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Research Article

Mathematical thinking processes for the pythagorean theorem of the secondary school students[1]

Esra Akarsu Yakar[2] and Suha Yilmaz[3]

Department of Mathematics and Science Education, Buca Faculty of Education, Dokuz Eylul University, Izmır, Turkiye

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Abstract

In this study, the mathematical thinking processes on geometry of 6th, 7th and 8th grade students who have the same geometric thinking levels were examined. Two students from each grade level were selected for the study. The geometric thinking levels of the students were determined as the third level. In addition, the algebraic thinking levels of the students were discussed. The three worlds of mathematics was used in the research. The mathematical thinking processes of the students were examined in terms of the embodied world, the proceptual world and the formal world. The Pythagorean Theorem was chosen as the geometry subject. Two-stage semi-structured interviews were conducted with the students. In the first part of the interviews, the verbal expression of the Pythagorean Theorem was directed to the students. In the second part, an activity was presented for them to discover the theorem in a real-life situation. As a result, while the students had difficulty in explaining the theorem in the verbal expression, they were able to express it more easily in a real-life situation. 7th and 8th grade students were more successful than 6th grade students in demonstrating the processes of the three worlds of mathematics.

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Introduction

Students' judgments about concepts in mathematics are related to mathematical thinking (Doğan & Güner, 2012). When the literature is examined, mathematical thinking is defined by some researchers as generalization, induction, deduction, logical thinking, using symbols, abstract thinking (Alkan & Bukova Güzel, 2005; Burton, 1984; Henderson et al., 2002; Liu, 2003; Liu & Niess, 2006; Mason, Burton & Stacey, 2010; Mubark, 2005; Polya, 1945; Stacey, 2006; Tall, 2002; Yeşildere & Türnüklü, 2007); it is defined by some researchers as the process of formation of concepts in the mind (Schoenfeld, 1992; Tall, 2006). Mathematical thinking is an important component in the concept formation process. Therefore, it is also important for mathematics teaching. During mathematical thinking, mathematical processes such as reasoning, problem solving and estimation are applied (Henderson et al., 2002). Mathematical thinking is individual. It is the process of creating new concepts within the knowledge structures of individuals. In other words, it is the act of adapting the new concepts encountered by the individual in his/her mind. It is the ability of an individual to use mathematical actions in the process of learning concepts. It is also the ability of the individual to explain the problem-solving processes. Studies that focus on the process in mathematical thinking (Dreyfus, 2002; Freudenthal, 1973; Tall, 1995) have examined mathematical thinking developed in concept formation. According to Bal & Dinç Artut (2020),

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[2] Corresponding Author: Ministry of National Education, Kocaeli, Türkiye, E-mail: es.akarsu@gmail.com, ORCID No: 0000-0002-4090-6419
[3] Dokuz Eylul University, Buca Faculty of Education, Izmir, Türkiye. E-mail: suha.yilmaz@deu.edu.tr, ORCID: 0000-0002-8330-9403

doing mathematics involves the application of operations and symbolic transformations. Mathematical thinking is more complex. It is also the ability to think in daily life.

Geometric thinking, which is part of mathematical thinking, is an individual's process of perceiving and interpreting shapes. The individual who encounters geometric concepts for the first time tries to make sense of them in his/her mind, analyzes the properties of shapes, divides shapes into groups. In short, he/she creates structures in his/her mind for geometric concepts and relationships. This process is the geometric thinking process. In geometric thinking, there is a transition from the physical world to the abstract world. According to Gündoğdu Alaylı (2012), geometric relations and the structuring of mathematical relations are related. All levels in mathematical thinking processes are also valid for geometric thinking.

In the literature (Alyeşil, 2005; Bobango, 1988; Fidan, 2009; Gündoğdu Alaylı, 2012; Olkun, Toluk & Durmuş, 2002; Özcan, 2012; Tuluk & Dağdelen, 2020, Usiskin, 1982) it is seen that the Van Hiele approach is generally based on the development of geometric thinking. Van Hiele (1986) states that geometric thinking in children occurs in five stages. These are visual level, analytical level, informal deduction (experience-based inference), formal deduction (inference) and the most advanced level. According to this theory, thinking proceeds sequentially. According to the first level, individuals can say the names of the shapes and make measurements. At the second level, the individual analyzes and explains the properties of shapes. At the third level, they can compare and classify shapes. At the fourth level, the individual can perform abstract thinking and make geometric proofs with the help of axioms. At the fifth level, he/she can put forward his/her own theorems. According to the studies in the literature (Breen, 2000; Gündoğdu Alaylı, 2012; Fidan, 2009; Fuys, 1985; Karakarçayıldız, 2016; Mistretta, 2000; NCTM, 2000; Özcan, 2012; Van de Walle, 2004), middle school students generally perform geometric thinking between the first and third levels.

In this study; the mathematical thinking processes of the students were examined with the theoretical framework of the three worlds of mathematics. The three worlds of mathematics deal with mathematical thinking in three phases. These phases are assumed to occur sequentially (Tall, 2007). Each thinking world represents the transition from concrete thinking to abstract thinking. According to this theory, the first stage of thinking is the embodied world. In this thinking world, the individual expresses the concrete properties of objects. The visual-spatial properties of the objects are in the foreground (Jukić & Brückler, 2014). The second stage of thinking is the proceptual world. In this world of thinking, there is concept thinking during operation. Expressing the concept with symbols includes algebraic thinking. The individual expresses the concept symbolically as a result of repetitive actions. The third stage of thinking is the formal world. We can find the proof here. The individual expresses the concepts with his/her own sentences, creates his/her own definitions of concepts as a result of mathematical proof.

Algebra and geometry are two related branches of mathematics. In the theoretical framework of the three worlds of mathematics, it is important to create symbolic expressions. Geometry topics also include topics that involve algebraic expression creation processes. One of them is the Pythagorean Theorem which is a subject where students need to use their knowledge of both geometry and algebra. According to Ramdhani & Suryadi (2018), students have difficulty in solving problems when they cannot comprehend the relationship between concepts in geometry and the Pythagorean Theorem. In this study, we focused on the Pythagorean Theorem which is among the 8th grade achievements in the Curriculum of the Ministry of National Education of the Republic of Turkey (MoNE, 2018). Therefore, examining the concept formation skills of 6th and 7th grade students about this theorem constitutes the originality of the research. In the study, the mathematical thinking processes of six secondary school students who have the same geometric thinking level were examined. The students were in 6th,7th and 8th grades. According to the theory of the three worlds of mathematics at the secondary school level, students cannot be expected to fully realize the formal world dimension. However, it is thought that students will be able to express the proof processes, albeit at a simple level. It is also envisaged that they can use their own definitions while expressing the concepts. This dimension was also considered in the study. It is thought that students have difficulties in transitioning from concrete thinking to abstract thinking. This idea forms the basis of the research. When the literature was examined, it was seen that the mathematical thinking processes of individuals were generally examined in terms of different theoretical frameworks such as APOS (Açan, 2015; Açıl, 2015; Akarsu, 2022; Hannah, Stewart & Thomas, 2016; Martínez-Planella & Triguerosb, 2019; Mudrikah, 2016), SOLO Taxonomy (Bağdat & Saban, 2014; Chan et al., 2002; Groth ve Bergner, 2006; Köse, 2018; Lucas & Mladenovic, 2008) and RBC Theory (Tsamir & Dreyfus, 2002; Türnüklü & Özcan; 2014; Yeşildere, 2006). When the studies which include three worlds of mathematics (Hannah, Stewart & Thomas, 2016; Jukić & Brückler, 2014; Kashefi, Ismail & Yusof, 2010; Kidron, 2008; Tall, Lima & Healy, 2014; Vandebrouck, 2011) were examined, no study was found that dealt with the theory of the three worlds of mathematics and geometric thinking together. Therefore, examining the theoretical framework of the three worlds of mathematics within the field of learning geometry emphasizes the importance of this research.

Method

Research Model

The research was designed with mixed pattern model. Quantitative and qualitative patterns were used together. Quantitative design was used to determine the participants of the study according to their geometric thinking levels. The singular screening method was determined as the quantitative research pattern. Qualitative design was used to determine students' mathematical thinking skills. The case study was chosen as a qualitative research pattern. Mathematical thinking skills of students were determined by semi-structured interview technique.

Participants, Sampling and Data Analysis

Purposive sampling method was used to determine the participants of the study. The criterion here is that students' geometric thinking levels are at the third level. In addition, considering that the algebraic thinking levels of the students may affect their mathematical thinking skills in geometry, the algebraic thinking levels of the students were examined.

"Algebraic Thinking Level Determination Test" developed by the researchers was used to determine the algebraic thinking levels of the students. This test is for middle school students and aims to measure the first three levels of algebraic thinking. The KR-20 reliability coefficient of the 27-question test, which was obtained as a result of validity and reliability studies, was determined as 0.86. The average difficulty index of the test was 0.60 and the average discrimination index was 0.54. The "Geometric Thinking Levels Scale" developed by Alyesil (2005) was used to determine the geometric thinking levels of the students. The scale was prepared using the Van Hiele Geometry Test and consisted of 20 questions. The alpha-confidence coefficient was 0.81. It was accepted that students correctly answered four of the five questions at each level, indicating that level of thinking. These scales were applied to 2808 secondary school students in Izmit District of Kocaeli Province. 1071 students were 6th grade students, 996 students were 7th grade students and 771 students were 8th grade students. In the study, 6 students whose geometric thinking level was determined as the third level were selected. The grades in the Year-End Report of the previous year were taken into account in determining the grade point averages of the mathematics courses. Information about these students is shown in Table 1.

Table 1. Information about the participants of the study

Student's Name	Grade Level	Algebraic thinking level	Grade Average
Rana	6	1	86
Sarp	6	3	100
Metin	7	1	67
Ilgın	7	3	100
Tan	8	1	55
Asaf	8	3	100

Note. Pseudonyms were used in student names.

Data Collection and Application

The processes of forming the Pythagorean Theorem were discussed in determining the mathematical thinking skills of the students. In this sense, semi-structured interview questions were prepared. Interviews were held with each student in two separate sessions. In the first session, the theorem was presented to the students verbally. The aim here was for students to show the proof process based on the verbal situation. In the second session, a real-life situation was presented to the students, in which they were expected to form the theorem, and they were expected to show the dimensions of the three worlds of mathematics in the process. Students were asked questions about each thinking dimension. The real life situation given is as follows:

"In Efe's project assignment, his teacher asked him to design a house model. Efe wants the house to be designed in a very different model and to attract attention. Efe thinks for days and decides to design a house with 4 rooms. The features of the house are as follows:

- Three rooms will be square
- One wall of each of the square-shaped rooms will be perpendicular to each other
- These three rooms will overlap each other two by two from one corner
- The walls of the fourth room should be in common with one wall of each of these three rooms and this room should be located between them."

In the evaluation of the interviews, classifications were made by the researchers about the levels of the three worlds of mathematics. Expert opinion was taken for these classifications and it was decided that they were suitable for the theoretical framework. The thinking processes determined according to the three worlds of mathematics are as in Table 2.

Table 2. General framework of mathematical thinking processes in terms of the theoretical framework of the three worlds of mathematics

Dimensions of the three worlds of mathematics	Abilities
The Embodied World	Focus on the properties of the data in the expression Describing the concept, expressions or objects Demonstrating expressions by testing them with numbers
The Proceptual World	Reaching algebraic expressions, equations and inequalities as a result of the generalization process of mathematics
The Formal World	Expressing the proof process

While determining the general framework for the thinking processes about the three world dimensions of mathematics, the literature (Tall, 2007) was adhered to. If students showed at least one process in each dimension, their thinking processes were evaluated in that dimension.

In this study, which was carried out by obtaining the necessary permissions, the audio recordings of the interviews were transferred to the computer by the researchers. During the document review, the level of the three worlds of mathematics that the student showed in each question in the thinking process was marked and thus the dimension he/she reached was determined. The diversification method (Creswell, 2013) was used in the process of ensuring the validity and reliability of the semi-structured interview process. Documents and audio recordings of the interviews were examined and evaluated together. The solution papers of the students became an element that increased the validity in the interview process. In the evaluations of the researchers regarding the questions, it was determined that the Miles & Huberman (1994) agreement percentages were over 70%. This situation has been interpreted as a high consensus among researchers.

Results

In the semi-structured interviews held in this section, the mathematical thinking processes of the answers given by the students were evaluated. Firstly, the Pythagorean Theorem was presented to the students verbally. The evaluation of the answers given by the students participating in the study is presented in Table 3.

Table 3. Mathematical thinking dimensions of students for verbal expression of the Pythagorean Theorem

Dimensions of the three worlds of mathematics	Abilities	Rana	Sarp	Metin	Ilgın	Tan	Asaf
The Embodied World	Focus on the properties of the data in the expression	X	X	X			X
	Describing the concept, expressions or objects		X			X	X
	Demonstrating expressions by testing them with numbers	X	X		X	X	X
The Proceptual World	Reaching algebraic expressions, equations and inequalities a result of the generalization process of mathematics	X		X	X	X	X
The Formal World	Expressing the proof process				X		

In this part of the study, Rana drew a right triangle and showed its right angle and right sides. She tried to explain the expression by giving her own number values to the perpendicular side lengths. However, she had difficulties because the number value she obtained for the third side length was not a perfect square number. Because she could not explain which number is the square of this number. This is a normal situation since there are no acquisitions of square root expressions in the 6th grade curriculum. Despite all this, Rana was able to create a symbolic expression suitable for verbal expression. Therefore, her thinking process was evaluated in the proceptual world dimension. The interview process is as follows.

Rana: When we draw a right triangle, these sides become perpendicular. The steep sides are here and here. Now, let's get 7 here, and this is 4. The thing of 7 to the square of 7, 7 times 7 is 49. 4 to the power square also equals 4 times 4, is 16. When we add 49 and 16, it becomes 65. That's 4 squared by 7.

Researcher: Can you show it algebraically?

Rana: We can say that a squared, a squared plus b squared is equal to c.

Researcher: Is it equal to c?

Rana: It should be equal to its square. c squared.

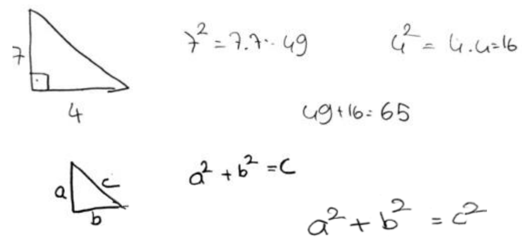

Figure 1. Symbolic representation of Rana for the verbally given Pythagorean Theorem

Sarp's level of thinking was limited to the embodied world dimension. Sarp was able to say that the concept of exponent was explained with the concept of square in the expression. He drew a right triangle and gave numerical values to the side lengths. While showing the expression in the question, he added the squares of the three side lengths and thought about how he should interpret the result he obtained. However, he could not explain. When he was asked to express the expression symbolically, he equated the sum of the squares of all the side lengths of a right triangle to the square of numbers. As a result, he made a mistake when he wanted to create a symbolic expression from the verbally given expression.

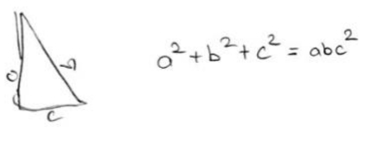

Figure 2. Symbolic representation of Sarp for the verbally given Pythagorean Theorem

Metin, illustrated by drawing the right triangle and right sides that explained to him in the verbal expression of the Pythagorean Theorem. He gave letters to the side lengths of the triangle. He was able to show the verbal expression symbolically. However, he did not give any explanation in terms of the accuracy of the statement. The thinking process was evaluated in the proceptual world dimension.

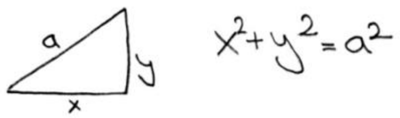

Figure 3. Symbolic representation of Metin for the verbally given Pythagorean Theorem

Ilgın, wanted to explain the expression by trying it with numbers first. Then she thought it would be more accurate to draw a right triangle and measure its sides. With the guidance of the researcher who conducted the interview, she drew a right triangle with vertical side lengths of 3 cm and 4 cm. She measured the third side of the triangle to be 5 cm. It demonstrated the numerical accuracy of the expression. She drew a right triangle again and gave letters to the side lengths. She was able to write a symbolic representation suitable for the expression. Thus, she developed a thinking process towards the formal world. Ilgın explained the verbal expression of the Pythagorean Theorem with the direction of the researcher. The interview process is as follows.

Ilgın: Can I draw a right triangle and show it? (She draws). This is 90 degrees. The steep sides are here and here. can I measure?

Researcher: Yes. But shall we do this? Let one of the perpendicular side lengths be 4 cm and the other 3 cm.

Ilgın: Okay, now let's draw it like this. This is 4 cm. This is 3 cm. If I measure here too, it's 5 cm. Now 3 times 3 I mean 3 squared is 9. 4 squared is 16. 16 plus 9 equals 25. And 5 squared makes 25. 25 equals 25 then this statement is true.

Researcher: Well, can you express it in algebraic notation?

Ilgın: The sum of a squared and b squared is equal to c squared.

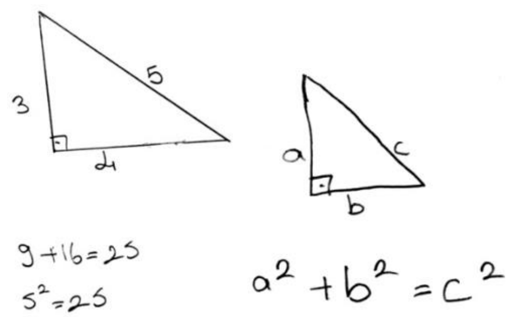

Figure 4. Symbolic representation of Ilgın for the verbally given Pythagorean Theorem

In this question about the Pythagorean Theorem, Ilgın wanted to develop a proof process that could explain the expression by drawing a right triangle and measuring the side lengths. The reason why the researcher directed Ilgın, who wanted to explain the accuracy of the expression, is because there is a possibility of reaching an expression with square roots. The subject of square root expressions is included in the 8th grade for the first time in the curriculum. In this sense, the researcher felt the need to guide the student.

Tan drew a right triangle and gave numerical values to the side lengths. He remembered that the expression told the Pythagorean Theorem. He continued to explain the statement instead of demonstrating its veracity. He was also able to write the appropriate relation for the expression. Therefore, the thinking process was evaluated in the proceptual world dimension.

Asaf stated that this statement was wrong, saying that the sum of the squares of the lengths of the right sides of a right triangle should be equal to the square of the length of the hypotenuse. He later realized that the length of the third side would be the hypotenuse. First, he wrote the appropriate correlation for the expression, then tried to explain it with a 3-4-5 triangle. However, he could not explain why the statement was true, and the thinking process was evaluated in the proceptual world dimension.

In the other part of the study, the real-life situation of discovering the Pythagorean Theorem was presented to the students. The mathematical thinking skills demonstrated by the students are presented in Table 4.

Table 4. Mathematical thinking dimensions of students to discover the Pythagorean Theorem in a real-life situation

Dimensions of the three worlds of mathematics	Abilities	Rana	Sarp	Metin	Ilgın	Tan	Asaf
The Embodied World	Focus on the properties of the data in the expression	X	X	X	X		X
	Describing the concept, expressions or objects	X	X	X	X	X	X
	Demonstrating expressions by testing them with numbers				X	X	X
The Proceptual World	Reaching algebraic expressions, equations and inequalities as a result of the generalization process of mathematics				X	X	X
The Formal World	Expressing the proof process				X	X	X

Rana first focused on the desired data in a real-life situation. Trying to draw a shape suitable for the data, she said that the part in between could be a triangle. However, she was unable to accurately draw the appropriate shape. Even though the part in between was a triangle as she drew, she could not show the other shapes as squares. Therefore, she could not develop a thinking process for the Pythagorean Theorem. The thinking process was evaluated in the embodied world dimension.

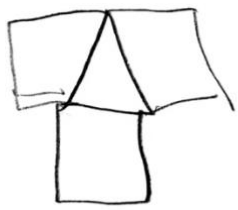

Figure 5. The process of creating the Pythagorean Theorem in Rana's real life situation

Sarp had to read the question several times to understand it. When he evaluated the data, he thought that the fourth room in between could be a right triangle. However, he gave up on this idea because he could not draw the right shape. Sarp, who thought that the room in between could be square, finally decided that this house could not be designed. The thinking process was evaluated in the embodied world dimension.

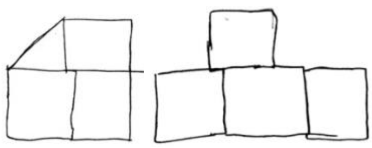

Figure 6. The process of creating the Pythagorean Theorem in Sarp's real life situation

Metin has read the statement many times, focusing on what is given. Then he wanted to draw the figure by trying to explain what was given. He tried to provide all the necessary conditions for the house by drawing more than one shape. He thought that the fourth room might be square, and he could not think about creating the Pythagorean Theorem. The thinking process has been limited to the embodied world dimension.

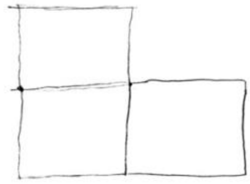

Figure 7. The process of creating the Pythagorean Theorem in Metin's real life situation

When Ilgın read the expression, she interpreted that the squares are perpendicular to each other as they must be parallel. Then she realized the mistake she made and developed a thinking to meet all the conditions. Ilgın was able to draw the correct shape. She stated that the fourth room in between would be a right triangle. Then, when asked to think of two perpendicular squares with a side of 3 cm and a side of 4 cm, she drew two squares of these lengths using a ruler.

She measured the side length of the third square as 5 cm. When she asked to find the area of the figure, she first calculated the areas of the squares. She explained the relationship between the areas of three squares as the sum of the areas of two squares equals the area of the third square. She was able to discover that the sum of the squares of the lengths of the two perpendicular sides of a triangle is equal to the square of the length of the third side. When she was asked what the third side length would be for a right triangle with side lengths of 6 cm and 8 cm, she said that the number 100 was obtained from the sum of its squares. She explained that the number 100 would be the square of this length. She was able to give the correct answer that it would be 10 cm. When she was asked to generalize this situation, she was able to express the Pythagorean Theorem symbolically. Thus, she developed a thinking process towards the formal world dimension.

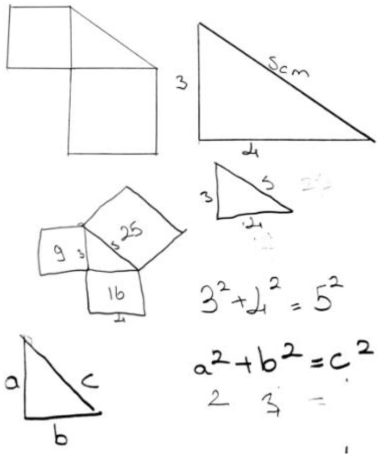

Figure 8. The process of creating the Pythagorean Theorem in Ilgın's real life situation

When Tan read the statement, he thought that the fourth room would be square. However; considering that the three rooms would overlap each other at the corners, he stated that the room in between could be an equilateral triangle. Then, when he evaluated that the two rooms should be perpendicular to each other, he said that the fourth room should be a right triangle. He was able to draw the correct shape. He stated that when one of the square rooms perpendicular to each other has a side length of 3 cm and the other a side length of 4 cm, he must use the Pythagorean Theorem to find the side length of the third square. He said that the side length of the third square would be 5 cm. When he asked why he took the squares, he replied that the sum of the areas of the given squares would equal the area of the third square. Tan, who explained the reason for a knowledge he had learned before and created the Pythagorean Theorem, carried out the formal world dimensional thinking process.

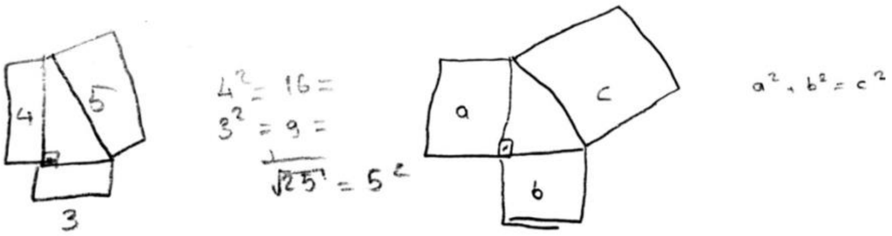

Figure 9. The process of creating the Pythagorean Theorem in Tan's real life situation

Asaf first explained what was requested. Then he drew the figure correctly and stated that the figure in between should be a triangle. When one of the squares perpendicular to each other has a side length of 3 cm and the other square has a side length of 4 cm, he measured the side length of the third square by drawing it with a ruler. He found the side length of the third square as 5 cm and said that this triangle is a right triangle since it is a special triangle of 3-4-5. Then, he developed a thinking process to explain the Pythagorean Theorem by making a connection between the side lengths of the triangle and the areas of the squares. The thinking process of Asaf, who formed the Pythagorean Theorem correctly, was evaluated in the formal world dimension.

The interview process is as follows:

Asaf: He will now design a 4 bedroom house. The three rooms of the house will be square and overlap one at a time. The fourth room will be in between. If I draw, the part in between would be a triangle. There are 4 rooms, three of them must be square and one of them must be triangles. I think it's a right triangle.

Researcher: What can you say about the side length of the third square if one of the perpendicular squares has a side of 3 cm and the side of the other is 4 cm?

Asaf: (Drawing and measuring). It should be 5 cm. Hmm, it's a 3-4-5 triangle.

Researcher: Is there a relationship between the areas of the squares?

Asaf: The side of this is 3 times 3, equals to 9. The area of this is 4 times 4, equals to 16. The area of this is 5 times 5, equals to 25. Hmm, the sum of the areas of the squares that are perpendicular is equal to the area of it. Then the sum of the squares of the right sides of the right triangle must be equal to the square of the other side. This is Pythagoras.

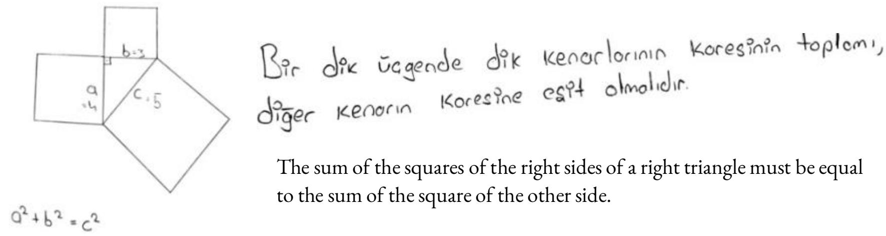

Figure 10. The process of creating the Pythagorean Theorem in Asaf's real life situation

Conclusion and Discussion

Students have difficulties in transitioning from concrete thinking to abstract thinking (Arslan & Yıldız, 2010; Keskin, Akbaba Dağ & Altun, 2013; Yeşildere & Türnüklü, 2007). One of the biggest challenges in mathematics education is the process of making sense of abstract concepts. In this process, we first use our concrete perceptions. Then, we evaluate abstract concepts within our existing mental schemes. The Pythagorean Theorem is a geometric construct involving abstract and symbolic thinking. In this study, the processes of students to form the Pythagorean Theorem were examined. In this process, mathematical thinking skills were evaluated. Mathematical thinking skills of 6th, 7th and 8th grade students with the same geometric thinking level were compared. When the relevant literature is examined (Açıl, 2015; Akarsu, 2022; Bağdat & Saban, 2014; Köse, 2018; Mudrikah, 2016; Türnüklü & Özcan; 2014), it is seen that the mathematical thinking processes of individuals are generally examined in terms of different theoretical frameworks such as APOS, SOLO Taxonomy and RBC Theory. In this study, the three worlds of mathematics were used as the theoretical framework.

In the study, geometric thinking levels were taken as a criterion in the selection of students. The geometric thinking levels of the six participants with whom the study was conducted together were determined as the third level. In addition, the algebraic thinking levels of the students were discussed. According to the theoretical framework of the three worlds of mathematics, mathematical thinking takes place in three stages (Tall, 2007). These stages follow a path from concrete to abstract. The reason why students' geometric thinking levels were chosen as third level in this study is to evaluate the progress of their thinking processes towards the formal world dimension. At secondary school level, students are not expected to be able to prove the equation fully. However, it was thought that they could show the proof process within their thinking process. Therefore, the results of the study also support this idea. In the thinking processes developed based on the verbal expression of the Pythagorean Theorem, the thinking levels of 6th and 8th grade students could not reach the formal world dimension. The 6th grade student, whose algebraic thinking level is at the first level, was able to create symbolic expression based on verbal expression. Since the 6th grade level is the grade level in which the subject of algebraic expressions is introduced, it is one of the important results of the research that the student can create symbolic expressions in accordance with the theorem. The 6th grade student, whose algebraic thinking level is at the third level, could not correctly show the symbolic expression for the verbal expression of the theorem. 7th and 8th grade students were able to create symbolic expressions suitable for verbal expression. In addition, the 7th grade student, whose algebraic and geometric thinking level was at the third level, was able to realize formal world dimensional thinking process.

In the process of creating the Pythagorean Theorem based on the real life situation, the thinking levels of the 6th grade students were limited to the embodied world dimension. The 7th grade and 8th grade students, whose algebraic thinking level was at the third level, were able to show all the thinking processes of the three worlds of mathematics. The result of the study shows that a 7th grade student with a third level of algebraic and geometric thinking was able to develop a thinking process towards the formal world dimension on a subject that he had not learned before. In the study, it was concluded that as the grade level increased, the thinking skills of the students also increased. In addition, students who could not develop a proof process in the oral expression of the Pythagorean Theorem were able to construct the theorem in real life situations. This showed that students were able to use their mathematical thinking skills more accurately in real-life situations.

Another result of the research is that 8th grade students have more problems in interpreting verbal expression despite being taught about the Pythagorean Theorem. They were able to construct the theorem in a real-life situation. Therefore, although the students stated that the verbal expression was the Pythagorean Theorem, their inability to explain the theorem led to the thought that they could not achieve permanent learning. The students memorized the application of the theorem in the context of the operational process. In the real-life situation, however, since no information about the Pythagorean Theorem was presented in the real-life situation, the process of discovering the theorem was able to be realized. In this sense, learning environments should be provided in schools where students can develop a thinking process towards the formal world dimension.

In this study, it was observed that the students were able to draw appropriate geometric shapes regarding the verbal instructions given in real life situations. This situation was interpreted as the fact that their geometric thinking levels were at the third level. As the grade level increased, the students' ability to create appropriate symbolic expressions increased.

In this study, it was observed that the difficulties experienced by the students in geometry affected their ability to develop the proof process. However, the study showed that middle school students were able to perform the proof process in geometry. In the study of Miyazaki et al. (2017), secondary school students were able to express the proof process in geometry. In particular, they stated that students with third and fourth level geometric thinking were more successful.

As a result of the research, it is seen that it is important to design activities and course environments that will reveal students' thinking skills for mathematical thinking processes. In the verbal expression, while the students were trying to understand the concepts, they carried out a thinking process to create the concepts in a real life situation. Verbal expression helped them create symbolic expressions about concepts. They had difficulty in carrying out the proof process. Therefore, it is thought that to enable students to discover the concepts in the activities instead of presenting them directly in the teaching process is important. According to Fidan and Türnüklü (2010), instead of giving the geometric concepts to the students directly, the students should be encouraged to find and create these concepts and they should be given education according to their level. Measurement and evaluation processes can also be assessed in this context.

Recommendations

With this research, the applicability of the theoretical framework of the three worlds of mathematics on geometry was discussed. In future studies, the applicability of the theoretical framework can be evaluated within other fields of mathematics. In addition, since thinking will progress in the process according to the three worlds of mathematics, studies that examine the thinking process in terms of grade levels can also be conducted.

Limitations of Study

The limitation of this study is that the fifth grade students, who are among the secondary school students, were not selected among the participants of the study. Because the achievements of algebra in the mathematics curriculum start from the sixth grade level. Since there is no algebraic achievement at the fifth grade level, fifth grade students were not selected as participants.

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Biodata of the Authors

Dr. **Esra Akarsu Yakar** completed her master's and doctorate in Dokuz Eylül University Institute of Educational Sciences, department of mathematics education. Her research interests are in mathematical thinking, the three worlds of mathematics, procept theory, mathematical language, algebraic and geometric thinking. Affiliation: Ministry of National Education. E-Mail: es.akarsu@gmail.com, ORCID: 0000-0002-4090-6419

Prof.Dr. **Suha Yilmaz** continues his academic life in Dokuz Eylül University, Buca Faculty of Education, Department of Mathematics and Science Education. His research interests are differential geometry, mathematical thinking, spatial thinking, mathematical modeling and technology in mathematics education. Affiliation: Dokuz Eylul University, Buca Faculty of Education, Department of Mathematics and Science Education. ORCID: 0000-0002-8330-9403

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