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Generation of Multistability through Unstable Systems

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ABSTRACT In this work, we propose an approach to generate multistability based on a class of unstable systems that have all their roots in the right complex half-plane. Multistability is the coexistence of multiple stable states for a set of system parameters. The approach is realized by using linear third order differential equations that consists of two parameters. The first bifurcation parameter transforms the unstable system with all its roots in the right complex half-plane into an unstable system with one root in the left complex half-plane and two roots remaining in the right complex half-plane. With this first transformation, the system is capable of generating attractors by means of a piecewise linear function and the system presents monostability. We then use the another bifurcation parameter to switch from a monostable multiscroll attractor to several multistable states showing a single-scroll attractor.

KEYWORDS
Chaos
Multistability
Switched systems
Attractors

INTRODUCTION

The coexistence of two or more attractors, for a given set of parameters, is called multistability and the convergence to one of the different attractors depends only on the initial condition. The coexistence of multiple behaviors is a universal phenomenon found in many area of science and in nature, from electronic devices and chemical reactions to weather and the brain. One of the pioneering studies on multistability was reported on visual perception (Attneave 1971). In electronic devices, the phenomenon of bistability has been widely explored and its applications in technological devices such as cell phones, computers, etc. Many studies have reported various multistability phenomena through different types of systems, for example, through coupled systems, delayed feedback systems, stochastic systems, among others (Feudel 2008).

There are different mechanisms for multistability emergence in dynamical systems. We can find in the reported literature the generation of multistable systems via Unstable Dissipative Systems (UDS). These dissipative systems with unstable dynamics defined in the space can be of type I or II. The system based on UDS-type I presents a one-dimensional stable manifold leading the trajectory to the equilibrium point and a two-dimensional unstable manifold leading the trajectory away from the equilibrium point. The UDS-type II presents a two-dimensional stable manifold and a one-dimensional unstable manifold, see Anzo-Hernández *et al.* (2018), Gilardi-Velázquez *et al.* (2017). In this work we focus on the creation of multistable systems from an proposed unstable system which is transformed into a system UDS-I.This class of systems UDS-I are useful for the generation of multiscroll attractors through a linear function by parts (PWL), and thus through a bifurcation parameter to obtain the multistability.

Recently, the fractional calculus has been used to make multistable systems based on fractional derivatives instead of integer derivatives in PWL systems that display multiple scrolls Echenausía-Monroy *et al.* (2022). The mechanisms that produce multistable behavior in integer and fractional PWL systems are currently a topic of research. In Gilardi-Velázquez *et al.* (2022) a PWL system showing multistable behavior was presented, in this system the nearest integer function was used to control the switching processes and the corresponding equilibrium between the individual switching surfaces was found.

In this paper our approach to generate chaotic systems is based on an transformation of unstable system to a class of *unstable dissipative systems* (UDS). Therefore, the chaotic self-excited attractor emerges from saddle equilibria. The manifolds of these hyperbolic equilibrium points are determined by the eigenvalues associated to the linear operator of the system. Therefore, the first requirement is to obtain a system based on a linear operator that has a complex conjugate eigenvalue pair with a positive real part and a negative real eigenvalue. With this first transformation, the system

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is capable of generating a monostable attractor. The next step is to switch the system from monostability to multistability.

The manuscript is organized in the following sections: In the first section , definitions are given for clarity in developing the approach to obtaining multistability. In the second section , we classify the types of unstable systems capable of generating families of systems that can generate multiscroll. In the third section by using another bifurcation parameter, we generate multistability in the given system. Finally, in the last section, the conclusions are given.

PRELIMINARIES

In this section, we give some basic definitions to understand the coexistence of multiple chaotic attractors. Two important features of chaotic system solutions are unpredictable behavior and trajectory divergence. A chaotic system usually displays the types of behavior listed above. The following definitions can be reviewed in Lynch (2004).

Definition 1 *A minimal closed invariant set A, $F(A) \subset A$, that attracts nearby trajectories that are in the basin of attraction B, $A \subset B$, towards it is called an* attractor*.*

Definition 2 *A* strange attractor *generated by a chaotic system is an attractor that shows fractal structure and sensitivity to initial conditions.*

There are many approaches to check that a system has chaotic behavior. For instance, *Lyapunov exponents* is a widely used method to verify whether a system is chaotic or not. A chaotic system presents a positive Lyapunov exponent and if the system presents two positive Lyapunov exponents it is called hyperchaotic. Other approach to generate chaotic behavior is by using homoclinic and heteroclinic orbits, this chaotic behavior is known as homoclinic and heteroclinic chaos. Two chaotic trajectories with very close initial conditions in the strange attractor will separate with a rate of divergence given by the positive Lyapunov exponent.

Lyapunov exponents can be computed by different methods and their performance can be consulted in Geist *et al.* (1990). Three-dimensional autonomous systems have been useful for modeling many phenomena of nature. For example, the highly simplified model of a convective fluid proposed by Edward Lorenz to generate meteorological data Lorenz (1963). A wide variety of behaviors was discovered in the simplified Lorenz model, finding that for some parameter values the system behave chaotically. Sparrow's work on the Lorenz system is an excellent reference for more details about the system, see Sparrow (1982).

The trajectories revolve around two equilibrium points C_1 and C_2 in an apparently stochastic way, which makes the trajectories unpredictable. This pioneer model has been widely study, for example, Guanrong Chen and Tetsushi Ueta introduced a variation on the Lorenz model. Another iconic chaotic system was introduced by Leon O. Chua in the mid-1980s, his system was implemented electronically and is known as the Chua circuit, and it exhibits a variety of behaviors. A review of Chua's circuit is presented in Madan (1993) and exhibits many interesting bifurcation and chaotic phenomena.

As the Lorenz system as the Chua system generate attractors that present a double scroll attractor. In this paper, one of our objectives is to *control* an attractor that present five scrolls, we called this attractor *multiscroll attractor*. One of the approaches used to generate a multiscroll attractor is by means the use of a piecewise linear function in the system $\dot{\mathbf{x}} = A\mathbf{x} + B$, see Campos-Cantón *et al.* (2010, 2012).

Dissipative systems and attractors

Let us consider a system given a set of nonlinear autonomous differential equations, as follows

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^3$ is the state vector. Important information is obtained by the equilibrium points $\mathbf{x}^*$ that satisfy $F(\mathbf{x}^*) = 0$. These points give qualitative information about the local behaviors of its solutions. The local behavior of a nonlinear system is obtained by the Jacobian matrix $DF(\mathbf{x}^*)$. For a *hyperbolic equilibrium point*, the eigenvalues of the Jacobian matrix $DF(\mathbf{x}^*)$ are nonzero. Hartman-Grobman Theorem states that in a vicinity of a hyperbolic equilibrium point $\mathbf{x}^*$ in which the phase portrait for the nonlinear system $\dot{\mathbf{x}} = F(\mathbf{x})$ resembles the linearization. The linearization is given as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= DF(\mathbf{x}^*) \\ &= A\mathbf{x}. \end{aligned} \tag{1}$$

What we have is that the phase portraits are qualitatively equivalent in the neighborhood of a hyperbolic critical point, see Hartman (1964). Other important concept is the volume contraction rate of a dynamical system

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $\mathbf{x} = (x, y, z)^T$ is the state vector and $F(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$ determines the evolution of the system, then the volume contraction rate is given by:

$$\Lambda = \nabla \cdot F(\mathbf{x}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Notice that the time evolution in phase space is determined by $V(t) = V_0 e^{\Lambda t}$, where $V_0 = V(0)$ and Λ is a constant. When a system is capable of dissipating energy, it is known as a dissipative system and is given for a negative value of Λ. When Λ is negative it leads to a fast exponential shrinks of the volume in state space. If the system is dissipative, it can develop attractors. Without loss of generality we analyze a jerk system. On the other hand, for the given system to be dissipative, it is necessary and sufficient that the sum of the roots of the characteristic polynomial be a negative quantity. That is, the eigenvalues associated with the matrix $A \in \mathbb{R}^3$ of the system (1) is $\sum_{i=1}^{3} \lambda_i < 0$. The saddle equilibria of a system in $\mathbb{R}^3$ can be characterized into two types according to the eigenvalues associated with matrix A.

Definition 3 *Let us consider a system defined by (1) in the space with eigenvalues λ_j, $j = 1, \ldots, 3$ associated with matrix A. The system is called* Unstable Dissipative System *(UDS) Type I if one eigenvalue $\lambda_1 \in \mathbb{R}^-$, the other two $\lambda_{2,3} \in \mathbb{C}^+$, and the sum of the eigenvalues is negative. Where $\mathbb{R}^-$ and $\mathbb{C}^+$ denote the negative real numbers and complex conjugate numbers with a positive real part, respectively.*

Definition 4 *Let us consider a system given by (1) in $\mathbb{R}^3$ with eigenvalues λ_i, $i = 1, 2, 3$ associated with matrix A. The system is said UDS Type II if one eigenvalue λ_1 is positive real number and the other two are complex conjugate numbers with a negative real part, and the sum of the eigenvalues is negative.*

Attractors in $\mathbb{R}^3$ are generated by several kind of dynamical systems, and particularly PWL systems based on the aforementioned two types of UDS have been employed to generate multiscroll attractors. Also some systems present the two types of UDS's to generate attractors, For example, the Chua system mentioned above considers two UDS Type I equilibria to generate the scrolls of the attractos and another UDS Type II equilibrium point between the two UDS Type I equilibria. This last equilibrium point does not generate a scroll in the attractor.

FROM UNSTABLE SYSTEMS TO SYSTEMS THAT GENERATE ATTRACTORS

In this section we generate attractors that present multiscroll. In this work we are interested in continuous piecewise functions as controllers for the generation of multiscroll atractors. We use a similar technique as in Díaz-González *et al.* (2017) where a bifurcation parameter is used to generate a family of multiscroll attractors. The idea is to destabilize Hurwitz polynomial for the generation of a class of systems that display multiscroll attractor based on unstable dissipative systems. One of our goals in this paper is to explain how to transform a totally unstable system into an unstable system capable of generating attractor through a piecewise linear function.

Unstable systems

Let us propose the following controlled linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u(r,\nu) + B(\nu)S, \tag{2}$$

where $A \in \mathbb{R}^{3\times 3}$ is a linear operator, $\mathbf{b} = (0,0,1)^T$ is a constant vector, and $B = (0,0,\nu)^T$ with $\nu \in \mathbb{R}$. We apply a feedback of the form

$$u(r,\nu) = \mathbf{c}^T(r,\nu)\cdot\mathbf{x},$$

where $S = S(\mathbf{x})$ is the following step function:

$$S = \begin{cases} s_1 & \text{for } c_1 < x, \\ s_2 & \text{for } c_2 < x \le c_1, \\ \vdots & \\ s_m & \text{for } x \le c_m, \end{cases}$$

leads

$$\dot{\mathbf{x}} = A(\nu)\mathbf{x} + B(\nu)S. \tag{3}$$

which describes a closed-loop or feedback system. The way to select the values of $c_i's$ in such a way that they help to develop UDS systems will be explained below. For the generation of multiscroll attractors we take $\nu = 1$ to analyze the location of the parameter r. Our goal is to set the appropriate value of the parameter r in such a way that the control linear system generates multiscroll attractors. Firstly we assume that $u(r,\nu)$ equal to 0, $B = 0$ and $\text{Re}(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of eigenvalues of A. If *all* eigenvalues of A have positive real part, then the system is totally unstable. For the system (2), the equilibrium point 0 has a neighborhood such that every nonzero solution that starts in the neighborhood must eventually leave the neighborhood and not return in the future time. Without loss of generality we consider the matrix form of the third-order jerk equation $x''' + a_1x'' + a_2x' + a_3x + \beta = 0$, where $a_1, a_2, a_3, \beta \in \mathbb{R}$. The generated system is of the form (2) with $u \equiv 0$ where the matrix A has the following form:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} \tag{4}$$

where $B = (b_1, b_2, b_3)^T$ is a vector with the following entries $B = (0,0,-\beta)^T$. The characteristic polynomial associated with A is defined by $p(t) = t^3 + a_1t^2 + a_2t + a_3$. We are going to characterize the unstable systems that can generate attractors through a bifurcation parameter. We begin with a test to characterize the roots of a polynomial of degree three.

Lemma 1 *The polynomial $p(t) = t^3 + a_1t^2 + a_2t + a_3$ has a positive real root and two complex conjugate roots $\alpha \pm \beta$ with $\alpha > 0$ and $\beta \neq 0$ if and only if $4a_2^3 + 27a_3^2 + 4a_1^3a_3 - a_1^2a_2^2 - 18a_1a_2a_3 > 0$.*

Proof 1 *Consider the polynomial $p(t)$given by the following form $p(t) = t^3 + a_1t^2 + a_2t + a_3$, we define $\Delta = 4a_2^3 + 27a_3^2 + 4a_1^3a_3 - a_1^2a_2^2 - 18a_1a_2a_3$. The proof is obtained from Cardano's formulas to obtain the roots of a cubic equation, see Uspensky (1987).*

Example 1 *If we consider the following polynomial $p(t) = t^3 - 0.86t^2 + 2.65t - 0.24$, we can see that it satisfies $4a_2^3 + 27a_3^2 + 4a_1^3a_3 - a_1^2a_2^2 - 18a_1a_2a_3 = 61.56 > 0$. That is, $p(t)$ satisfies condition from Lemma 1, so it has one real positive root and two roots in the form $\alpha + i\beta$ with $\alpha > 0$ and $\beta \neq 0$.*

Instability parameter to generate unstability and multiscrolls attractors

We will use a polynomial approach that will help us to find bounds that will allow us to obtain the necessary instability in UDS-I to generate attractors by using an instability parameter. The unstability parameter of $p(t)$ is set according to the following definition.

Definition 5 *Let $p(t)$ be the characteristic polynomial of A and t_1, t_2, $\dots$, t_n are its zeros in the complex right half-plane ($\mathbb{C}^+$). The* abscissa of instability *σ_p of the polynomial $p(t)$ is defined as follows*

$$\sigma_p = \min_{1\le i\le n}\{\text{Re}\,(t_i)\}. \tag{5}$$

If $\underline{\sigma_p}$ and $\overline{\sigma_p}$ are numbers such that $\underline{\sigma_p} \le \sigma_p \le \overline{\sigma_p}$, then they are named lower and upper bound, respectively.

Now we are going to follow a similar approach to Aguirre-Hernández *et al.* (2015) where a polynomial approach is given. The characteristic polynomial of the system (3) is given by $f_r(t) = p(t-r)$, note that $f_r(t)$ is a set of polynomials such that $f_0(t) = p(t)$ is an unstable polynomial. Now by Taylor's theorem $f_r(t)$ can be rewritten as

$$\begin{aligned} f_r(t) &= t^n + \frac{p^{(n-1)}(-r)}{(n-1)!}t^{n-1} + \dots + \frac{p'(-r)}{1!}t + p(-r) \\ &= t^n + A_{n-1}(r)t^{n-1} + \dots + A_1(r)t + A_0(r). \end{aligned} \tag{6}$$

Our goal is to generate the chaotic behavior with a translation of the characteristic polynomial with an upper bound of the abscissa of instability. The roots of $f_r(t)$ are in the imaginary axis when $r = -\sigma_p$. Therefore the system (2) could generate multiscroll attractors for r in an interval contained in $(-\overline{\sigma_p}, -\sigma_p)$, where $\overline{\sigma_p}$ is an upper bound of the abscissa of instability.

Maximal instability interval

It is important to know the maximum range of the r parameter in order to control that the system remains UDS-I. In addition, a necessary condition for generating attractors is that system (2) satisfies the condition of dissipativity. Thus we have the following lemma for a system of dimension three.

Lemma 2 *Let $p(t) = t^3 + a_1t^2 + a_2t + a_3$ be a real unstable characteristic polynomial with roots $t_1 = \alpha_1$, $t_2 = \alpha_2 + i\beta_2$ and $t_3 = \alpha_2 - i\beta_2$, ($\alpha_2 > \alpha_1$), in the complex right half-plane. If $f_r(t) = p(t-r)$ is unstable and dissipative, then we have that the following conditions are fulfilled.*

i) $r < \frac{a_1}{3}$.

ii) $r > -\alpha_2$.

Proof 2 *If the sum of its roots, $r + t_j$, is negative, then the family $f_r(t)$ is dissipative if the sum of its roots. So, $2\alpha_2 + \alpha_1 + 3r < 0$ and hence $r < \frac{-2\alpha_2-\alpha_1}{3}$. On the other hand*

$$\begin{aligned} p(t) &= t^3 - (\alpha_1 + 2\alpha_2)t^2 + (2\alpha_1\alpha_2 + \alpha_2^2 - \beta_2^2)t - \alpha_1\alpha_2^2 + \alpha_1\beta_2^2 \\ &= t^3 + a_1 t^2 + a_2 t + a_3. \end{aligned}$$

Therefore we have that $r < \frac{a_1}{3}$. (ii) The roots of $f_r(t)$ are in the imaginary axis for $r = -\sigma_u$, if $\alpha_2 + r > 0$, then $f_r(t)$ is UDS-I. So $r > -\alpha_2$.

Let us summarize all of this in a theorem.

Theorem 1 *Let $p(t)$ be the unstable polynomial of degree three with a pair of conjugate complex roots and one real root as in Lemma 2. Then $f_r(t)$ is UDS-type I if and only if $r \in (-\alpha_2, \frac{a_1}{3})$.*

Proof 3 *The proof follows from Lemma 2.*

Example 2 *Consider the system given by (2) for*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.15 & -2.36 & 0.687 \end{pmatrix}$$

whose characteristic polynomial is $p(t) = t^3 - 0.687t^2 + 2.36t - 0.15$, from Lemma (1) we have that $4a_2^3 + 27a_3^2 + 4a_1^3 a_3 - a_1^2 a_2^2 - 18a_1 a_2 a_3 = 46.3728 > 0$. That is, $p(t)$ satisfies the condition from Lemma 1, so it has one real positive root and two roots in the form $\alpha + i\beta$ with $\alpha > 0$ and $\beta \neq 0$. The abscissa of instability of the polynomials is $\sigma_u = 0.0647$. Now we will use the result of Theorem 1 taking the value of $r = -0.3$ for the created $f_r(t)$. We obtain the polynomial $f_{-0.3}(t) = t^3 + 0.663t^2 + 1.1342t + 1.1422$ whose eigenvalues are $\lambda_1 = -0.8695$, $\lambda_{2,3} = 0.1032 \pm i1.1415$.

Therefore, for $f_{-0.3}(t)$ we have an UDS type I capable of generating multiscroll attractors by using a piecewise linear function.

Generation of multiscroll

Now, let us consider a control PWL system to generate multiscroll attractors as follows

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u(r) + BS \tag{7}$$

where $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ is the state vector, $\mathbf{b} = B = (0, 0, 1)^T$ is a constant vector, $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{3\times 3}$ with $i, j = 1, 2, 3$ denotes a nonsingular linear matrix.

We have that $p_A(t)$ is the characteristic polynomial of the system and S is a step function defined as follows

$$S = \begin{cases} s_1 & \text{for } c_1 < x, \\ s_2 & \text{for } c_2 < x \leq c_1, \\ \vdots & \\ s_m & \text{for } x \leq c_m. \end{cases}$$

Define the linear control $u = c^T(r) \cdot \mathbf{x} = [a_3 - A_0(r), a_2 - A_1(r), a_1 - A_2(r)] \cdot \mathbf{x}$, where $A_j(r) = \frac{p^j(-r)}{j!}$. Therefore the controlled system can be given as follows

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -A_0(r) & -A_1(r) & -A_2(r) \end{pmatrix} \mathbf{x} + BS = A_c\mathbf{x} + BS. \tag{8}$$

Then, the closed-loop characteristic polynomial is given by:

$$\begin{aligned} f_r(t) &= t^3 + A_2(r)t^2 + A_1(r)t + A_0(r), \\ &= p_A(t - r). \end{aligned}$$

The equilibrium points of the system (8) are $\mathbf{x}_i^* = -A_c^{-1}BS$, with $i = 1, \ldots, m$, and each entry s_i of the PWL system is considered to preserve bounded trajectories of system and let the generation of an attractor. Therefore, the choice of $c_i's$ determines the atoms $D_i's$ in step function S. Each atom $D_i's$ of the partition of the space contains an equilibrium $\mathbf{x}_i^*$. The design of the s_i depends on the region we want to place the equilibrium point and the switching surfaces, we choose them so that the equilibrium point is in the center of these varieties, that is, we calculate the Euclidean distance

$$d(\mathbf{x}_0^*, \mathbf{x}_1^*) = \sqrt{(x_0^* - x_1^*)^2 + (y_0^* - y_1^*)^2 + (z_0^* - z_1^*)^2},$$

which has to be the same between each equilibrium point. Consider the following system to illustrate the generation of multiscroll attractors.

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.15 & -2.36 & 0.687 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} S. \tag{9}$$

For this example, we define S as follows:

$$S(x) = \begin{cases} 1.0280, & \text{for } 0.7500 < x; \\ 0.6853, & \text{for } 0.4500 < x \leq 0.7500; \\ 0.3427, & \text{for } 0.1500 < x \leq 0.4500; \\ 0, & \text{for } -0.1500 < x \leq 0.1500; \\ -0.3427 & \text{for } x \leq -0.1500. \end{cases}$$

and $u(r) = (-0.15 - p(-r), 2.36 - \frac{p'(-r)}{1!}, -0.687 - \frac{p''(-r)}{2!})x$ where $p(t) = t^3 - 0.687t^2 + 2.36t - 0.15$ is unstable.

The controlled system is

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p(-r) & -\frac{-p'(-r)}{1!} & -\frac{-p''(-r)}{2!} \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} S. \tag{10}$$

We have that $f_r(t) = t^3 + \frac{p''(-r)}{2!}t^2 + \frac{p'(-r)}{1!}t + p(-r)$. For $r = 0$, $f_0(t) = t^3 - 0.687t^2 + 2.36t - 0.15$ is a unstable polynomial and there is not multiscroll. The instability parameter of $f_0(t)$ is

$\sigma_{f_0} = 0.0647$. Then other behavior could appear when $r \in (-U_{diss(p_A)}, -\sigma_{p_A})$. By example 2 for $r = -0.3$ we have that $f_{-0.3}(t) = t^3 + 0.663t^2 + 1.1342t + 1.1422$. Hence $\sum_{j=1}^{3} t_j < 0$ consequently the system (10) is dissipative when $r = -0.3$.

Next, in the following Figure 1 the generation of the attractor from the system (10) is illustrated. The following Figure 2 shows

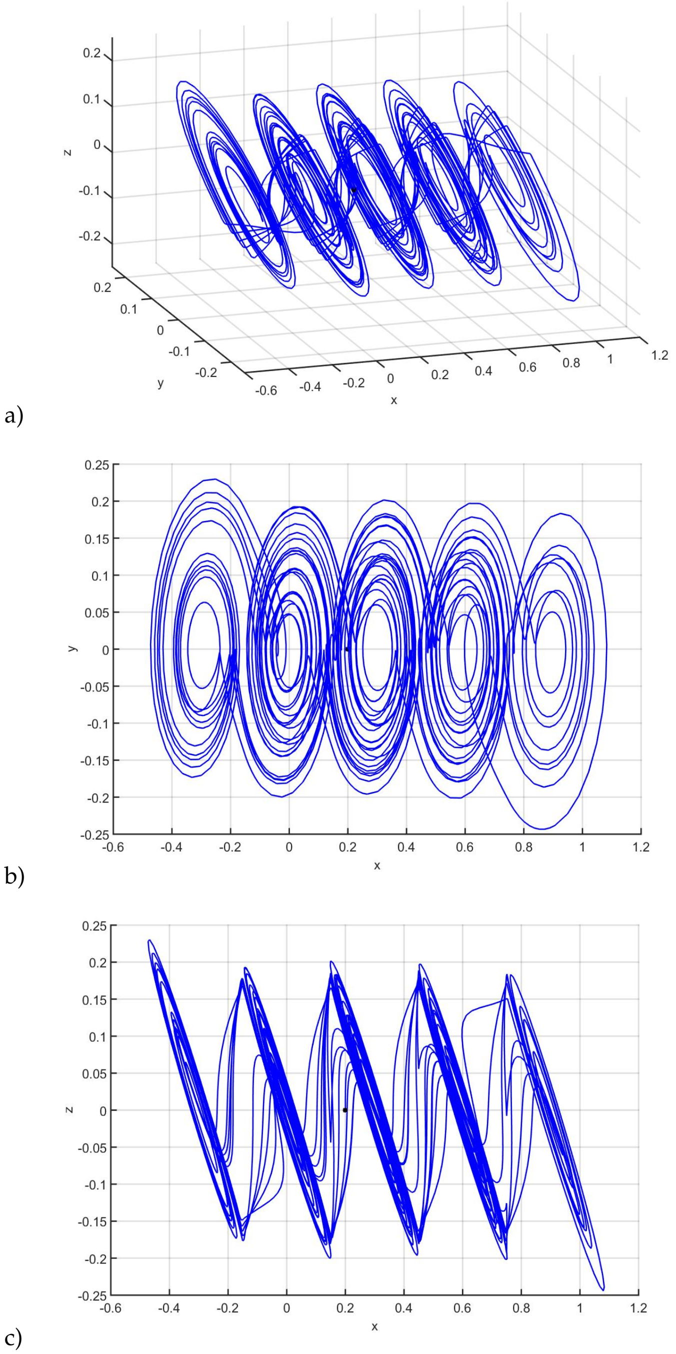

Figure 1 Attractor generated by the system (10) for $r = -0.30$. a) Solution of the system (10) with initial condition $\mathbf{x}_0 = (0.2, 0.0, 0.0)^T$. b) Projections of the attractor on the planes: b) (x, y). And c) (x, z).

the generation of attractors for five different conditions: a) $\mathbf{x}_0 = (-0.4, 0, 0)^T$, b) $\mathbf{x}_0 = (-0.1, 0.0, 0.0)^T$, c) $\mathbf{x}_0 = (0.2, 0.0, 0.0)^T$, d) $\mathbf{x}_0 = (0.5, 0.0, 0.0)^T$ and e) $\mathbf{x}_0 = (0.8, 0.0, 0.0)^T$.

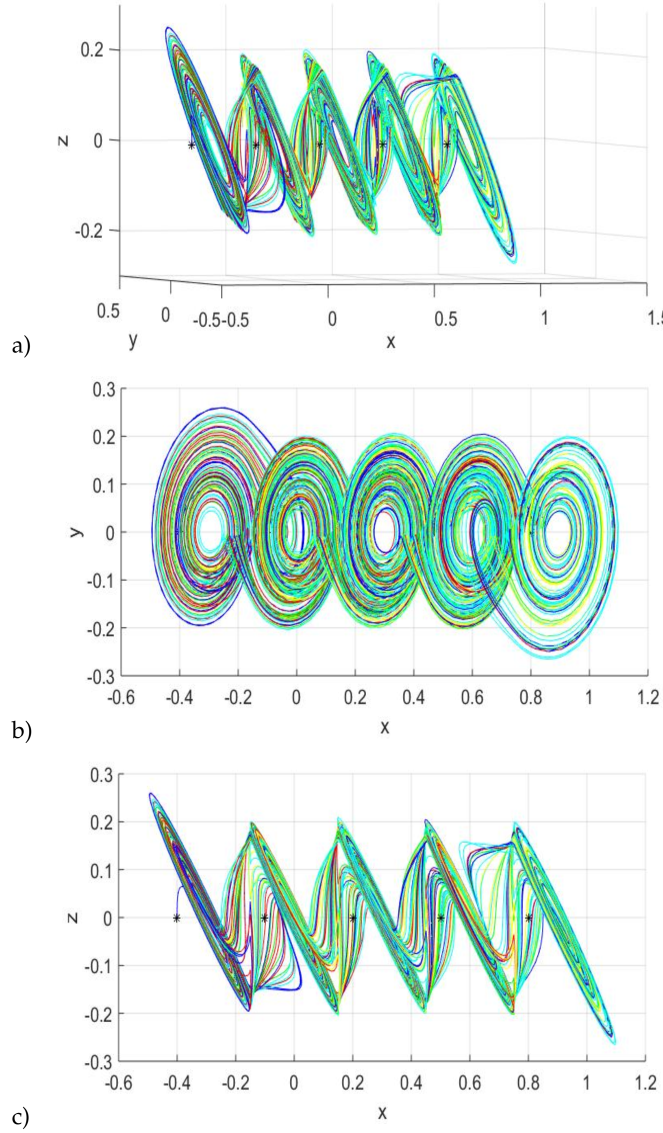

Figure 2 Generation of five attractors with five different conditions. a) [Blue, $\mathbf{x}_0 = (-0.4, 0.0, 0.0)^T$; red, $\mathbf{x}_0 = (-0.1, 0.0, 0.0)^T$; green, $\mathbf{x}_0 = (0.2, 0.0, 0.0)^T$; cyan, $\mathbf{x}_0 = (0.5, 0.0, 0.0)^T$ and yellow, $\mathbf{x}_0 = (0.8, 0.0, 0.0)^T$. Projections of the attractor on the planes: b) (x, y). And c) (x, z).

GENERATION OF MULTISTABILITY FROM INSTABILITY

In this section, we present the way to move from a system with monostability to a system that generates multistability through moving the stable and unstable varieties. The phenomenon of having two attractors coexisting generated by a nonlinear system has been reported by Arecchi *et al.* (1985), who called this behavior generalized multistability. Two problems related to the coexistence of attractions have been studied. The first problem is about choosing a desired attractor to which the system should converge, and the second problem is about excluding certain unwanted attractors from the dynamics (Pisarchik and Feudel 2014).

By moving the stable and unstable varieties of the system we can trap the trajectory of the system in different attractors, where any initial condition belonging to some basin of attraction will always converge to the same attractor, that is, a single attractor of the different ones that coexist. Depending on the initial condition that is split, the trajectory remains oscillating around of any of the equilibrium points of the system, and all the dynamics will be maintained in the attractor where the system's trajectory is enclosed. The region of coexistence of these attractors is critical, since a small noise can commute the physical system, adding a new characteristic to usual chaotic scenarios. In such cases, the properties of the areas of attraction are largely determined by the structure of saddle-type equilibrium points.

Description of the model

Consider the system UDS-I

$$\dot{\mathbf{x}} = A(\nu)\mathbf{x} + B(\nu)S \tag{11}$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ is vector states, $B = (0, 0, \nu)^T$ with $\nu \in \mathbb{R}$, S function linear piecewise and $A(\nu) = (a_{ij}) \in \mathbb{R}^3$ is of the form

$$A(\nu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{p_{A_c}(-r)}{0!}\cdot\nu & -\frac{p'_{A_c}(-r)}{1!}\cdot\nu & -\frac{p''_{A_c}(-r)}{2!}\cdot\nu \end{pmatrix}. \tag{12}$$

The mission of the parameter ν, better known as the bifurcation parameter, is to control the stable and unstable varieties in each s_i to catch the trajectories in a single attraction. This parameter can affect the dissipativity of the system since the dissipativity is given by $-\frac{p''_{A_c}(-r)}{2!}\cdot\nu$. For $\nu = 1$ system (11) is a system capable of generating attractor multiscroll, when ν varies we need the system to remain dissipative, this is true if $-\frac{p''_{A_c}(-r)}{2!}\cdot\nu < 0$, which allows us to obtain qualitative information about the interval where ν can vary and generate multistability. If we take $-\frac{p''_{A_c}(-r)}{2!}\cdot\nu = -1$ we obtain that $\nu = 1.5383$ and for this value the system (11) can generate multistability as shown in the following example.

Example 3 *Consider the system*

$$\dot{\boldsymbol{x}} = A(\nu)\boldsymbol{x} + B(\nu)S \tag{13}$$

with

$$S(x) = \begin{cases} 1.028, & \text{for } 0.75 < x; \\ 0.6853, & \text{for } 0.45 < x \leq 0.75; \\ 0.3427, & \text{for } 0.15 < x \leq 0.45; \\ 0, & \text{for } -0.15 < x \leq 0.15; \\ -0.3427 & \text{for } x \leq -0.15. \end{cases}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.1422\cdot\nu & -1.1342\cdot\nu & -0.663\cdot\nu \end{pmatrix}$$

with characteristic polynomial $p(t) = t^3 + 0.663t^2 + 1.1342t + 1.1422$ *the system (13) is UDS-I for* $\nu = 1$*. As mentioned, our first challenge is that we need the sum of the eigenvalues of the system to be negative. Taking the value of* ν *from the equation* $-\frac{p''_{A_c}(-r)}{2!}\cdot\nu = -1$ *we have that* $\nu = 1.5083$*. For this value the system has the following form:*

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.7228 & -1.7107 & -1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 \\ 0 \\ 1.5083 \end{pmatrix} S. \tag{14}$$

We can observe in the Figure 3 the graphical representation of multistability.

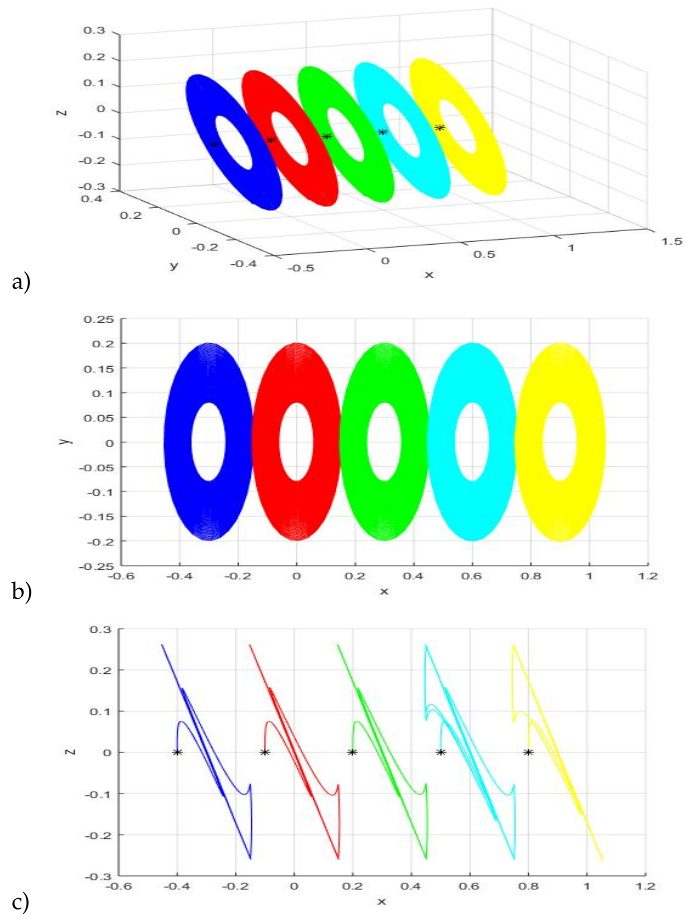

Figure 3 a) Attractors for five different initial conditions: (blue) $\mathbf{x}_0 = (-0.4, 0.0, 0.0)^t$, (red) $\mathbf{x}_0 = (-0.1, 0.0, 0.0)^t$, (green) $\mathbf{x}_0 = (0.2, 0.0, 0.0)^t$,(cian) $\mathbf{x}_0 = (0.5, 0.0, 0.0)^t$,(yellow) $\mathbf{x}_0 = (0.8, 0.0, 0.0)^t$. Projections of the attractors on the planes: b) (x, y); c) (x, z).

In Figure 3 we can see that for five different initial conditions, the dynamics of the system remains trapped in a single attractor of the five attractors coexisting, this depends on the initial condition is within the attraction basin of one of the five attractors that coexist. In the graph of the Figure 3 c) we can see how to move the bifurcation parameter ν such that the stable and unstable manifolds can be controlled, that is, at the moment trajectory of the system leaves by the unstable manifolds W^u, the stable manifolds W^s manages to catch the trajectory to again maintain it within the domain of the equilibrium point.

The initial conditions to generate multistability are the same as those used in the figure 2. Thus the dynamics of the system remain trapped in some region depending on the initial condition that is chosen.

CONCLUSION

By using a system of linear differential equations we generate multistability starting from a totally unstable system. First, we use a parameter for moving the eigenvalues of the associated matrix and to obtain a UDS-I system. Next, by means of another parameter we control the stable and unstable manifolds of the system to catch the generated trajectory from a given initial condition only in an attractor.

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Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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