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On The Para-Octonions; a Non-Associative Normed Algebra

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ABSTARCT

In this paper, para-octonions and their algebraic properties are provided by using the Cayley-Dickson multiplication rule between the octonionic basis elements. The trigonometric form of a para-octonion is similar to the trigonometric form of dual number and quasi-quaternion. We study the De-Moivre's theorem for para-octonions, extending results obtained for real octonions and defining generalize Euler's formula for para-octonions.

Keywords: Alternativity, Cayley-Dickson construction, De-Moivre's formula, Para-octonion

Para-Ktonyonlar Üzerine; Bir İlişkisel Olmayan Normlu Cebir

ÖZET

Bu çalışmada, octonyonik baz elemanları arasında Cayley-Dickson çarpım kuralı kullanılarak para-octonyonlar ve cebirsel özellikleri verilmiştir. Bir para-octonyonun trigonometrik formu bir dual-sayının ve bir quasi-kuaterniyonun trigonometrik formuna benzerdir. Para-octonyonlar içn De-Moivre'nin teoremi ele alınarak reel-octonyonlar için elde edilen sonuçlar genelleştirilmiştir. Ayrıca, para-octonyonlar için genel Euler formülleri tanımlanmıştır.

Anahtar kelimeler: Alternatiflik, Cayley-Dickson yapı, De-Moivre formu, para-oktoniyon

1. INTRODUCTION

The real octonions algebra as the ordered couple of real quaternions, was invented by J. T. Graves (1843) and A. Cayley (1845) independently. In mathematics, the real octonions form a normed division algebra over the real numbers, usually represented by $\mathbb{O}$. In our previous works, we studied some algebraic properties of real, split, complex, semi-octonions, and quasi-octonions.

In this paper, we study some algebraic properties of para-octonions, which is called $\frac{1}{8}-$ octonions in [9]. A pare-octonions can be written in form a dual quasi-quaternions. We review the generalized octonions algebra, and show that if put $\alpha=\beta=\gamma=0,$ we obtain para-octonions algebra. Like real octonions, para-octonions form a non-associative algebra, but unlike real octonions, they are not division algebra. We investigate the De Moivre's formula for these octonions and by using this formula, we obtain any power of a para-octonion. We hope that this work will contribute to the study of physics and other sciences.

2. THEORETICAL BACKGROUND

In this section, we give a brief summary of the generalized octonions. For detailed information about these octonions, we refer the reader to [1].

Definition 1. A generalized octonion x is defined as

$$x=a_0\,e_0+a_1\,e_1+a_2\,e_2+a_3\,e_3+a_4\,e_4+a_5\,e_5+a_6\,e_6+a_7\,e_7,$$

where $a_0,\ldots,a_7$ are real numbers and e_i, $(0\le i\le 7)$ are octonionic units satisfying the equalities that are given in the following table;

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The multiplication rules among the basis elements of octonions e_i can be expressed in the form:

.	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\alpha$	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	$-e_7$	αe_6
e_2	$-e_3$	$-\beta$	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	αe_2	$-\beta e_1$	$-\alpha\beta$	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-\gamma$	γe_1	γe_2	γe_3
e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	$-\alpha\gamma$	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	$-\beta\gamma$	$\beta\gamma$
e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	$-\alpha\beta\gamma$

Special Cases:

1. If $\alpha = \beta = \gamma = 1$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of real octonions O [5].
2. If $\alpha = \beta = 1,\ \gamma = -1$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of split octonions (Psoudo-octonions) O' [4].
3. If $\alpha = \beta = 1,\ \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of semi-octonions O_S [3].
4. If $\alpha = \beta = -1,\ \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of split semi-octonions O'_S [5].
5. If $\alpha = 1,\ \beta = \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of quasi-octonions O_q [6].
6. If $\alpha = -1,\ \beta = \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of split quasi-octonions O'_q [8].
7. If $\alpha = \beta = \gamma = 0$, is considered, then $O(\alpha,\beta,\gamma)$ is the algebra of para-octonions O_p.

The generalized octonions algebra, $O(\alpha,\beta,\gamma)$, is a non-commutative, non-associative, alternative, flexible and power-associative [1].

3. PARA-OCTONIONS ALGEBRA

Definition 2. A para-octonion x is expressed as a real linear combination of the unit octonions $(e_0, e_1, ..., e_7)$, *i.e.*

$$x = (x_0, x_1, ..., x_7) = x_0 e_0 + \sum_{i=1}^{7} x_i e_i,$$

where $x_0, ..., x_7$ are real numbers and e_i, $(0 \le i \le 7)$ are imaginary octonion units satisfying the non-commutative multiplication rules;

$$e_k^2 = 0, \quad k = 0, ..., 7$$

$$e_1e_2 = e_3 = -e_2e_1, \qquad e_2e_4 = e_6 = -e_4e_2$$

$$e_1e_4 = e_5 = -e_4e_1, \qquad e_2e_5 = e_7 = -e_5e_2$$

$$e_1e_6 = -e_7 = -e_6e_1, \qquad e_3e_4 = e_7 = -e_4e_3$$

The above multiplication rules are given in the following Table;

.	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	0	e_5	0	$-e_7$	0
e_2	$-e_3$	0	0	e_6	e_7	0	0
e_3	0	0	0	e_7	0	0	0
e_4	$-e_5$	$-e_6$	$-e_7$	0	0	0	0
e_5	0	$-e_7$	0	0	0	0	0
e_6	e_7	0	0	0	0	0	0
e_7	0	0	0	0	0	0	0

This form, $x = x_0e_0 + \sum_{i=1}^{7} x_ie_i$, is called the standard form of a para-octonion. By using the Cayley-Dickson construction, a para-octonion x can also be written as

$$x=(a_0\,e_0+a_1\,e_1+a_2\,e_2+a_3\,e_3)+(a_4+a_5\,e_1+a_6\,e_2+a_7\,e_3)e_4=q+q'l,$$

where $l^2=0$ and q,q' are quasi-quaternions (1/4 -quaternions) [2], *i.e.*

$$q,q'\in \mathrm{H}_q^{\circ}=\left\{q=a_0+a_1e_1+a_2e_2+a_3e_3 \;\middle|\; e_1^{\,2}=e_2^{\,2}=e_3^{\,2}=0,\ e_1e_2=e_3,\ e_1e_3=0=e_2e_3,\ a_i\in\mathrm{R}\right\},$$

This construction lets us view the para-octonions as a two dimensional vector space over quasi-quaternions.

A para-octonion x can be decomposed in terms of its scalar (S_x) and vector $(\vec{V}_x)$ parts as

$$S_x=a_0,\quad \vec{V}_x=a_1\,e_1+a_2\,e_2+a_3\,e_3+a_4\,e_4+a_5\,e_5+a_6\,e_6+a_7\,e_7.$$

For two para-octonions $x=\sum_{i=0}^{7}a_i\,e_i$ and $w=\sum_{i=0}^{7}b_i\,e_i$, the summation and substraction processes are given as $x\pm w=\sum_{i=0}^{7}(a_i\pm b_i)\,e_i$.

The product of two para-octonions $x=S_x+\vec{V}_x, w=S_w+\vec{V}_w$ is expressed as

$$x.w=S_x\,S_w-\left\langle \vec{V}_x,\vec{V}_w\right\rangle+S_x\,\vec{V}_w+S_w\,\vec{V}_x+\vec{V}_x\times\vec{V}_w$$

This product can be described by a matrix-vector product as

$$x.w=\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0\\ a_2 & 0 & a_0 & 0 & 0 & 0 & 0 & 0\\ a_3 & -a_2 & a_1 & a_0 & 0 & 0 & 0 & 0\\ a_4 & 0 & 0 & 0 & a_0 & a_1 & 0 & 0\\ a_5 & -a_4 & 0 & 0 & a_1 & a_0 & 0 & 0\\ a_6 & 0 & -a_4 & 0 & a_2 & 0 & a_0 & 0\\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}\begin{bmatrix} b_0\\ b_1\\ b_2\\ b_3\\ b_4\\ b_5\\ b_6\\ b_7 \end{bmatrix}.$$

Para-octonions multiplication is not associative, since

$$e_1(e_2\,e_4)=e_1e_6=-e_7,$$
$$(e_1e_2)\,e_4=e_3e_4=e_7.$$

But it has the property of *alternativity*, that is, any two elements in it generate an associative subalgebra isomorphic to R, $\mathbb{C}^0$, H^0.

e_0 and $e_i\ (1\le i\le 7)$ generate a subalgebra isomorphic to dual numbers $\mathbb{C}^0$,

Subalgebra with bases $e_0,\ e_i,\ e_j,\ e_k\ (1\le i,j,k\le 7)$ is isomorphic to quasi-quaternions algebra H_q^0.

2.1 Some Properties of Para-octonions

The *conjugate* of para-octonion $x=\sum_{i=0}^{7}a_i\,e_i=S_x+\vec{V}_x$ is

$$\overline{x}=a_0\,e_0-\sum_{i=1}^{7}a_i\,e_i=S_x-\vec{V}_x.$$

Conjugate of product of two para-octonions and its own are described as

$$\overline{xy}=\overline{y}\,\overline{x},\ \ \overline{\overline{x}}=x$$

It is clear that the scalar and vector parts of x is denoted by $S_x=\dfrac{x+\overline{x}}{2}$ and $\vec{V}_x=\dfrac{x-\overline{x}}{2}$.

1) The *norm* of x is

$$N_x=x\,\overline{x}=\overline{x}\,x=\left|x\right|^2=a_0^2.$$

It satisfies the following property

$$N_{xy}=N_x\,N_y=N_y\,N_x$$

If $N_x=1$, then x is called a unit para-octonion. We will use to denote the set of unit para-octonions.

2) The *inverse* of x with $N_x\neq 0$, is

$$x^{-1}=\frac{1}{N_x}\overline{x}.$$

4) The *trace* of element x is defined as $t(x)=x+\overline{x}.$

The para-octonions algebra is not division algebra, because for every nonzero $x \in O_P$ the relation $N_x = 0$, implies $x \neq 0$.

Example 1. Consider the para-octonions

$$x_1 = 1 + (1,-1,2,-2,0,1,1),$$
$$x_2 = 0 + (1,-1,1,-2,0,1,1) \text{ and}$$
$$x_3 = -2 + (1,-1,\sqrt{2},-2,2,1,1);$$

1. The vector parts of x_1, x_2 are

$$\vec{V}_{x_1} = (1,-1,2,-2,0,1,1),\ \vec{V}_{x_2} = (1,-1,1,-2,0,1,1).$$

2. The conjugates of x_1, x_2 are

$$\overline{x}_1 = 1 - (1,-1,2,-2,0,1,1),\ \overline{x}_2 = 0 - (1,-1,1,-2,0,1,1).$$

3. The norms are given by

$$N_{x_1} = 1,\ N_{x_2} = 0,\ N_{x_3} = 4.$$

4. The inverses are

$$x_1^{-1} = 1 - (1,-1,2,-2,0,1,1),\quad x_3^{-1} = \frac{1}{4}[-2-(1,-1,\sqrt{2},-2,2,1,1)],$$

and x_2 not invertible.

5. One can realize the following operations

$$x_1 + x_2 = 1 + (2,-2,3,-4,0,2,2)$$
$$x_1 - x_2 = 2 + (0,0,1,0,0,0)$$
$$x_1 x_2 = 0 + (1,-1,1,-2,0,1,-1)$$
$$x_2 x_1 = 0 + (1,-1,1,-2,0,1,3)$$

$$N_{x_1 x_2} = N_{x_1} N_{x_2} = N_{x_2 x_1} = 0.$$

Theorem 1.4. The set O_P^1 of unit split semi-octonions is a subgroup of the group O_P^0 where $O_P^0 = O_P - [0 - \vec{0}]$.

Proof: Let $x, y \in O_P^1$. We have $N_{xy} = 1$, *i.e.* $xy \in O_P^1$ and thus the first subgroup requirement is satisfied. Also, by the property

$$N_x = N_{\overline{x}} = N_{x^{-1}} = 1,$$

the second subgroup requirement $x^{-1} \in O_P^1$.

3.2 Trigonometric form and De Moivre's theorem

The trigonometric (polar) form of a nonzero para-octonion $x = \sum_{i=0}^{7} a_i e_i$ is

$$x = r(\cos\varphi + \vec{w}\sin\varphi)$$

where $r = |x| = \sqrt{N_x}$ is the modulus of x,

$$\cos\varphi = \frac{a_0}{r},\quad \sin\varphi = \varphi = \frac{\left(\sum_{i=1}^{7} a_i^2\right)^{1/2}}{r}$$

and

$$\vec{w} = (w_1, w_2, ..., w_7) = \frac{1}{(\sum_{i=1}^{7} a_i^2)^{1/2}}(a_1, a_2, ..., a_7).$$

This is similar to polar coordinate expression of a quasi-quaternion and dual number.

Example 2. The trigonometric forms of the para-octonions

$$x_1 = 1 + (1,-1,0,1,1,1,-1) \text{ is } x_1 = \cos\varphi + \vec{w}_1 \sin\varphi,$$
$$x_2 = -2 + (2,-1,0,1,-1,2,1) \text{ is } x_2 = 2[\cos\varphi + \vec{w}_2 \sin\varphi]$$

where

$$\vec{w}_1 = \frac{1}{\sqrt{6}}(1,-1,0,1,1,1,-1),\ \vec{w}_2 = \frac{1}{\sqrt{12}}(2,-1,0,1,-1,2,1)$$

and $N_{\vec{w}_1} = N_{\vec{w}_2} = 0.$

Theorem 1.5. (De Moivre's theorem) If $x = r(\cos\varphi + \vec{w}\sin\varphi)$ be a para-octonion and n is any positive integer, then

$$x^n = r^n (\cos n\varphi + \vec{w}\sin n\varphi)$$

Proof: The proof easily follows by induction on n. ■

The Theorem holds for all integers n, since

$$q^{-1} = \cos\varphi - \vec{w}\sin\varphi,$$

$$q^{-n} = \cos(-n\varphi) + \vec{w}\sin(-n\varphi)$$
$$= \cos n\varphi - \vec{w}\sin n\varphi.$$

Example 3. Let $x = 1 + (1,-1,2,1,2,2,-1)$. Find x^{10} and x^{-45}.

Solution: First write x in trigonometric form.

$$x = \cos\varphi + \vec{w}\sin\varphi,$$

where $\cos\varphi = 1,\ \sin\varphi = 4,\ \vec{w} = \frac{1}{4}(1,-1,2,1,2,2,-1).$

Applying de Moivre's theorem gives:

$$x^{10} = \cos 10\varphi + \vec{w}\sin 10\varphi = 1 + 40\,\vec{w} = 1 + 10(1,-1,2,1,2,2,-1)$$
$$x^{-45} = \cos(-45\varphi) + \vec{w}\sin(-45\varphi) = 1 - 45\,(1,-1,2,1,2,2,-1).$$

Corollary 1.5. The equation 1, doesn't have solution for a unit para-octonion.

Example 3.5. Let $x = -1 + (1,-1,2,1,1,0,-1)$ be a unit para-octonion. There is no *n* (*n* > 0) such that $x^n = 1.$

For any unit para-octonion $x = \cos\varphi + \vec{w}\sin\varphi,$ since $\vec{w}^2 = 0,$ a natural generalization of Euler's formula is

$$e^{\vec{w}\varphi} = 1 + \vec{w}\varphi + \frac{(\vec{w}\varphi)^2}{2!} + \frac{(\vec{w}\varphi)^3}{3!} + \ldots = 1 + \vec{w}\varphi = \cos\varphi + \vec{w}\sin\varphi = x,$$

3.3 Roots of Para-octonion

Theorem 1.6. Let $x = r(\cos\varphi + \vec{w}\sin\varphi)$ be a para-octonion. The equation $a^n = x$ has only one root and this is

$$a = \sqrt[n]{r}\left(\cos\frac{\varphi}{n} + \vec{w}\sin\frac{\varphi}{n}\right)$$

Proof: We assume that $a = M(\cos\lambda + \vec{w}\sin\lambda)$ is a root of the equation $a^n = x$, since the vector parts of x and a are the same. From Theorem 4.5, we have

$$a^n = M^n(\cos n\lambda + \vec{w}\sin n\lambda).$$

Now, we find

$$M = \sqrt[n]{r},\quad \cos\varphi = \cos n\lambda,\qquad \sin\varphi = \sin n\lambda.$$

So, $a = \sqrt[n]{r}\left(\cos\frac{\varphi}{n} + \vec{w}\sin\frac{\varphi}{n}\right)$ is a root of equation $a^n = x.$ If we suppose that there are two roots satisfying the equality, we obtain that these roots must be equal to each other.

■

Example 1.6. Let $x = 8 + (1,0,-\sqrt{2},0,2,-1,0)$ be a para-octonion. The cube root of the octonion x is

$$x^{1/3} = \sqrt[3]{8}\left(\cos\frac{\varphi}{3} + \vec{w}\sin\frac{\varphi}{3}\right)$$
$$= 2\left(1 + \frac{1}{3\sqrt{8}}\vec{w}\right).$$

Conclusion

In this paper, we defined and gave some of algebraic properties of para-octonions and showed that the trigonometric form of para-octonions is similar to quasi-quaternions and dual numbers. The De Moivre's formulas for these octonions is obtained. We gave some examples for clarification.

Further Work

We will give a complete investigation to real matrix representations of para-octonions, and give any powers of these matrices.

REFERENCES

[1] Flaut, C., & Shpakivskyi V., (2015). An efficient method for solving equations in generalized quaternion and octonion algebras, *Advance in Applied Clifford algebra*, 25 (2), 337–350.

[2] Jafari M., On the Properties of Quasi-Quaternions Algebra, (2014). *Communications, faculty of science, university of Ankara, Series A1: Mathematics and statistics*, 63(1), 1-10.

[3] Jafari M., (2015). A viewpoint on semi-octonion algebra, *Journal of Selcuk university natural and applied science*, 4(4), 46-53.

[4] Jafari M., (2015). Split Octonion Analysis, Representation Theory and Geometry, Submitted for publication.

[5] Jafari M., Azanchiler H., (2015). On the structure of the octonion matrices, Submitted for publication.

[6] Jafari M., (2015). An Introduction to Quasi-Octonions and Their Representation, DOI: 10.131 40/RG.2.1.3833.0082

[7] Mortazaasl H., Jafari M., (2013). A study on Semi-quaternions Algebra in Semi-Euclidean 4-Space, *Mathematical Sciences and Applications E-Notes*, 1(2) 20-27.

[8] Jafari M., (2015). The Fundamental Algebraic Properties of Split Quasi-Octonions, DOI: 10. 13140/RG .2.1.3348.2728

[9] Rosenfeld B. A., Geometry of Lie Groups, Kluwer Academic Publishers, Dordrecht , 1997.