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On Caputo fractional elliptic equation with nonlocal condition

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Abstract

The paper consider the Caputo elliptic equation with nonlocal condition. We obtain the upper bound of the mild solution. The second contribution is to provide the lower bound of the solution at terminal time. We show that the convergence results between Caputo modified Helmholtz equation and Caputo Poisson equation. The main tool is the use of upper and lower bounds of the Mittag-Lefler function, combined with analysis in Hilbert scales space.

Keywords: Fractional evolution equation, Caputo derivative, Mittag-Lefler functions.
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1. Introduction

Today, the topic of fractional differential equation has received the attention of many scientists in different fields. In some real simulation models, there are some interference phenomena due to viscous factors. Therefore, models that involve memory are more descriptive than some classical models. A good choice is to use derivatives of order to create some models, to better explain the phenomena that classical derivatives are missing, see [6, 7, 8, 4, 5]. Mathematicians have spent a lot of time studying different types of derivatives, but it seems that the two types of derivatives Caputo and Riemann-Liouville are of most interest to them. The reason they are of interest comes from the model sticking to memory effects. We temporarily list some articles about Caputo or Riemann-Liouville [1, 2, 3, 18, 10, 12, 14, 9, 11, 15, 20, 13, 19] and some other

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derivatives, see [21, 22**?** , 23, 24, 25].
In this paper, we are interested in the following evolution equation with a time-fractional derivative

$$\begin{cases} {}_C D_t^\alpha w + w_{xx} = kw, & \text{in} \quad (0,\pi)\times(0,T], \\ w(0,t) = w(\pi,t) = 0, & 0 < t < T, \\ w_t(x,0) = 0, & 0 < t < T, \end{cases} \tag{1}$$

with the nonlocal condition as follows

$$aw(x,0) + bw(x,T) + \int_0^T \psi(t) w(x,t) dt = f(x), \quad 0 < x < \pi. \tag{2}$$

In (2), the functions f and ψ are defined later. The number T is a positive constant. In our problem, k is a constant $k \geq 0$ and a, b are two constants which are not negative. If $\alpha = 2$ and $k > 0$, Problem (1) is called classical modified Helmholtz equation which is well-known in physics. If $\alpha = 2$ and $k = 0$, Problem (1) is called classical Poisson equation.

The term ${}_C D_t^\alpha w$ appears on (1) is called the Caputo derivative with respect to t which is defined by (see [27, 26])

$$\begin{cases} {}_C D_t^\alpha w(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-r)^{1-\alpha} \dfrac{\partial^2 w}{\partial r^2}(x,r) \mathrm{d}r, & \text{for} \quad 1 < \alpha < 2, \\ {}_C D_t^\alpha w(t,x) = \dfrac{\partial^\alpha w}{\partial t^\alpha}(x,t), & \text{for} \quad \alpha = 1, 2, \end{cases} \tag{3}$$

and Γ is the Gamma function.
As is known, there are a number of physical phenomena described by classical elliptic equations that do not satisfy some explanations, thus requiring the appearance of the Caputo derivative. This is also the reason why Problem (1) is called Caputo elliptic equation. When we encounter some observation or physical phenomenon involving viscoelasticity, we need the Caputo derivative rather than the classical derivative. And by using Caputo derivative in the (1) model we will obtain the better simulate.

Let us try to discuss on some fractional elliptic equations. In [14], the authors considered an elliptic equation associated with the Riemann–Liouville derivative. Some other articles for an elliptic equation with fractional order have been studied in [29, 30]. Let us refer to the interesting paper [28]. The authors [28] provided the ill-posedness of the Problem (1) in the case $k = 0$ without giving its approximate solution. Motivated by the work [28], the author [33] studied Cauchy problem for a semilinear fractional elliptic equation. Under some assumptions of the sought solution, they proposed the Fourier truncation method for approximating the problem. Some estimates of logarithmic type between the sought solution and regularized solution are established. Further development work of [33] has been completed in detail in [34].

To the best of our knowledge, this is the first work to survey about Problem (1) with the nonlocal condition (2). This condition (2) is probably the first to appear. For some papers on nonlocal condition, we can take a closer look at some problems related to this model (2), such as [37]. It can be immediately recognized that the condition in this paper is more complicated than the conditions in [37].

There are three main results in this paper. The first result shows the well-posedness when the input data is in Hilbert scale space. The second result shows that the lower bound of the mild solution at the final time $t = T$. According to our research, there are very few results on the lower bound of the solution. In addition, we show that the solution to Caputo modified Hemholtz equation will tends to the mild solution to Caputo Poisson equation. In order to overcome some difficult things for the proof, we need to use some techniques related to the bounds of the Mittag Lefler functions.

2. Preliminaries

Definition 2.1. *(Hilbert scale space).* Let us define the Hilbert scale space $\mathbb{H}^\theta(0,\pi)$ given as follows

$$\mathbb{H}^\theta(0,\pi) = \left\{ v \in L^2(0,\pi) \;\Big|\; \sum_{j=1}^{\infty} j^{4\theta} \Big(\int_\Omega f(x) e_j(x) \mathrm{d}x \Big)^2 < \infty \right\},$$

for any $s \geq 0$. It is well-known that $\mathbb{H}^s(\Omega)$ is a Hilbert space corresponding to the norm

$$\|v\|_{\mathbb{H}^\theta(0,\pi)} = \left(\sum_{j=1}^{\infty} j^{4\theta} \Big(\int_\Omega v(x) e_j(x) \mathrm{d}x \Big)^2 \right)^{1/2}, \quad v \in \mathbb{H}^\theta(0,\pi).$$

Here $e_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$.

Definition 2.2. The Mittag-Leffler function is defined by

$$E_{\alpha,\theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \theta)}, \quad z \in \mathbb{C}, \tag{4}$$

where $\alpha > 0$ and $\theta \in \mathbb{R}$ are arbitrary constants.

The following lemmas provide upper and lower bounds of the Mittag-Lefler functions $E_{\alpha,1}(z)$, $E_{\alpha,2}(z)$, $E_{\alpha,\alpha}(z)$ by the exponential functions.

Lemma 2.1 (See [31]). *Let $0 < \alpha_0 < \alpha_1 < 2$ and $\alpha \in [\alpha_0, \alpha_1]$. Then there exists two constants $\mu_1, \mu_2 > 0$ and $z > 0$ such that*

$$\frac{\mu_1}{\alpha} \exp\left(z^{\frac{1}{\alpha}} \right) \leq E_{\alpha,1}(z) \leq \frac{\mu_2}{\alpha} \exp\left(z^{\frac{1}{\alpha}} \right). \tag{5}$$

In addition, there exists two constants $\mu_3, \mu_4 > 0$ such that

$$\frac{\overline{\mu}_3}{\alpha} \exp\left(z^{\frac{1}{\alpha}} \right) \leq z^{\frac{\alpha-1}{\alpha}} E_{\alpha,\alpha}(z) \leq \frac{\mu_4}{\alpha} \exp\left(z^{\frac{1}{\alpha}} \right), \tag{6}$$

for any $z > 0$.

Lemma 2.2. *(see [31]) Let $z \in \mathbb{R}$. Then we have*

$$\frac{d}{dz} E_{\alpha,1}(z) = \frac{E_{\alpha,\alpha}(z)}{\alpha} \tag{7}$$

and

$$\frac{d}{dz} E_{\alpha,1}(\lambda z^\alpha) = \frac{1}{z} E_{\alpha,0}(\lambda z^\alpha). \tag{8}$$

Theorem 2.1. *Let the function $f \in H^\theta(0,\pi)$. Then if $a \geq 0, b \geq 0$ and $\psi(t) > 0$ then problem* (1)-(2) *has a unique solution w which satisfies*

$$\Big\| w \Big\|_{L^\infty(0,T;\mathbb{H}^\theta(0,\pi))} \leq \frac{\mu_2}{b\mu_1} \Big\| f \Big\|_{\mathbb{H}^\theta(0,\pi)}. \tag{9}$$

Let us assume that $f \in \mathbb{H}^{\theta+\frac{1}{\alpha}}(0,\pi)$. If $a = b = 0$ and $\psi(t) > M_0 > 0$ for any $0 \leq t \leq T$ then

$$\Big\| w \Big\|_{L^\infty(0,T;\mathbb{H}^\theta(0,\pi))} \lesssim \Big\| f \Big\|_{\mathbb{H}^{\theta+\frac{1}{\alpha}}(0,\pi)}. \tag{10}$$

Proof. First, we need to give the explicit fomula of the mild solution to (1)-(2). Let us assume that Problem (1)-(2) has a solution

$$w(x,t)=\sum_{j=1}^{\infty} w_j(t)e_j(x), \quad w_j(t)=\int_0^{\pi} w(x,t)e_j(x)\mathrm{d}x.$$

It is obvious to see that z_n satisfies the following system

$$_C D_t^{\alpha} w_j(t)=j^2 w_j(t)+k w_j(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t} w_j(0)=0, \tag{11}$$

and

$$\int_0^T \psi(t) w_j(t) dt=\int_{\Omega} f(x)e_j(x)\mathrm{d}x. \tag{12}$$

In view of the previous result of [26, Section 2], we give the solution of (11) in the following

$$w_j(t)=E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) w_j(0). \tag{13}$$

The condition (2) provide us to get that

$$a w_j(0)+b w_j(T)+\int_0^T \psi(t) w_j(t) dt=f_j, \quad 0<x<\pi. \tag{14}$$

Thus, we get that the following estimate

$$a w_j(0)+bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right) w_j(0)+\left(\int_0^T \psi(t) E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) dt\right) w_j(0)=f_j. \tag{15}$$

Hence, we have immediately that

$$w_j(0)=\frac{f_j}{a+bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right)+\int_0^T \psi(t) E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) dt}. \tag{16}$$

Combining (13) and (16), we get

$$w(x,t)=\sum_{j=1}^{\infty} \frac{E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) f_j}{a+bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right)+\int_0^T \psi(t) E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) dt} e_j(x). \tag{17}$$

If $a\geq 0, b>0$ and $\psi(t)>0$ then we get

$$\begin{aligned} \left\|w(.,t)\right\|^2_{\mathbb{H}^{\theta}(0,\pi)} &= \sum_{j=1}^{\infty} j^{4\theta}\left(\frac{E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) f_j}{a+bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right)+\int_0^T \psi(t) E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) dt}\right)^2 \\ &\leq \sum_{j=1}^{\infty} j^{4\theta}\left(\frac{E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) f_j}{bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right)}\right)^2. \end{aligned} \tag{18}$$

Here, since the condition $a\geq 0, b>0$ and $\psi(t)>0$, we note that

$$a+bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right)+\int_0^T \psi(t) E_{\alpha,1}\left((j^2+k)t^{\alpha}\right) dt \geq bE_{\alpha,1}\left((j^2+k)T^{\alpha}\right).$$

Using Lemma (2.1), we obtain that

$$E_{\alpha,1}\left((j^2+k)t^{\alpha}\right)\leq \frac{\mu_2}{\alpha}\exp\left((j^2+k)^{\frac{1}{\alpha}} t\right) \tag{19}$$

and

$$E_{\alpha,1}\left((j^2+k)T^\alpha\right) \geq \frac{\mu_1}{\alpha}\exp\Big((j^2+k)^{\frac{1}{\alpha}}T\Big) \tag{20}$$

From some previous observations, we deduce that

$$\Big\|w(.,t)\Big\|^2_{\mathbb{H}^\theta(0,\pi)} \leq \frac{\mu_2^2}{b^2\mu_1^2}\sum_{j=1}^\infty j^{4\theta}\exp\Big(2(j^2+k)^{\frac{1}{\alpha}}(t-T)\Big)f_j^2 \leq \frac{\mu_2^2}{b^2\mu_1^2}\sum_{j=1}^\infty j^{4\theta}f_j^2. \tag{21}$$

This implies that

$$\Big\|w(.,t)\Big\|_{\mathbb{H}^\theta(0,\pi)} \leq \frac{\mu_2}{b\mu_1}\Big\|f\Big\|_{\mathbb{H}^\theta(0,\pi)}. \tag{22}$$

Thus, we infer that $w \in L^\infty(0,T;\mathbb{H}^\theta(0,\pi))$ and

$$\Big\|w\Big\|_{L^\infty(0,T;\mathbb{H}^\theta(0,\pi))} \leq \frac{\mu_2}{b\mu_1}\Big\|f\Big\|_{\mathbb{H}^\theta(0,\pi)}. \tag{23}$$

We show the second case. If $\psi(t) \geq M_0$ then we get

$$\int_0^T \psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt \geq M_0\int_0^T E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt \geq \frac{M_0\mu_1}{\alpha}\int_0^T \exp\Big((j^2+k)^{\frac{1}{\alpha}}t\Big)dt.$$

Thus, we obtain

$$\int_0^T \psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt \geq \frac{M_0\mu_1}{\alpha(j^2+k)^{\frac{1}{\alpha}}}\Big[\exp\Big((j^2+k)^{\frac{1}{\alpha}}T\Big)-1\Big]. \tag{24}$$

This implies that

$$\begin{aligned}&\left(\frac{E_{\alpha,1}\left((j^2+k)t^\alpha\right)f_j}{a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)+\int_0^T\psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt}\right)^2\\ &\leq \frac{\mu_2^2(j^2+k)^{\frac{2}{\alpha}}\exp\Big(2(j^2+k)^{\frac{1}{\alpha}}t\Big)}{M_0^2\mu_1^2\Big[\exp\Big((j^2+k)^{\frac{1}{\alpha}}T\Big)-1\Big]^2} \lesssim (j^2+k)^{\frac{2}{\alpha}}.\end{aligned} \tag{25}$$

From above observation and noting that $a=b=0$, we deduce that

$$\begin{aligned}\Big\|w(.,t)\Big\|^2_{\mathbb{H}^\theta(0,\pi)} &= \sum_{j=1}^\infty j^{4\theta}\left(\frac{E_{\alpha,1}\left((j^2+k)t^\alpha\right)f_j}{a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)+\int_0^T\psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt}\right)^2\\ &\lesssim \sum_{j=1}^\infty j^{4\theta}(j^2+k)^{\frac{2}{\alpha}}|f_j|^2 \lesssim \Big\|f\Big\|^2_{\mathbb{H}^{\theta+\frac{1}{\alpha}}(0,\pi)}.\end{aligned} \tag{26}$$

The above estimate allows us to get the desired result (10). □

The following theorem gives us an interesting conclusion that solution of the Caputo modified Helmholtz equation ($k>0$) will converge to the solution of Caputo Poisson equation ($k=0$).

Theorem 2.2. *Let $\psi=0$. Let w^k be the solution to problem (1)-(2) with $k>0$. Let w^* be the solution problem (1)-(2) with $k=0$. Then if $f\in\mathbb{H}^\theta(0,\pi)$ then we get*

$$\|w^k-w^*\|_{L^\infty(0,T;\mathbb{H}^\theta(0,\pi))} \lesssim k^{\frac{1}{\alpha}}\|f\|_{\mathbb{H}^\theta(0,\pi)}. \tag{27}$$

Proof. Since (17), we infer that

$$w^k(x,t) = \sum_{j=1}^{\infty} \frac{E_{\alpha,1}\left((j^2+k)t^\alpha\right) f_j}{a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)} e_j(x), \quad w^*(x,t) = \sum_{j=1}^{\infty} \frac{E_{\alpha,1}\left(j^2 t^\alpha\right) f_j}{a + bE_{\alpha,1}\left(j^2 T^\alpha\right)} e_j(x). \tag{28}$$

From above observation, we find that

$$\begin{aligned} & w^k(x,t) - w^*(x,t) \\ &= \sum_{j=1}^{\infty} \frac{E_{\alpha,1}\left((j^2+k)t^\alpha\right)\Big(a + bE_{\alpha,1}\left(j^2T^\alpha\right)\Big) - E_{\alpha,1}\left(j^2 t^\alpha\right)\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a + bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} f_j e_j(x). \end{aligned} \tag{29}$$

The above expression is rewritten as follows

$$\begin{aligned} w^k(x,t) - w^*(x,t) &= \sum_{j=1}^{\infty} \frac{a\left(E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2t^\alpha\right)\right)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a + bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} f_j e_j(x) \\ &+ \sum_{j=1}^{\infty} \frac{bE_{\alpha,1}\left(j^2T^\alpha\right)\left(E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2t^\alpha\right)\right)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a + bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} f_j e_j(x) \\ &- \sum_{j=1}^{\infty} \frac{bE_{\alpha,1}\left(j^2t^\alpha\right)\left(E_{\alpha,1}\left((j^2+k)T^\alpha\right) - E_{\alpha,1}\left(j^2T^\alpha\right)\right)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a + bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} f_j e_j(x) \\ &= \mathcal{J}_1(k) + \mathcal{J}_2(k) + \mathcal{J}_3(k). \end{aligned} \tag{30}$$

Step 1. Estimation of $\mathcal{J}_1(k)$.
Let us first consider the term $\mathcal{J}_1(k)$. Indeed, we get

$$\left\|\mathcal{J}_1(k)\right\|^2_{\mathbb{H}^\theta(0,\pi)} = \sum_{j=1}^{\infty} j^{4\theta} \left(\frac{a\left(E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2t^\alpha\right)\right)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a + bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} \right)^2 |f_j|^2. \tag{31}$$

In view of the equality $\frac{d}{dz}E_{\alpha,1}(z) = \frac{E_{\alpha,\alpha}(z)}{\alpha}$, we know that

$$\begin{aligned} \Big|E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2t^\alpha\right)\Big| &= \Big|\frac{1}{\alpha}\int_{j^2t^\alpha}^{(j^2+k)t^\alpha} E_{\alpha,\alpha}(\xi)d\xi\Big| \\ &\le \frac{\mu_4}{\alpha^2}\int_{j^2t^\alpha}^{(j^2+k)t^\alpha} \xi^{\frac{1-\alpha}{\alpha}} \exp(\xi^{\frac{1}{\alpha}})d\xi \\ &= \frac{\mu_4}{\alpha^2}\exp(\xi^{\frac{1}{\alpha}})\Big|_{j^2t^\alpha}^{(j^2+k)t^\alpha} = \frac{\mu_4}{\alpha^2}\Big[\exp(t(j^2+k)^{\frac{1}{\alpha}}) - \exp(t(j^2)^{\frac{1}{\alpha}})\Big]. \end{aligned} \tag{32}$$

In addition, using the inequality $1 - e^{-z} \le z$ for any $z > 0$, we have

$$\begin{aligned} \exp(t(j^2+k)^{\frac{1}{\alpha}}) - \exp(t(j^2)^{\frac{1}{\alpha}}) &= \exp(t(j^2+k)^{\frac{1}{\alpha}})\Big[1 - e^{tj^{\frac{2}{\alpha}} - t(j^2+k)^{\frac{1}{\alpha}}}\Big] \\ &\le T\exp((j^2+k)^{\frac{1}{\alpha}}t)\left((j^2+k)^{\frac{1}{\alpha}} - j^{\frac{2}{\alpha}}\right). \end{aligned} \tag{33}$$

Using the inequality $(a+b)^m \le a^m + b^m$ for $0 < m < 1$ and $a, b > 0$, we obtain

$$(j^2+k)^{\frac{1}{\alpha}} - j^{\frac{2}{\alpha}} \le k^{\frac{1}{\alpha}} \tag{34}$$

since we note that $\frac{1}{\alpha} \le 1$. Due to some previous results, we know that

$$\exp(t(j^2+k)^{\frac{1}{\alpha}}) - \exp(t(j^2)^{\frac{1}{\alpha}}) \le \exp((j^2+k)^{\frac{1}{\alpha}} t). \tag{35}$$

This follows from (32) that

$$\Big| E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2 t^\alpha\right) \Big| \le \frac{T\mu_4}{\alpha^2} k^{\frac{1}{\alpha}} \exp((j^2+k)^{\frac{1}{\alpha}} t), \tag{36}$$

for any $0 \le t \le T$. In addition, since (44), we find that

$$a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right) \ge \frac{b\mu_1}{\alpha} \exp\Big((j^2+k)^{\frac{1}{\alpha}} T\Big). \tag{37}$$

Combining (31) and (37), we get

$$\Big\|\mathcal{I}_1(k)\Big\|^2_{\mathbb{H}^\theta(0,\pi)} \le \left(\frac{T\mu_4}{\alpha b \mu_1}\right)^2 k^{\frac{2}{\alpha}} \sum_{j=1}^\infty j^{4\theta} |f_j|^2. \tag{38}$$

Thus, we deduce that

$$\Big\|\mathcal{I}_1(k)\Big\|_{\mathbb{H}^\theta(0,\pi)} \le \frac{T\mu_4}{\alpha b\mu_1} k^{\frac{1}{\alpha}} \Big\|f\Big\|_{\mathbb{H}^\theta(0,\pi)}. \tag{39}$$

Step 2. Estimation of $\mathcal{I}_2(k)$.
Using Parseval's equality, we get that

$$\begin{aligned}\Big\|\mathcal{I}_2(k)\Big\|^2_{\mathbb{H}^\theta(0,\pi)} &= \sum_{j=1}^\infty j^{4\theta} \left(\frac{bE_{\alpha,1}\left(j^2T^\alpha\right)\left(E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2 t^\alpha\right)\right)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a+bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} \right)^2 |f_j|^2 \\ &\le \sum_{j=1}^\infty j^{4\theta} \left(\frac{\left(E_{\alpha,1}\left((j^2+k)t^\alpha\right) - E_{\alpha,1}\left(j^2 t^\alpha\right)\right)}{a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)} \right)^2 |f_j|^2. \end{aligned} \tag{40}$$

Combining (36) and (37) and (40), we infer that

$$\Big\|\mathcal{I}_2(k)\Big\|_{\mathbb{H}^\theta(0,\pi)} \le \frac{T\mu_4}{\alpha b\mu_1} k^{\frac{1}{\alpha}} \Big\|f\Big\|_{\mathbb{H}^\theta(0,\pi)}. \tag{41}$$

Step 3. Estimation of $\mathcal{I}_3(k)$.
Using Parseval's equality, we get that

$$\Big\|\mathcal{I}_3(k)\Big\|^2_{\mathbb{H}^\theta(0,\pi)} = \sum_{j=1}^\infty j^{4\theta} \left(\frac{bE_{\alpha,1}\left(j^2t^\alpha\right)\left(E_{\alpha,1}\left((j^2+k)T^\alpha\right) - E_{\alpha,1}\left(j^2 T^\alpha\right)\right)}{\Big(a + bE_{\alpha,1}\left((j^2+k)T^\alpha\right)\Big)\Big(a+bE_{\alpha,1}\left(j^2T^\alpha\right)\Big)} \right)^2 |f_j|^2. \tag{42}$$

Using Lemma (2.1), we obtain that

$$E_{\alpha,1}\left(j^2t^\alpha\right) \le \frac{\mu_2}{\alpha} \exp\left(j^{\frac{2}{\alpha}} t\right), \tag{43}$$

and

$$E_{\alpha,1}\big(j^2T^\alpha\big) \ge \frac{\mu_1}{\alpha} \exp\left(j^{\frac{2}{\alpha}} T\right) \tag{44}$$

Thus, we provide that

$$\frac{bE_{\alpha,1}\left(j^2t^\alpha\right)}{a+bE_{\alpha,1}\left(j^2T^\alpha\right)} \le \frac{E_{\alpha,1}\left(j^2t^\alpha\right)}{E_{\alpha,1}\left(j^2T^\alpha\right)} \le \frac{\mu_2}{\mu_1}\exp\left(j^{\frac{2}{\alpha}}(t-T)\right) \le \frac{\mu_2}{\mu_1}. \tag{45}$$

Moreover, using (36) and (37), we get that

$$\frac{\left|E_{\alpha,1}\left((j^2+k)T^\alpha\right)-E_{\alpha,1}\left(j^2T^\alpha\right)\right|}{a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)} \le \frac{T\mu_4}{\alpha b\mu_1}k^{\frac{1}{\alpha}} \tag{46}$$

Combining (42), (45) and (46), we deduce that

$$\left\|\mathcal{I}_3(k)\right\|^2_{\mathbb{H}^\theta(0,\pi)} \le \left(\frac{T\mu_4\mu_2}{\alpha b|\mu_1|^2}\right)^2 k^{\frac{2}{\alpha}}\sum_{j=1}^\infty j^{4\theta}|f_j|^2. \tag{47}$$

Therefore

$$\left\|\mathcal{I}_3(k)\right\|_{\mathbb{H}^\theta(0,\pi)} \le \frac{T\mu_4\mu_2}{\alpha b|\mu_1|^2}k^{\frac{1}{\alpha}}\left\|f\right\|_{\mathbb{H}^\theta(0,\pi)}. \tag{48}$$

In view of three results as in three steps, we conclude that

$$\left\|w^k(.,t)-w^*(.,t)\right\|_{\mathbb{H}^\theta(0,\pi)} \le \sum_{j=1}^3\left\|\mathcal{I}_j(k)\right\|_{\mathbb{H}^\theta(0,\pi)} \lesssim k^{\frac{1}{\alpha}}\left\|f\right\|_{\mathbb{H}^\theta(0,\pi)}. \tag{49}$$

From the right hand side of the above expression independent of t, we immediately obtain the assertion (27).

□

Theorem 2.3. *Let the function $f \in H^\theta(0,\pi)$. Then if $a \ge 0, b \ge 0$ and $0 < \psi(t) \le M_1$, $M_1 > 0$. Then problem (1)-(2) has a unique solution w which satisfies*

$$\left\|w(.,T)\right\|_{\mathbb{H}^\theta(0,\pi)} \ge \frac{\mu_1}{\alpha\left(a+\frac{b\mu_2}{\alpha}+\frac{M_1\mu_2}{\alpha(1+k)^{\frac{1}{\alpha}}}\right)}\left\|f\right\|_{\mathbb{H}^\theta(0,\pi)}. \tag{50}$$

Proof. In view of (17), we get that

$$w(x,T) = \sum_{j=1}^\infty \frac{E_{\alpha,1}\left((j^2+k)T^\alpha\right)f_j}{a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)+\int_0^T\psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt}e_j(x). \tag{51}$$

Using the Parseval's equality, we get that

$$\left\|w(.,T)\right\|^2_{\mathbb{H}^\theta(0,\pi)} = \sum_{j=1}^\infty j^{4\theta}\left(\frac{E_{\alpha,1}\left((j^2+k)T^\alpha\right)}{a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)+\int_0^T\psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt}\right)^2|f_j|^2. \tag{52}$$

We recall that from (44)

$$\frac{\mu_2}{\alpha}\exp\left((j^2+k)^{\frac{1}{\alpha}}T\right) \ge E_{\alpha,1}\left((j^2+k)T^\alpha\right) \ge \frac{\mu_1}{\alpha}\exp\left((j^2+k)^{\frac{1}{\alpha}}T\right). \tag{53}$$

Since the condition $\psi(t) \le M_1$, we know that

$$\int_0^T \psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt \le M_1\int_0^T E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt$$
$$\le \frac{M_1\mu_2}{\alpha}\int_0^T \exp\left((j^2+k)^{\frac{1}{\alpha}}t\right)dt$$
$$= \frac{M_1\mu_2}{\alpha(j^2+k)^{\frac{1}{\alpha}}}\Big[\exp\left((j^2+k)^{\frac{1}{\alpha}}T\right)-1\Big]. \tag{54}$$

Since $j^2+k \ge 1+k$, we follows from two above observations that

$$a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)+\int_0^T \psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt$$
$$\le \left(a+\frac{b\mu_2}{\alpha}+\frac{M_1\mu_2}{\alpha(1+k)^{\frac{1}{\alpha}}}\right)\exp\left((j^2+k)^{\frac{1}{\alpha}}T\right). \tag{55}$$

Hence, we deduce that

$$\frac{E_{\alpha,1}\left((j^2+k)T^\alpha\right)}{a+bE_{\alpha,1}\left((j^2+k)T^\alpha\right)+\int_0^T \psi(t)E_{\alpha,1}\left((j^2+k)t^\alpha\right)dt} \ge \frac{\mu_1}{\alpha\Big(a+\frac{b\mu_2}{\alpha}+\frac{M_1\mu_2}{\alpha(1+k)^{\frac{1}{\alpha}}}\Big)}. \tag{56}$$

Thus, we follows from (57) that

$$\left\|w(.,T)\right\|^2_{\mathbb{H}^\theta(0,\pi)} \ge \frac{\mu_1^2}{\alpha^2\Big(a+\frac{b\mu_2}{\alpha}+\frac{M_1\mu_2}{\alpha(1+k)^{\frac{1}{\alpha}}}\Big)^2}\sum_{j=1}^{\infty} j^{4\theta}|f_j|^2. \tag{57}$$

The proof is completed. □

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