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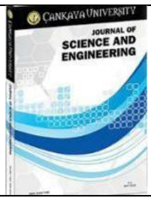

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A Generalized Series Solution of nth Order Ordinary Differential Equations

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Abstract

Differential equations in general play major role in finding solutions to many problems in real life. These real-life problems are modeled by either ordinary differential equations (with uni-variate independent variable) or partial differential equations (with multi-variate independent variables). The solution method adopted is determined by the nature of the differential equation. In this paper, the solution of an nth order Ordinary Differential Equation (ODE) is considered. The power series and the conditions for its convergence or otherwise is examined. Also, the index shift in the summation is applied in the simplification of the resulting algebraic expression and with the introduction of the factorial notation, the number of operations required to solve the problem is minimized. The resulting model therefore simplifies the solution method without the rigour of index shit in the summands and algebraic manipulations of the expression obtained. This makes the model applicable to the solution of ordinary differential equation of any order n. The generalized model is thereafter applied to an ordinary differential equation of order seven without recourse to index shift. This simplified form gives the solution considered and a simple and generalized solution is obtained.

1. Introduction

The importance of Differential Equations in modeling real life problems arising from several areas of human endeavor ranging from science and technology to the humanities cannot be overemphasized. As such various solution methods have been proposed and applied in solving differential equations [1].

A differential equation with only one independent variable is called Ordinary Differential Equation (ODE) while one with at least two independent variables is called Partial Differential Equation (PDE).

The solution methods applicable to ODEs with constant coefficients can be solved by means of algebraic methods. Such solutions can be expressed in terms of elementary functions such as trigonometric, exponential, or polynomial functions [2]. Solutions of ODEs with variable coefficients on the other hand can be obtained by other methods. These can be solved by other means and their solutions can be expressed in non-elementary functions. One of such methods is the power series method, which is equally applicable to the solution of Legendre's and Bessel's equations [3, 4, 5].

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2. Details

Consider the Taylor series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x - x_0)^n \qquad (1).$$

If there exist derivatives of all orders of $f(x)$at $x = x_0$, then (1) defines the Taylor series about $x = x_0$. If however $x = 0$, then (1) becomes a Maclaurin series [6, 7].

Again, let.

$$a_n = \frac{f^{(n)}(x)}{n!} \qquad (2),$$

such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (3)$$

then (1) is called power series where a_n and x_0 are numbers [3,8].

Let the nth order homogenous linear differential equation be defined as

$$p_0(x)y^n + p_1(x)y^{n-1} + p_2(x)y^{n-2} + p_3(x)y^{n-3} + \cdots + p_n(x) = 0 \qquad [5]$$

where $p_n(x)$, $k = 1(1)n$ are either polynomials or convergent power series about $x = x_0$, with no common polynomial divisor, then $x = x_0$ is called an ordinary point of (1) provided $P(x_0) \neq 0$, and is called a singular point if $p_0(x) = 0$ [9].

Theorem: Let the sequence $\left|\frac{a_n}{a_{n+1}}\right|, n = 1,2,3,\cdots$ converge with limit L^* and let R be the radius of convergence. If:

i. $L^* = 0$ then $R = \infty$ that is (1) converges for all x.
ii. $L^* \neq 0$ (i.e., $L^* > 0$) then $R = \frac{1}{L^*} = \frac{a_n}{a_{n+1}}$
iii. $\left|\frac{a_n}{a_{n+1}}\right| \to \infty$, then $R = 0$ (convergence is only at x_0) [10].

Theorem: The power series (1) is said to be convergent if there exists $0 \leq \rho \leq \infty$ called the radius of convergence such that $|x - x_0| < \rho$.

Also, the convergence or otherwise of series (1) can be determined by the ratio test, that is,

$$L = |x - x_0| \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \qquad (4)$$

That is the given a power series (2) converges if $L < 1$, diverges if $L > 1$ and may or may not converge if $L = 1$

Theorem: The power series (2.3) has a positive radius of convergence [3, 7].

2.1. Shift of Summation Index in Power Series

Repeatedly differentiating the power series (1) yields

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1} \tag{5}$$

$$f''^{(x)} = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \tag{6}$$

$$f'''(x) = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n (x - x_0)^{n-3} \tag{7}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n (x - x_0)^{n-k} \quad [3-5] \tag{8}$$

But n cannot take on the values of 0, 1, 2, …, $n-k+1$ respectively in the above, hence the summation indices are shifted, thus.

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \tag{9}$$

$$f''^{(x)} = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \tag{10}$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x - x_0)^{n-3} \tag{11}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n (x - x_0)^{n-k} \tag{12}$$

But (2.3) can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{n!}{(n-0)!} a_n (x - x_0)^n \tag{13}$$

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \quad = \sum_{n=1}^{\infty} \frac{n!}{(n-1)!} a_n (x - x_0)^{n-1} \tag{14}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \quad = \sum_{n=2}^{\infty} \frac{n!}{(n-2)!} a_n (x - x_0)^{n-2} \tag{15}$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x - x_0)^{n-3} = \sum_{n=3}^{\infty} \frac{n!}{(n-3)!} a_n (x - x_0)^{n-3} \tag{16}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n (x - x_0)^{n-k} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k} \tag{17}$$

2.2. The Factorial Notation and Index Change

Now, suppose.

$$f(x) = (x - x_0)^c \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{18}$$

is expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{n!}{(n-0)!} (x - x_0)^{n+c} \tag{19}$$

Then the successive derivatives of $f(x)$ will be

$$f'(x) = \sum_{n=1}^{\infty} a_n \frac{n!}{(n-1)!} (x - x_0)^{n+c-1} \tag{20}$$

$$f''(x) = \sum_{n=2}^{\infty} a_n \frac{n!}{(n-2)!} (x - x_0)^{n+c-2} \tag{21}$$

$$f'''(x) = \textstyle\sum_{n=3}^{\infty} a_n \frac{n!}{(n-3)!} (x - x_0)^{n+c-3} \tag{22}$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n+c-k} \tag{23}$$

$k = 5$

$$y^{(5)}(x) = \sum_{n=0}^{\infty} \frac{(n+5)!}{n!} a_{n+5} x^n$$

$$y^{(5)}(x) = \sum_{n=0}^{\infty} \frac{(n+5)(n+4)(n+3)(n+2)(n+1)n!}{n!} a_{n+5} x^n$$

$$= \sum_{n=0}^{\infty} (n+5)(n+4)(n+3)(n+2)(n+1) a_{n+5} x^n$$

From the given example

$$\sum_{n=0}^{\infty} (n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1) a_{n+7} x^n - \sum_{n=0}^{\infty} (n+5)(n+4)(n+3)(n+2)(n+1) a_{n+5} x^n = 0$$

$$\sum_{n=0}^{\infty} \{(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1) a_{n+7} - (n+5)(n+4)(n+3)(n+2)(n+1) a_{n+5}\} x^n = 0$$

Since $x^n \neq 0$, then

$$(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1) a_{n+7} - (n+5)(n+4)(n+3)(n+2)(n+1) a_{n+5} = 0$$

$$(n+7)(n+6)(n+5)(n+4)(n+3)(n+2)(n+1) a_{n+7} = (n+5)(n+4)(n+3)(n+2)(n+1) a_{n+5}$$

$$a_{n+7} = \frac{1}{(n+7)(n+6)} a_{n+5}$$

$$n = 0: \; a_7 = \frac{1}{(7)(6)} a_5$$

$$n = 1:\ a_8 = \frac{1}{(8)(7)} a_6$$

$$n = 2:\ a_9 = \frac{1}{(9)(8)} a_7 = \frac{1}{(9)(8)(7)(6)} a_5$$

$$n = 3:\ a_{10} = \frac{1}{(10)(9)} a_8 = \frac{1}{(10)(9)(8)(7)} a_6$$

$$n = 4:\ a_{11} = \frac{1}{(11)(10)} a_9 = \frac{1}{(11)(10)(9)(8)(7)(6)} a_5$$

$$n = 5:\ a_{12} = \frac{1}{(12)(11)} a_{10} = \frac{1}{(12)(11)(10)(9)(8)(7)} a_6$$

$$n = 6:\ a_{13} = \frac{1}{(13)(12)} a_{11} = \frac{1}{(13)(12)(11)(10)(9)(8)(7)(6)} a_5$$

$$n = 7:\ a_{14} = \frac{1}{(14)(13)} a_{12} = \frac{1}{(14)(13)(12)(11)(10)(9)(8)(7)} a_6$$

$$n = 8:\ a_{15} = \frac{1}{(15)(14)} a_{13} = \frac{1}{(15)(14)(13)(12)(11)(10)(9)(8)(7)(6)} a_5$$

$$a_{2k+5} = \frac{5!}{(2k+5)!} a_5$$

$$a_{2k+6} = \frac{6!}{(2k+6)!} a_6$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + \frac{5!}{(2k+5)!} a_5 x^{2k+5} + \frac{6!}{(2k+6)!} a_6 x^{2k+6} + \cdots$$

3. Conclusions

In this paper, we have discussed the solution of ODE about a singular point. The power series was employed with its associated and the conditions for convergence. The index shift in the summation enabled the simplification of the algebraic expression obtained via the factorial notation. This in turn reduced the required number of operations in the solution of the problem. In other words, the model simplified the solution method without recourse to index shit and further simplifying the expression obtained.

While most illustrative examples have been limited to problems of first, second and third orders, we have proposed a generalized model for all order problems which eliminates index shift.

The model can be applied in solving ordinary differential equation of any order n. The application of the generalized model to an ordinary differential equation of order seven and the result obtained, without necessarily shifting the index makes the model a robust one with reduced computational steps. Thus, for an ODE of any order. This therefore simplifies solution method of any order of a homogenous linear ODE.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Adebisi A. Ibrahim: Writing, Reviewing, and editing.

Emmanuel O. Adeyefa: Reviewing, Supervision and editing.

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