theorem 6.7. *Let $\mathbb{V}$ be a three dimensional vector space over a field of order $q \in \{2, 3, 5, 8, 17\}$ and $\mathbb{A}$ be the set of all two dimensional nonzero subspaces of $\mathbb{V}$. Then Cay$(S^*(\mathbb{V}), \mathbb{A})$ is Hamiltonian.*

Proof. By [11, Theorem 6.10], Cay$(S^{**}(\mathbb{V}), \mathbb{A})$ is Hamiltonian. Since $\mathbb{V}$ is adjacent to all the elements in Cay$(S^*(\mathbb{V}), \mathbb{A})$, we see that Cay$(S^*(\mathbb{V}), \mathbb{A})$ is Hamiltonian. □

Theorem 6.8. *Let $\mathbb{V}$ be a three dimensional vector space over a field of order q and $\mathbb{A}$ be the set of all one dimensional subspaces of $\mathbb{V}$. Then the domination number $\gamma(Cay(S^*(\mathbb{V}), \mathbb{A})) = q + 2$.*

Proof. Note that $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ is a bipartite graph with vertex partition

$$V_1 = \mathbb{A} \cup \{\mathbb{V}\}$$

and

$$V_2 = \{\text{all two dimensional subspaces of } \mathbb{V}\}.$$

Consider the set $D = \{\langle \alpha_2, \alpha_1 + a\alpha_3 \rangle, \langle \alpha_2, \alpha_3 \rangle, \mathbb{V} \mid a \in \mathbb{F}\}$. Then the following are true.

- $\langle \alpha_i \rangle$ is dominated by $\langle \alpha_i, \alpha_j \rangle$ for $i, j = 1, 2, 3$ and $i \neq j$;
- $\langle \alpha_1 + a\alpha_2 \rangle$ and $\langle \alpha_2 + a\alpha_3 \rangle$ are dominated by $\langle \alpha_1, \alpha_2 \rangle$ and $\langle \alpha_2, \alpha_3 \rangle$ respectively;
- $\langle \alpha_1 + a\alpha_3 \rangle$ is dominated by $\langle \alpha_2, \alpha_1 + a\alpha_3 \rangle$;
- $\langle \alpha_1 + a\alpha_2 + b\alpha_3 \rangle$ is dominated by $\langle \alpha_2, \alpha_1 + b\alpha_3 \rangle$;
- Set of all two dimensional subspace are dominated by $\mathbb{V}$.

This shows that D is a dominating set of $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$ with $|D| = q + 2$. To conclude the proof, one has to show that $q + 1$ elements are not sufficient for a dominating set in $\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})$. Since $\mathbb{V}$ dominates all two dimensional subspaces, for a minimal dominating set, one has to choose elements in V_2 which dominate all the elements in $V_1 \setminus \mathbb{V}$. By Remark 6.1, $\mathrm{Cay}(S^{**}(\mathbb{V}), \mathbb{A})$ is a $q + 1$-regular graph. Further $|V_1 \setminus \mathbb{V}| = q^2 + q + 1$ and $\frac{q^2+q+1}{q+1} = q + \frac{1}{q+1}$. This indicates that at least $q + 1$ elements from V_2 are needed to dominate all the elements in $V_1 \setminus \mathbb{V}$. Hence $\gamma(\mathrm{Cay}(S^*(\mathbb{V}), \mathbb{A})) = q + 2$. □

Now, we have the following corollary.

Corollary 6.9. *Let $\mathbb{V}$ be a three dimensional vector space over a finite field of order q with basis $\mathfrak{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathbb{A} = \{\langle \alpha_1 \rangle, \langle \alpha_2 \rangle, \langle \alpha_3 \rangle\}$. Then $\gamma(Cay(S^*(\mathbb{V}), \mathbb{A})) = q + 2$.*

From Corollary 5.2, $\mathbb{V}$ is adjacent to all the vertices and hence we have the following corollary regarding domination for two dimensional case.

Corollary 6.10. *Let $\mathbb{V}$ be a three dimensional vector space over a field of order q with basis $\mathfrak{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathbb{A}$ be the set of all two dimensional of $\mathbb{V}$. Then $\gamma(Cay(S^*(\mathbb{V}), \mathbb{A})) = 1$.*

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