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Research Article

On the Leonardo quaternions sequence

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Abstract

In the present work, a new sequence of quaternions related to the Leonardo numbers – named the Leonardo quaternions sequence – is defined and studied. Binet's formula and certain sum and binomial-sum identities, some of which derived from the mentioned formula, are established. Tagiuri-Vajda's identity and, as consequences, Catalan's identity, d'Ocagne's identity and Cassini's identity are presented. Furthermore, applying Catalan's identity, and the connection between composition algebras and vector cross product algebras, Gelin-Cesàro's identity is also stated and proved. Finally, the generating function, the exponential generating function and the Poisson generating function are deduced. In addition to the results on Leonardo quaternions, known results on Leonardo numbers and on Fibonacci quaternions are extended.

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1. Introduction

The problem of classifying all composition algebras with identity (that is, Hurwitz algebras, [6, 7]) became known as the Hurwitz's problem. Briefly speaking, what are all pairs consisting of a not necessarily associative algebra A with identity and a nondegenerate quadratic form Q on A such that $Q(xy) = Q(x)Q(y)$? The problem was solved by Hurwitz, for the real and complex cases, and by Jacobson, for a field of characteristic different from 2. In fact, the complete answer was given by the generalized Hurwitz Theorem which asserts that, over a field of characteristic different from 2, A is isomorphic either to the base field, a separable quadratic extension of the base field, a generalized quaternion algebra or a generalized octonion algebra, repectively, of dimensions $1, 2, 4$ and 8, [28]. With respect to quaternions, as can be seen by the number and the variety of publications produced over the years, the interest in this subject remains very much alive. For instance, among many others, some recent works on quaternions are related to: algebra, as [6] and [7]; analysis on manifolds and differential geometry, as [23]; differential and difference equations, as [8]; spatial analytic geometry, as [45]; numerical analysis,

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as [27]; quaternionic analysis, as [38]; spectral theory and quaternion groups, as [11]; algebraic number theory, as [9]. The present work belongs to the latter area, concerning, more concretely, a new sequence, which is a quaternions sequence, named the Leonardo quaternions sequence. This sequence is related to the Leonardo numbers, whose associated sequence is part of the smoothsort algorithm due to Dijkstra, [25].

The sequence of Leonardo numbers was studied by Catarino and Borges in [15] and [16]. In the former reference, several properties and identities were established. An incomplete version of the Leonardo numbers was defined in the latter reference. In [2], Alves and Vieira studied the roots of the characteristic polynomial derived from the sequence of Leonardo, performing a root search through the Newton fractal. In [53], Vieira, Alves and Catarino discussed the bidimensional recurrent relations of the Leonardo numbers starting with its unidimensional model, establishing also the connection between this sequence and the sequence of Fibonacci numbers. In [54], Vieira, Mangueira, Alves and Catarino undertook research on generating matrices for the Leonardo numbers and, considering its nth powers, obtained new relations related to the sequence of Leonardo numbers. More recently, in [5], Alp and Koçer deduced several identities involving Leonardo numbers. In general, integer sequences have been the subject of several research studies. A vast bibliography covers several topics related to integer sequences. One can find in studies presented in [1, 13, 15–17, 19, 26, 34, 42, 44, 53, 54] some properties of these sequences, which are a very small proportion of articles devoted to the various number sequences. Sequences of polynomials have also been considered by several researchers. For instance, Cação, Malonek and Tomaz, in [12], work with shifted generalized Pascal matrices in the context of Appell sequences.

Earlier papers on distinct sequences involving quaternions are [3, 4, 9, 10, 14, 18, 20, 21, 29–32, 39, 42, 43, 47–51, 55], among others. Motivated by a rational sequence, quaternions whose components are Vietoris' numbers are studied in [18]. In [3, 41, 51, 55], a split version of quaternions is considered. More concretely, in the latter references we find the study of split Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas quaternions and their generalizations, as well as the split generalized Fibonacci quaternions. In [4], the non split version of generalized quaternions is introduced, while in [10] and [43] the authors study other generalizations of Fibonacci and Lucas quaternions. Another kind of Pell and Pell-Lucas quaternions are presented in [9], where some identities satisfied by these sequences of quaternions and their respective generating functions are stated. In [14], a new sequence of quaternions is defined, as well as in references [20, 21, 29–31, 47, 48, 50], where also some properties of these sequences are established. From a generalization for Fibonacci quaternions due to İpek in [33], (p, q)-Lucas quaternions are introduced in [39] and, afterwards, the authors derive some identities for both of these quaternions sequences. In reference [49], some matrix representations associated with the Horadam quaternions are presented. Furthermore, using the matrix technique, many identities related to the mentioned quaternions and some binomial-sum identites, as an application, are derived.

The structure of the present work, divided into two more sections, is as follows. In section 2, where some background is presented, known definitions and results related to the real division algebra of quaternions, to the Fibonacci numbers sequence, to the Fibonacci quaternions sequence and to the Leonardo numbers sequence are collected. In addition, known results on Leonardo numbers and on Fibonacci quaternions are extended. The connection between composition algebras and vector cross product algebras, used in a later section, is recalled in section 2 too. After this introductory section, section 3 is dedicated to the definition of a new sequence of quaternions related to the Leonardo numbers – named the Leonardo quaternions sequence – and to the study of its properties. In particular, beyond some basic properties, Binet's formula is deduced. Some sum and binomial-sum identities, some of which derived from Binet's formula, are also established.

Several identities, namely Tagiuri-Vajda's identity and, as consequences, Catalan's identity, d'Ocagne's identity and Cassini's identity, are presented. Moreover, applying Catalan's identity, and the connection between composition algebras and vector cross product algebras, specifically for quaternions, Gelin-Cesàro's identity is also stated and proved. Lastly, the generating function, the exponential generating function and the Poisson generating function are deduced. All the cited references may be found in the corresponding list at the end of the paper.

2. Some background

2.1. The real division algebra of quaternions

To start with, recall that the real algebra of quaternions can be defined as $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ with multiplication determine by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Recall also that $\mathbb{H}$ is a non-commutative associative division algebra that is unital. Its identity denoted by 1, whenever clear from the context, is omitted.

This algebra splits as $\mathbb{H} = \mathbb{R}1 \oplus \mathrm{Im}\mathbb{H}$, where $\mathrm{Im}\mathbb{H} = \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ is known as the subspace of purely imaginary quaternions. In this sense, every quaternion q can be written as $q = q_0 1 + \tilde{q}$ with $\tilde{q} \in \mathrm{Im}\mathbb{H} \simeq \mathbb{R}^3$. The map $\overline{} : \mathbb{H} \to \mathbb{H}$ defined as

$$q = q_0 + \tilde{q} \mapsto \overline{q} = q_0 - \tilde{q} \tag{2.1}$$

is an involution, that is, for all $p, q \in \mathbb{H}$, $\overline{\overline{q}} = q$, $\overline{pq} = \overline{q}\,\overline{p}$, $\overline{p+q} = \overline{p} + \overline{q}$. The mentioned map is commonly called the standard involution.

The 4-dimensional algebra $\mathbb{H}$ possesses the structure of composition algebra, provided by the non-degenerate quadratic form $N : \mathbb{H} \to \mathbb{R}$ defined as

$$q \mapsto N(q) = q\overline{q} = \overline{q}q, \tag{2.2}$$

which is mutliplicative, that is, for all $p, q \in \mathbb{H}$, $N(pq) = N(p)N(q)$. Usually, in the sense of composition algebras, $N(q)$ is called the norm of the quaternion q.

As can be seen in [8], [37] and references there in, there is a connection between composition algebras and vector cross product algebras. In particular, the multiplication of $\mathbb{H}$, as recalled in the following result, can be written in terms of the Euclidean inner product and the usual vector cross product in $\mathbb{R}^3$, hereinafter denoted by $\langle \cdot, \cdot \rangle$ and $\times$, respectively.

Theorem 2.1 ([37]). *Let $p, q \in \mathbb{H}$. Then*

$$pq = p_0 q_0 - \langle \tilde{p}, \tilde{q} \rangle + p_0 \tilde{q} + q_0 \tilde{p} + \tilde{p} \times \tilde{q}.$$

This theorem enhances the proof of part of a result – consisting of two equalities which involve quaternions – presented by Daşdemir in [24], where he mentions a proof by employing the multiplication table of $\mathbb{H}$. For completness, making use of the relation among the multiplication of $\mathbb{H}$, $\langle \cdot, \cdot \rangle$ and $\times$ expressed in Theorem 2.1, we present a different proof of one of the equalities.

Proposition 2.2 ([24]). *Let $p, q \in \mathbb{H}$. Then*

$$p^2 = 2p_0 p - N(p), \tag{2.3}$$

$$pq = qp + 2\tilde{p} \times \tilde{q}. \tag{2.4}$$

Proof. Let $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}, q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$. As, by straightforward calculations,

$$\begin{aligned}
&p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 + 2(p_0 p_1 \mathbf{i} + p_0 p_2 \mathbf{j} + p_0 p_3 \mathbf{k}),\\
&2p_0 p = 2p_0^2 + 2(p_0 p_1 \mathbf{i} + p_0 p_2 \mathbf{j} + p_0 p_3 \mathbf{k}),\\
&N(p) = \overline{p}p = p_0^2 + p_1^2 + p_2^2 + p_3^2,
\end{aligned}$$

then we arrive at (2.3). Concerning (2.4), Theorem 2.1 allows us to write

$$pq = p_0q_0 - \langle \tilde{p}, \tilde{q} \rangle + p_0\tilde{q} + q_0\tilde{p} + \tilde{p} \times \tilde{q},$$
$$qp = q_0p_0 - \langle \tilde{q}, \tilde{p} \rangle + q_0\tilde{p} + p_0\tilde{q} + \tilde{q} \times \tilde{p},$$

and, subtracting member by member, to obtain (2.4). □

For other more aspects related to quaternions see, for instance, [22] – a general reference for the present section.

2.2. The Fibonacci numbers sequence

Although the Fibonacci sequence of numbers is one of the most well known sequences, for completeness, it is recalled.

Definition 2.3 ([36]). The sequence $\{F_n\}_{n=0}^{\infty}$ of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, n \in \mathbb{N}_0 \text{ such that } n \geq 2, \tag{2.5}$$

with initial conditions given by $F_0 = 0$ and $F_1 = 1$.

In the next theorem, Binet's formula for Fibonacci numbers is also recalled.

Theorem 2.4 (Binet's formula, [36]). *For $n \in \mathbb{N}_0$,*

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}, \tag{2.6}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Assuming that Binet's formula (2.6) holds for negative subscripts, it is possible to extend Definition 2.3 to negative subscripts as follows.

Definition 2.5 ([36]). For $n \in \mathbb{N}$,

$$F_{-n} = (-1)^{n+1} F_n. \tag{2.7}$$

In the following proposition, several properties related to Fibonacci numbers, one also related to Lucas numbers, are collected.

Proposition 2.6. *For $s, t \in \mathbb{Z}$, the following identities hold,*

(a) [35]
$$F_{s+t} = F_t F_{s+1} + F_{t-1} F_s; \tag{2.8}$$

(b) [36]
$$F_{2s} = F_{s+1}^2 - F_{s-1}^2; \tag{2.9}$$

(c) [36]
$$F_{2s} = F_s L_s; \tag{2.10}$$

(d) [36]
$$(-1)^t F_{s-t} = F_s F_{t+1} - F_{s+1} F_t; \tag{2.11}$$

(e) [52]
$$(-1)^s F_t F_r = F_{s+t} F_{s+r} - F_s F_{s+t+r}. \tag{2.12}$$

2.3. The Fibonacci quaternions sequence

The Fibonacci quaternions were introduced by Horadam in [32], where he presented the definition and some properties.

Definition 2.7 ([32]). The sth element of the Fibonacci quaternions sequence $\{QF_n\}_{n=0}^{\infty}$ is given by

$$QF_s = F_s + F_{s+1}\mathbf{i} + F_{s+2}\mathbf{j} + F_{s+3}\mathbf{k}, \quad s \in \mathbb{N}_0. \tag{2.13}$$

Halici, in [29], continued Horadam's study and presented the subsequent results that we recall, namely the following recurrence relation and Binet's formula, respectively.

Proposition 2.8 ([29]). *For $n \in \mathbb{N}_0$,*

$$QF_{n+1} = QF_n + QF_{n-1}. \tag{2.14}$$

Theorem 2.9 (Binet's formula, [29]). *For $n \in \mathbb{N}_0$,*

$$QF_n = \frac{\underline{\phi}\phi^n - \underline{\psi}\psi^n}{\phi - \psi}, \tag{2.15}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, $\underline{\phi} = 1 + \phi\mathbf{i} + \phi^2\mathbf{j} + \phi^3\mathbf{k}$ *and* $\underline{\psi} = 1 + \psi\mathbf{i} + \psi^2\mathbf{j} + \psi^3\mathbf{k}$.

Assuming that Binet's formula (2.15) holds for negative subscripts and taking into account Definition 2.5, it is possible to extend Definition 2.7 to negative subscripts as follows.

Definition 2.10. For $s \in \mathbb{N}$,

$$QF_{-s} = F_{-s} + F_{-s+1}\mathbf{i} + F_{-s+2}\mathbf{j} + F_{-s+3}\mathbf{k}. \tag{2.16}$$

The next lemma and theorem establish Tagiuri-Vajda's identity for the Fibonacci quaternions.

Lemma 2.11. *For $r \in \mathbb{Z}$,*

$$\underline{\phi\psi}\psi^r - \underline{\psi\phi}\phi^r = (\phi - \psi)[-2QF_r + 3(2F_{r+1} - F_r)\mathbf{k}], \tag{2.17}$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ *and* $\psi = \dfrac{1-\sqrt{5}}{2}$.

Proof. Straightforward calculations lead to

$$\underline{\phi\psi} = 2 + 2\psi\mathbf{i} + (2 + 2\psi)\mathbf{j} + (5 - 2\psi)\mathbf{k}$$

and

$$\underline{\psi\phi} = 2 + 2\phi\mathbf{i} + (2 + 2\phi)\mathbf{j} + (5 - 2\phi)\mathbf{k}.$$

Invoking Binet's formula (2.6) for the Fibonacci numbers in Theorem 2.4,

$$\begin{aligned}
&\underline{\phi\psi}\psi^r - \underline{\psi\phi}\phi^r \\
&\quad = 2(\psi^r - \phi^r) + 2(\psi^{r+1} - \phi^{r+1})\mathbf{i} + 2(\psi^r + \psi^{r+1} - \phi^r - \phi^{r+1})\mathbf{j} \\
&\qquad + (5\psi^r - 2\psi^{r+1} - 5\phi^r + 2\phi^{r+1})\mathbf{k} \\
&\quad = -2(\phi - \psi)\left[\frac{\phi^r - \psi^r}{\phi - \psi} + \frac{\phi^{r+1} - \psi^{r+1}}{\phi - \psi}\mathbf{i} + \left(\frac{\phi^r - \psi^r}{\phi - \psi} + \frac{\phi^{r+1} - \psi^{r+1}}{\phi - \psi}\right)\mathbf{j}\right] \\
&\qquad + (\phi - \psi)\left(-5\frac{\phi^r - \psi^r}{\phi - \psi} + 2\frac{\phi^{r+1} - \psi^{r+1}}{\phi - \psi}\right)\mathbf{k} \\
&\quad = -2(\phi - \psi)(F_r + F_{r+1}\mathbf{i} + F_r\mathbf{j} + F_{r+1}\mathbf{j}) + (\phi - \psi)(-5F_r + 2F_{r+1})\mathbf{k} \\
&\quad = -2(\phi - \psi)(F_r + F_{r+1}\mathbf{i} + F_{r+2}\mathbf{j} + F_{r+3}\mathbf{k}) + 2(\phi - \psi)F_{r+3}\mathbf{k} \\
&\qquad + (\phi - \psi)(-5F_r + 2F_{r+1})\mathbf{k} \\
&\quad = (\phi - \psi)(-2QF_r + 2F_{r+3}\mathbf{k} - 5F_r\mathbf{k} + 2F_{r+1}\mathbf{k}) \\
&\quad = (\phi - \psi)(-2QF_r + 4F_{r+1}\mathbf{k} + 2F_{r+2}\mathbf{k} - 5F_r\mathbf{k}) \\
&\quad = (\phi - \psi)(-2QF_r + 6F_{r+1}\mathbf{k} - 3F_r\mathbf{k}),
\end{aligned}$$

and the lemma follows. □

Theorem 2.12 (Tagiuri-Vajda's identity)**.** *For $n, k, r \in \mathbb{Z}$,*

$$QF_{n+k}QF_{n+r} - QF_nQF_{n+k+r} = (-1)^n F_k[2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}]. \tag{2.18}$$

Proof. Applying Binet's formula (2.15) for the Fibonacci quaternions in Theorem 2.9, we have, on the one hand,

$$\begin{aligned} &QF_{n+k}QF_{n+r} \\ &= \left(\frac{\underline{\phi}\phi^{n+k} - \underline{\psi}\psi^{n+k}}{\phi - \psi}\right)\left(\frac{\underline{\phi}\phi^{n+r} - \underline{\psi}\psi^{n+r}}{\phi - \psi}\right) \\ &= \frac{1}{(\phi-\psi)^2}\left(\phi^{2n+k+r}\underline{\phi}^2 - \psi^{n+r}\phi^{n+k}\underline{\phi\psi} - \psi^{n+k}\phi^{n+r}\underline{\psi\phi} + \psi^{2n+k+r}\underline{\psi}^2\right), \end{aligned}$$

and, on the other hand,

$$\begin{aligned} &QF_nQF_{n+k+r} \\ &= \left(\frac{\underline{\phi}\phi^{n} - \underline{\psi}\psi^{n}}{\phi - \psi}\right)\left(\frac{\underline{\phi}\phi^{n+k+r} - \underline{\psi}\psi^{n+k+r}}{\phi - \psi}\right) \\ &= \frac{1}{(\phi-\psi)^2}\left(\phi^{2n+k+r}\underline{\phi}^2 - \phi^{n}\psi^{n+k+r}\underline{\phi\psi} - \psi^{n}\phi^{n+k+r}\underline{\psi\phi} + \psi^{2n+k+r}\underline{\psi}^2\right). \end{aligned}$$

Hence, Binet's formula (2.6) for the Fibonacci numbers in Theorem 2.4 and, once again, (2.15) lead to

$$\begin{aligned} &QF_{n+k}QF_{n+r} - QF_nQF_{n+k+r} \\ &= \frac{(\phi\psi)^n}{(\phi-\psi)^2}(-\psi^r\phi^k\underline{\phi\psi} - \psi^k\phi^r\underline{\psi\phi} + \psi^{k+r}\underline{\phi\psi} + \phi^{k+r}\underline{\psi\phi}) \\ &= \frac{(-1)^n}{(\phi-\psi)^2}(\phi^k - \psi^k)(\underline{\psi\phi}\phi^r - \underline{\phi\psi}\psi^r) \\ &= \frac{(-1)^n}{\phi-\psi}F_k(\underline{\psi\phi}\phi^r - \underline{\phi\psi}\psi^r) \\ &= \frac{(-1)^{n+1}}{\phi-\psi}F_k(\underline{\phi\psi}\psi^r - \underline{\psi\phi}\phi^r) \\ &= \frac{(-1)^{n+1}}{\phi-\psi}F_k(\phi-\psi)[-2QF_r + 3(2F_{r+1} - F_r)\mathbf{k}], \end{aligned}$$

which, after simplification, is the right hand side of (2.18). □

As consequences of Tagiuri-Vajda's identity, the last results of this section establish, respectively, Catalan's identity, d'Ocagne's identity and Cassini's identity for the Fibonacci quaternions. We point out that Halici presented Cassini's identity for the Fibonacci quaternions in [29], which she obtained, for non-negative subscripts, in a different way – from the Fibonacci quaternion matrix. In addition, as a particular case of Catalan's identity for k-Fibonacci quaternions due to Polath and Kesim, [40], Daşdemir presented Catalan's identity, with positive subscripts, for the Fibonacci quaternions in [24].

Corollary 2.13 (Catalan's identity)**.** *For $n, r \in \mathbb{Z}$,*

$$QF_{n-r}QF_{n+r} - QF_n^2 = (-1)^{n-r+1}(2F_rQF_r - 3F_{2r}\mathbf{k}). \tag{2.19}$$

Proof. Taking $k = -r$ in (2.18) gives

$$\begin{aligned} &QF_{n-r}QF_{n+r} - QF_n^2 \\ &\quad = (-1)^n F_{-r}[2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}] \\ &\quad = (-1)^{n+r+1}F_r[2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}] \\ &\quad = (-1)^{n-r+1}[2F_rQF_r - 3F_r(2F_{r+1} - F_r)\mathbf{k}] \\ &\quad = (-1)^{n-r+1}[2F_rQF_r - 3(F_{r+1} - F_{r-1})(F_{r+1} + F_{r-1})\mathbf{k}], \end{aligned}$$

which, by (2.9) in Proposition 2.6, leads to (2.19). □

Corollary 2.14 (d'Ocagne's identity). *For $n, m \in \mathbb{Z}$,*

$$QF_{n+1}QF_m - QF_nQF_{m+1} = (-1)^n[2QF_{m-n} - 3(2F_{m-n+1} - F_{m-n})\mathbf{k}]. \tag{2.20}$$

Proof. It suffices to consider $r = m - n$ and $k = 1$ in (2.18). □

Corollary 2.15 (Cassini's identity). *For $n \in \mathbb{Z}$,*

$$QF_{n-1}QF_{n+1} - QF_n^2 = (-1)^n(2QF_1 - 3\mathbf{k}). \tag{2.21}$$

Proof. With $r = -k = 1$ in (2.18), we get (2.21). In fact,

$$\begin{aligned} QF_{n-1}QF_{n+1} - QF_n^2 &= (-1)^nF_{-1}[2QF_1 - 3(2F_2 - F_1)\mathbf{k}] \\ &= (-1)^n(-1)^2F_1[2QF_1 - 3(2F_2 - F_1)\mathbf{k}]. \end{aligned}$$

□

2.4. The Leonardo numbers sequence

The sequence of Leonardo numbers is entry A001595 of the On-line Encyclopedia of Integers Sequences, [46], and was studied by Catarino and Borges in [15].

Definition 2.16 ([15]). The sequence $\{Le_n\}_{n=0}^{\infty}$ of Leonardo numbers is defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \in \mathbb{N}_0 \text{ such that } n \geq 2, \tag{2.22}$$

with initial conditions given by $Le_0 = Le_1 = 1$.

As recalled in the following results, the sequence of Leonardo numbers can be related to the well-known sequence of Fibonacci numbers, as well as to the Lucas sequence $\{L_n\}_{n=0}^{\infty}$. The latter sequence of numbers has been presented in several research papers in the literature.

Proposition 2.17 ([15]). *For $n \in \mathbb{N}_0$,*

$$Le_n = 2F_{n+1} - 1. \tag{2.23}$$

Proposition 2.18 ([15]). *For $n \in \mathbb{N}_0$,*

$$Le_n = 2\left(\frac{L_n + L_{n+2}}{5}\right) - 1, \tag{2.24}$$

$$Le_{n+3} = \frac{L_{n+1} + L_{n+7}}{5} - 1, \tag{2.25}$$

$$Le_n = L_{n+2} - F_{n+2} - 1. \tag{2.26}$$

In the next result, the Binet's formula for Leonardo numbers is also recalled.

Proposition 2.19 (Binet's formula, [15]). *For $n \in \mathbb{N}_0$,*

$$Le_n = 2\frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} - 1, \tag{2.27}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ e $\psi = \frac{1-\sqrt{5}}{2}$.

Assuming that Binet's formula (2.27) holds for negative subscripts, it is possible to extend Definition 2.16 to negative subscripts. In fact,

$$\begin{aligned} Le_{-n} &= 2\frac{\phi^{-n+1}-\psi^{-n+1}}{\phi-\psi}-1 \\ &= 2\frac{(-\psi)^{n-1}-(-\phi)^{n-1}}{\phi-\psi}-1 \\ &= 2(-1)^n\frac{\phi^{n-1}-\psi^{n-1}}{\phi-\psi}-1 \\ &= 2(-1)^n F_{n-1}-1, \end{aligned}$$

which leads to the next definition. This definition appears also in [5], although introduced in a different way – making use of the definition of Fibonacci numbers with negative subscripts.

Definition 2.20. For $n \in \mathbb{N}$,

$$Le_{-n} = (-1)^n(Le_{n-2}+1)-1. \tag{2.28}$$

In the following theorem, Tagiuri-Vajda's identity for the Leonardo numbers is established. An alternative form of presenting this identity was given by Alp and Koçer in [5], where the authors expressed it, only for non-negative subscripts, in terms of Fibonacci numbers and Leonardo numbers.

Theorem 2.21 (Tagiuri-Vajda's identity)**.** *For $n, k, r \in \mathbb{Z}$,*

$$\begin{aligned} &Le_{n+k}Le_{n+r} - Le_nLe_{n+k+r} \\ &\quad = (-1)^{n+1}(Le_{k-1}+1)\left[\frac{1}{2}(-1)^{n+1}(Le_{n+r-1}-Le_{n-1}) + Le_{r-1}+1\right] \\ &\qquad + \frac{1}{2}(Le_k-1)(Le_{n+r}-Le_n). \end{aligned} \tag{2.29}$$

Proof. From (2.23) in Proposition 2.17, (2.8) and (2.12) in Proposition 2.6,

$$\begin{aligned} &Le_{n+k}Le_{n+r} - Le_nLe_{n+k+r} \\ &\quad = (2F_{n+k+1}-1)(2F_{n+r+1}-1) - (2F_{n+1}-1)(2F_{n+k+r+1}-1) \\ &\quad = 4(F_{n+1+k}F_{n+1+r} - F_{n+1}F_{n+1+k+r}) \\ &\qquad +2(F_{n+1}+F_{n+1+k+r}-F_{n+k+1}-F_{n+r+1}) \\ &\quad = 4(-1)^{n+1}F_kF_r + 2(F_{n+1}+F_{k-1}F_{n+1+r}+F_kF_{n+r+2}-F_{n+k+1}-F_{n+r+1}) \\ &\quad = 4(-1)^{n+1}F_kF_r \\ &\qquad +2[F_{n+1+r}(F_{k-1}-1)+F_k(F_{n+r+1}+F_{n+r})+F_{n+1}-F_{n+k+1}] \\ &\quad = 4(-1)^{n+1}F_kF_r + 2[F_{n+1+r}(F_{k-1}+F_k-1)+F_kF_{n+r}+F_{n+1}-F_{n+k+1}] \\ &\quad = 4(-1)^{n+1}F_kF_r \\ &\qquad +2[F_{n+1+r}(F_{k+1}-1)+F_kF_{n+r}+F_{n+1}-F_kF_n-F_{k+1}F_{n+1}] \\ &\quad = 4(-1)^{n+1}F_kF_r + 2[F_{n+1+r}(F_{k+1}-1)+F_k(F_{n+r}-F_n)+F_{n+1}(1-F_{k+1})] \\ &\quad = 4(-1)^{n+1}F_kF_r + 2(F_{k+1}-1)(F_{n+1+r}-F_{n+1}) + 2F_k(F_{n+r}-F_n) \\ &\quad = (-1)^{n+1}(Le_{k-1}+1)(Le_{r-1}+1) + \tfrac{1}{2}(Le_k-1)(Le_{n+r}-Le_n) \\ &\qquad +\frac{1}{2}(Le_{k-1}+1)(Le_{n+r-1}-Le_{n-1}), \end{aligned}$$

which easily leads to (2.29). □

As consequences of Tagiuri-Vajda's identity, Catalan's identity, d'Ocagne's identity and Cassini's identity for Leonardo numbers are, respectively, obtained in the following results. The latter three identities, without using Tagiuri-Vajda's identity and only for non-negative subscripts, have been previously presented by Catarino and Borges in [15].

Corollary 2.22 (Catalan's identity)**.** *For $n, r \in \mathbb{Z}$,*

$$Le_{n-r}Le_{n+r} - Le_n^2 = \quad -Le_{n-r} - Le_{n+r} + 2Le_n + (-1)^{n-r}(Le_{r-1}+1)^2. \tag{2.30}$$

Proof. Taking $k = -r$ in (2.29) leads to

$$\begin{aligned}
&(-1)^{n+1}(Le_{-r-1}+1)[Le_{r-1}+1+\frac{1}{2}(-1)^{n+1}(Le_{n+r-1}-Le_{n-1})]\\
&\quad +\frac{1}{2}(Le_{-r}-1)(Le_{n+r}-Le_n)\\
&\quad = (-1)^{n+1}[(-1)^{r+1}(Le_{r-1}+1)][Le_{r-1}+1+(-1)^{n+1}\tfrac{1}{2}(Le_{n+r-1}-Le_{n-1})]\\
&\quad +\frac{1}{2}[(-1)^r(Le_{r-2}+1)-2](Le_{n+r}-Le_n)\\
&\quad = (-1)^{n+r}(Le_{r-1}+1)[Le_{r-1}+1+(-1)^{n+1}\tfrac{1}{2}(Le_{n+r-1}-Le_{n-1})]\\
&\quad +\frac{1}{2}[(-1)^r(Le_{r-2}+1)-2](Le_{n+r}-Le_n)\\
&\quad = (-1)^{n+r}(Le_{r-1}+1)^2+\frac{1}{2}(-1)^{r+1}(Le_{r-1}+1)(Le_{n+r-1}-Le_{n-1})]\\
&\quad +\left[\frac{1}{2}(-1)^r(Le_{r-2}+1)-1\right](Le_{n+r}-Le_n)\\
&\quad = (-1)^{n-r}(Le_{r-1}+1)^2+\frac{1}{2}(-1)^{r}(Le_{r-1}+1)(Le_{n-1}-Le_{n+r-1})]\\
&\quad +\frac{1}{2}(-1)^r(Le_{r-2}+1)(Le_{n+r}-Le_n)-Le_{n+r}+Le_n
\end{aligned}$$

As, applying (2.11) in Proposition 2.6 and (2.23) in Proposition 2.17,

$$\begin{aligned}
&\frac{1}{2}(-1)^r(Le_{r-1}+1)(Le_{n-1}-Le_{n+r-1})]+\frac{1}{2}(-1)^r(Le_{r-2}+1)(Le_{n+r}-Le_n)\\
&\quad = \frac{1}{2}(-1)^r[2F_r(2F_n-2F_{n+r})+2F_{r-1}(2F_{n+r+1}-2F_{n+1})]\\
&\quad = 2(-1)^r(F_nF_r-F_{n+1}F_{r-1}+F_{r-1}F_{n+r+1}-F_rF_{n+r})\\
&\quad = 2(-1)^r[(-1)^{r-1}F_{n-r+1}+(-1)^{n+r}F_{-n-1}]\\
&\quad = 2(-1)^r[(-1)^{r-1}F_{n-r+1}+(-1)^{n+r}(-1)^{n+2}F_{n+1}]\\
&\quad = 2(F_{n+1}-F_{n-r+1})\\
&\quad = Le_n - Le_{n-r},
\end{aligned}$$

then (2.30) holds. □

Corollary 2.23 (d'Ocagne's identity)**.** *For $m, n \in \mathbb{Z}$,*

$$Le_{n+1}Le_m - Le_nLe_{m+1} = \quad 2(-1)^{n+1}(Le_{m-n-1}+1) + Le_{m-1} - Le_{n-1}. \tag{2.31}$$

Proof. Taking $r = m - n$ and $k = 1$ in (2.29) leads to

$$\begin{aligned}
&Le_{n+1}Le_m - Le_nLe_{m+1}\\
&\quad = (-1)^{n+1}(Le_0+1)\left[\frac{1}{2}(-1)^{n+1}(Le_{m-1}-Le_{n-1})+Le_{m-n-1}+1\right]\\
&\quad +\frac{1}{2}(Le_1-1)(Le_m-Le_n)\\
&\quad = Le_{m-1}-Le_{n-1}+2(-1)^{n+1}Le_{m-n-1}+2(-1)^{n+1},
\end{aligned}$$

which is easily seen to be equal to the right hand side of (2.31). □

Corollary 2.24 (Cassini's identity)**.** *For $n \in \mathbb{Z}$,*

$$Le_{n-1}Le_{n+1} - Le_n^2 = -Le_{n-1} + Le_{n-2} - 4(-1)^n. \tag{2.32}$$

Proof. With $r = -k = 1$ in (2.29), we get

$$
\begin{aligned}
&Le_{n-1}Le_{n+1} - Le_n^2 \\
&\quad = (-1)^{n+1}(Le_{-2}+1)\left[\frac{1}{2}(-1)^{n+1}(Le_n - Le_{n-1}) + Le_0 + 1\right] \\
&\qquad + \frac{1}{2}(Le_{-1}-1)(Le_{n+1}-Le_n) \\
&\quad = Le_n - Le_{n-1} + 4(-1)^{n+1} - Le_{n+1} + Le_n \\
&\quad = -4(-1)^n + Le_{n-2} - Le_{n-1},
\end{aligned}
$$

and (2.32) follows. □

3. The Leonardo quaternions sequence

In order to introduce a new quaternion sequence involving Leonardo's numbers, we begin this section by presenting the definition of a general element of this sequence.

Definition 3.1. The sth element of the Leonardo quaternions sequence $\{QLe_n\}_{n=0}^{\infty}$ is given by

$$QLe_s = Le_s + Le_{s+1}\mathbf{i} + Le_{s+2}\mathbf{j} + Le_{s+3}\mathbf{k}, \quad s \in \mathbb{N}_0. \tag{3.1}$$

If we denote the sth element of the the complex sequence with Leonardo' numbers by

$$QLe_s^c = Le_s + Le_{s+1}\mathbf{i}, \tag{3.2}$$

one can rewrite the sth element of $\{QLe_n\}_{n=0}^{\infty}$ as

$$QLe_s = Le_s + Le_{s+1}\mathbf{i} + (Le_{s+2} + Le_{s+3}\mathbf{i})\,\mathbf{j} = QLe_s^c + QLe_{s+2}^c\mathbf{j}.$$

Taking into account (2.22), through easy calculations, the sth element of $\{QLe_n\}_{n=0}^{\infty}$ can also be written as

$$QLe_s = Le_s\,(1+\mathbf{j}+\mathbf{k}) + Le_{s+1}\,(\mathbf{i}+\mathbf{j}+2\mathbf{k}) + \mathbf{j} + 2\mathbf{k}. \tag{3.3}$$

Binet's formula for Leonardo quaternions is given in the the following result.

Theorem 3.2 (Binet's formula)**.** *For $n \in \mathbb{N}_0$,*

$$QLe_n = 2\left(\frac{\phi^{n+1}\widehat{\phi} - \psi^{n+1}\widehat{\psi}}{\phi - \psi}\right) - q_u, \tag{3.4}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, $\widehat{\phi} = 1 + \phi\mathbf{i} + (1+\phi)\,\mathbf{j} + (1+2\phi)\,\mathbf{k}$, $\widehat{\psi} = 1 + \psi\mathbf{i} + (1+\psi)\,\mathbf{j} + (1+2\psi)\,\mathbf{k}$ and $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. From (3.3), applying Binet's formula (2.27) for Leonardo numbers in Theorem 2.19, we obtain

$$
\begin{aligned}
&QLe_n \\
&\quad = 2\frac{\phi^{n+1}-\psi^{n+1}}{\phi-\psi}(1+\mathbf{j}+\mathbf{k}) + 2\frac{\phi^{n+2}-\psi^{n+2}}{\phi-\psi}(\mathbf{i}+\mathbf{j}+2\mathbf{k}) - 1 - \mathbf{j} - \mathbf{j} - \mathbf{k} \\
&\quad = 2\frac{\phi^{n+1}(1+\phi\mathbf{i}+(1+\phi)\mathbf{j}+(1+2\phi)\mathbf{k}) - \psi^{n+1}(1+\psi i+(1+\psi)\mathbf{j}+(1+2\psi)\mathbf{k})}{\phi-\psi} \\
&\qquad - q_u.
\end{aligned}
$$

Hence, this leads to (3.4). □

Assuming that Binet's formula (3.4) holds for negative subscripts and taking into account Definition 2.20, it is possible to extend Definition 3.1 to negative subscripts as follows.

Definition 3.3. For $s \in \mathbb{N}$,

$$QLe_{-s} = Le_{-s} + Le_{-s+1}\mathbf{i} + Le_{-s+2}\mathbf{j} + Le_{-s+3}\mathbf{k}. \tag{3.5}$$

3.1. Basic properties

In this subsection, several properties of the Leonardo quaternions sequence are studied.

Proposition 3.4. *Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. The following identities hold:*

(a)
$$QLe_n + \overline{QLe_n} = 2Le_n; \tag{3.6}$$

(b)
$$QLe_{n+2} = QLe_{n+1} + QLe_n + q_u; \tag{3.7}$$

(c)
$$QLe_n = 2QF_{n+1} - q_u; \tag{3.8}$$

(d)
$$QLe_{n+1} - QLe_n = 2QF_n; \tag{3.9}$$

(e)
$$q_u QLe_s - QLe_s q_u = 2[(Le_{s+1} + 1)\mathbf{i} - (Le_{s+2} + 1)\mathbf{j} + (Le_s + 1)\mathbf{k}]. \tag{3.10}$$

Proof. As $QLe_n = Le_n + \widetilde{QLe_n}$ and $\overline{QLe_n} = Le_n - \widetilde{QLe_n}$, it is straightforward to get (3.6).

Since $QLe_{n+1} + QLe_n = Le_{n+2} + Le_{n+3}\mathbf{i} + Le_{n+4}\mathbf{j} + Le_{n+5}\mathbf{k} - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$, we obtain (3.7).

Concerning (3.8), as $QLe_n = Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\mathbf{j} + Le_{n+3}\mathbf{k}$, then, by (2.23) in Proposition 2.17, we get

$$QLe_n = 2(F_{n+1} + F_{n+2}\mathbf{i} + F_{n+3}\mathbf{j} + F_{n+4}\mathbf{k}) - (1 + \mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Taking into account that the recurrence relation (3.7) is inhomogeneous, Proposition 2.17 leads to (3.9) as

$$QLe_{n+1} - QLe_n = QLe_{n-1} + q_u = 2(F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k}) = 2QF_n.$$

Finally, in order to prove (3.10), observe that

$$\begin{aligned} q_u QLe_s &= Le_s - Le_{s+1} - Le_{s+2} - Le_{s+3} + (Le_{s+1} + Le_s + Le_{s+3} - Le_{s+2})\mathbf{i} \\ &\quad +(Le_{s+2} - Le_{s+3} + Le_s + Le_{s+1})\mathbf{j} \\ &\quad +(Le_{s+3} + Le_{s+2} - Le_{s+1} + Le_s)\mathbf{k} \end{aligned}$$

and

$$\begin{aligned} QLe_s q_u &= Le_s - Le_{s+1} - Le_{s+2} - Le_{s+3} + (Le_s + Le_{s+1} + Le_{s+2} - Le_{s+3})\mathbf{i} \\ &\quad +(Le_s - Le_{s+1} + Le_{s+2} + Le_{s+3})\mathbf{j} \\ &\quad +(Le_s + Le_{s+1} - Le_{s+2} + Le_{s+3})\mathbf{k}. \end{aligned}$$

Hence,

$$q_u QLe_s - QLe_s q_u = 2(Le_{s+3} - Le_{s+2})\mathbf{i} + 2(Le_{s+1} - Le_{s+3})\mathbf{j} + 2(Le_{s+2} - Le_{s+1})\mathbf{k},$$

which is easily seen to be equal to the second member in (3.10). □

In the next result, properties related to the norm of a Leonardo quaternion, and some relations with the Fibonacci sequence, the Fibonacci quaternions sequence and the Lucas sequences, are presented.

Proposition 3.5. *For $n \in \mathbb{N}_0$,*

(a)
$$N(QLe_n) = 4\left(N(QF_{n+1}) - L_{n+4} + 1\right), \tag{3.11}$$

where $N(QF_{n+1}) = 3F_{2n+5}$;

(b)
$$N(QLe_n) - N(QLe_{n+1}) = 4L_{n+3}\left(1 - 3F_{n+3}\right); \tag{3.12}$$

(c)

$$N(QLe_n) = (Le_{n+3} - 1)(Le_{n+4} - 1) - (Le_{n-1} - 1)(Le_n - 1) + 4. \tag{3.13}$$

Proof. One starts by proving (a) applying (2.23) in Proposition 2.17, which leads to

$$\begin{aligned} N(QLe_n) &= Le_n^2 + Le_{n+1}^2 + Le_{n+2}^2 + Le_{n+3}^2 \\ &= (2F_{n+1} - 1)^2 + (2F_{n+2} - 1)^2 + (2F_{n+3} - 1)^2 + (2F_{n+4} - 1)^2 \\ &= 4[(F_{n+1}^2 + F_{n+2}^2 + F_{n+3}^2 + F_{n+4}^2) \\ &\quad -(F_{n+1} + F_{n+2} + F_{n+3} + F_{n+4}) + 1] \\ &= 4[N(QF_{n+1}) - (2F_{n+3} + F_{n+4}) + 1] \\ &= 4[N(QF_{n+1}) - (Le_{n+2} + 1 + F_{n+4}) + 1]. \end{aligned}$$

From here and (2.26) in Proposition 2.18, the right-hand side in (3.11) is obtained. Moreover, by [32], $N(QF_{n+1})$ can be written as stated.

In what concerns (3.12), taking into account (a),

$$\begin{aligned} N(QLe_n) - N(QLe_{n+1}) &= 4(N(QF_{n+1}) - N(QF_{n+2}) + L_{n+5} - L_{n+4}) \\ &= 4[3(F_{2n+5} - F_{2n+7}) + L_{n+3})], \end{aligned}$$

where, by (2.10) in Proposition 2.6, $F_{2n+5} - F_{2n+7} = -F_{2n+6} = -F_{n+3}L_{n+3}$.

As

$$\begin{aligned} N(QLe_n) &= QLe_n\overline{QLe_n} \\ &= Le_n^2 + Le_{n+1}^2 + Le_{n+2}^2 + Le_{n+3}^2 \\ &= \sum_{s=0}^{n+3} Le_s^2 - \sum_{s=0}^{n-1} Le_s^2, \end{aligned}$$

then, invoking [15, Proposition 3.3], the latter expression is precisely (3.13). □

3.2. Sum and binomial-sum identities

In the next result, formulae for the sum of the first n Leonardo quaternions, of the first n Leonardo quaternions with odd subscripts, and of the first n Leonardo quaternions with even subscripts, are presented.

Theorem 3.6. *Let* $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. *For* $n \in \mathbb{N}_0$,

$$\sum_{s=1}^{n} QLe_s = QLe_{n+2} - QLe_2 - nq_u, \tag{3.14}$$

$$\sum_{s=1}^{n} QLe_{2s-1} = QLe_{2n} - QLe_0 - nq_u, \tag{3.15}$$

$$\sum_{s=1}^{n} QLe_{2s} = QLe_{2n+1} - QLe_1 - nq_u, \tag{3.16}$$

Proof. For (3.14), we use induction on n. From (3.7), it is straightforward to see that the base case holds. As for the induction step, we have

$$\begin{aligned} \sum_{s=1}^{n+1} QLe_s &= \sum_{s=1}^{n} QLe_s + QLe_{n+1} \\ &= QLe_{n+2} - QLe_2 - nq_u + QLe_{n+1} \\ &= QLe_{n+3} - QLe_2 - (n+1)q_u. \end{aligned}$$

In a similar way, (3.15) and (3.16) can be proved by induction on n. □

In the following result, certain binomial-sum identities are established.

Theorem 3.7. *Let* $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. *For* $n \in \mathbb{N}_0$,

$$QLe_{2n} = \sum_{i=0}^{n} \binom{n}{i} (QLe_i + q_u) - q_u, \tag{3.17}$$

$$QLe_{2n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} (QLe_{i-1} + q_u) - q_u. \tag{3.18}$$

Proof. Applying the Binet's formula (3.4) for the Leonardo quaternions in Theorem 3.2, we arrive at

$$\begin{aligned} QLe_{2n} &= 2\frac{\phi^{2n+1}\widehat{\phi} - \psi^{2n+1}\widehat{\psi}}{\phi - \psi} - q_u \\ &= 2\frac{(1+\phi)^n \phi\widehat{\phi} - (1+\psi)^n \psi\widehat{\psi}}{\phi - \psi} - q_u \\ &= \frac{2}{\phi - \psi}\left(\sum_{i=0}^{n}\binom{n}{i}\phi^i\phi\widehat{\phi} - \sum_{i=0}^{n}\binom{n}{i}\psi^i\psi\widehat{\psi}\right) - q_u \\ &= \sum_{i=0}^{n}\binom{n}{i}\left(2\frac{\phi^{i+1}\widehat{\phi} - \psi^{i+1}\widehat{\psi}}{\phi - \psi} - q_u\right) + \sum_{i=0}^{n}\binom{n}{i} q_u - q_u, \end{aligned}$$

and (3.17) follows. Using a similar reasoning, we get (3.18). □

3.3. Tagiuri-Vajda's identity and related identities

The first theorem of this section, preceded by an auxiliary result, establishes Tagiuri-Vajda's identity for the Leonardo quaternions.

Lemma 3.8. *For* $r \in \mathbb{Z}$,

$$\psi^r \widehat{\phi}\widehat{\psi} - \phi^r \widehat{\psi}\widehat{\phi} = -(\phi - \psi)[QLe_{r-1} + q_u - (6F_{r+1} - 3F_r)\mathbf{k}]$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$, $\psi = \dfrac{1-\sqrt{5}}{2}$, $\widehat{\phi} = 1+\phi\mathbf{i}+(1+\phi)\,\mathbf{j}+(1+2\phi)\,\mathbf{k}$, $\widehat{\psi} = 1+\psi\mathbf{i}+(1+\psi)\,\mathbf{j}+(1+2\psi)\,\mathbf{k}$ *and* $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. As $\widehat{\phi}\widehat{\psi} = 2 + 2\psi\mathbf{i} + (2+2\psi)\mathbf{j} + (5-2\psi)\mathbf{k}$ and $\widehat{\psi}\widehat{\phi} = 2 + 2\phi\mathbf{i} + (2+2\phi)\mathbf{j} + (5-2\phi)\mathbf{k}$, then Binet's formula (2.6) for the Fibonacci numbers in Theorem 2.4 allows to arrive at

$$\begin{aligned} \psi^r \widehat{\phi}\widehat{\psi} - \phi^r \widehat{\psi}\widehat{\phi} &= 2(\psi^r - \phi^r) \\ &\quad + 2(\psi^{r+1} - \phi^{r+1})\mathbf{i} + 2[(\psi^r - \phi^r) + (\psi^{r+1} - \phi^{r+1})]\mathbf{j} \\ &\quad + [5(\psi^r - \phi^r) - 2(\psi^{r+1} - \phi^{r+1})]\mathbf{k} \\ &= -2(\phi - \psi)F_r - 2(\phi - \psi)F_{r+1}\mathbf{i} - 2(\phi - \psi)(F_r + F_{r+1})\mathbf{j} \\ &\quad + (\phi - \psi)(-5F_r + 2F_{r+1})\mathbf{k} \\ &= -(\phi - \psi)[2(F_r + F_{r+1}\mathbf{i} + F_{r+2}\mathbf{j} + F_{r+3}\mathbf{k}) \\ &\quad - (-5F_r + 2F_{r+1})\mathbf{k} - 2F_{r+3}\mathbf{k}] \\ &= -(\phi - \psi)[2QF_r - (-5F_r + 2F_{r+1} + 2F_{r+3})\mathbf{k}] \\ &= -(\phi - \psi)[2QF_r - 3(2F_{r+1} - F_r)\mathbf{k}] \\ &= -(\phi - \psi)[QLe_{r-1} + q_u - (6F_{r+1} - 3F_r)\mathbf{k}]. \end{aligned}$$

□

Theorem 3.9 (Tagiuri-Vajda's identity)**.** *For* $n, k, r \in \mathbb{Z}$,

$$\begin{aligned} &QLe_{n+k}QLe_{n+r} - QLe_nQLe_{n+k+r} \\ &\quad = 2(-1)^{n+1}(Le_{k-1} + 1)(QLe_{r-1} - 3(2F_{r+1} - F_r)\mathbf{k} + q_u) \\ &\qquad + q_u(QLe_{n+k+r} - QLe_{n+r}) + (QLe_n - QLe_{n+k})q_u. \end{aligned} \tag{3.19}$$

Proof. On the one hand, from the Binet's formula (3.4) for the Leonardo quaternions in Theorem 3.2,

$$\begin{aligned}
&QLe_{n+k}QLe_{n+r}\\
&=\left(2\frac{\phi^{n+k+1}\widehat{\phi}-\psi^{n+k+1}\widehat{\psi}}{\phi-\psi}-q_u\right)\left(2\frac{\phi^{n+r+1}\widehat{\phi}-\psi^{n+r+1}\widehat{\psi}}{\phi-\psi}-q_u\right)\\
&=\frac{4}{(\phi-\psi)^2}\left(\phi^{2n+k+r+2}\widehat{\phi}^2-\phi^{n+k+1}\psi^{n+r+1}\widehat{\phi}\widehat{\psi}-\psi^{n+k+1}\phi^{n+r+1}\widehat{\psi}\widehat{\phi}+\psi^{2n+k+r+2}\widehat{\psi}^2\right)\\
&\quad-\frac{2}{\phi-\psi}\left(\phi^{n+k+1}\widehat{\phi}q_u-\psi^{n+k+1}\widehat{\psi}q_u+q_u\phi^{n+r+1}\widehat{\phi}-q_u\psi^{n+r+1}\widehat{\psi}\right)+q_u^2.
\end{aligned}$$

On the other hand, once again by Binet's formula (3.4) in Theorem 3.2,

$$\begin{aligned}
&QLe_{n}QLe_{n+k+r}\\
&=\left(2\frac{\phi^{n+1}\widehat{\phi}-\psi^{n+1}\widehat{\psi}}{\phi-\psi}-q_u\right)\left(2\frac{\phi^{n+k+r+1}\widehat{\phi}-\psi^{n+k+r+1}\widehat{\psi}}{\phi-\psi}-q_u\right)\\
&=\frac{4}{(\phi-\psi)^2}\left(\phi^{2n+k+r+2}\widehat{\phi}^2-\phi^{n+1}\psi^{n+k+r+1}\widehat{\phi}\widehat{\psi}-\psi^{n+1}\phi^{n+k+r+1}\widehat{\psi}\widehat{\phi}+\psi^{2n+k+r+2}\widehat{\psi}^2\right)\\
&\quad-\frac{2}{\phi-\psi}\left(\phi^{n+1}\widehat{\phi}q_u-\psi^{n+1}\widehat{\psi}q_u+q_u\phi^{n+k+r+1}\widehat{\phi}-q_u\psi^{n+k+r+1}\widehat{\psi}\right)+q_u^2.
\end{aligned}$$

Hence, from Binet's formula (2.27) for the Leonardo quaternions in Proposition 2.19,

$$\begin{aligned}
&QLe_{n+k}QLe_{n+r}-QLe_nQLe_{n+k+r}\\
&\quad=\frac{4(\phi\psi)^{n+1}}{(\phi-\psi)^2}[(\phi^k-\psi^k)(-\psi^r\widehat{\phi}\widehat{\psi}+\phi^r\widehat{\psi}\widehat{\phi})]\\
&\quad\quad-\frac{2}{\phi-\psi}[(\phi^{n+k+1}\widehat{\phi}-\psi^{n+k+1}\widehat{\psi})q_u+q_u(\phi^{n+r+1}\widehat{\phi}-\psi^{n+r+1}\widehat{\psi})\\
&\quad\quad-(\phi^{n+1}\widehat{\phi}-\psi^{n+1}\widehat{\psi})q_u-q_u(\phi^{n+k+r+1}\widehat{\phi}-\psi^{n+k+r+1}\widehat{\psi})]\\
&\quad=\frac{2(-1)^{n+1}}{\phi-\psi}[-(Le_{k-1}+1)(\psi^r\widehat{\phi}\widehat{\psi}-\phi^r\widehat{\psi}\widehat{\phi})]\\
&\quad\quad-[(QLe_{n+k}-QLe_n)q_u+q_u(QLe_{n+r}-QLe_{n+k+r})],
\end{aligned}$$

which, by Lemma 3.8, leads to (3.19). □

As consequences of Tagiuri-Vajda's identity, Catalan's identity, d'Ocagne's identity and Cassini's identity for Leonardo quaternions are, respectively, obtained in the following results.

Corollary 3.10 (Catalan's identity)**.** *For $n, r \in \mathbb{Z}$,*

$$\begin{aligned}
&QLe_{n-r}QLe_{n+r}-QLe_n^2\\
&\quad=2(-1)^{n+r}(Le_{r-1}+1)(QLe_{r-1}-3(2F_{r+1}-F_r)\mathbf{k}+q_u)\\
&\quad\quad+q_u(QLe_n-QLe_{n+r})+(QLe_n-QLe_{n-r})q_u.
\end{aligned}\tag{3.20}$$

Proof. In order to arrive at (3.20), take $k=-r$ in (3.19). More concretely,

$$\begin{aligned}
&QLe_{n-r}QLe_{n+r}-QLe_n^2\\
&\quad=2(-1)^{n+1}[Le_{-r-1}+1][QLe_{r-1}-3(2F_{r+1}-F_r)\mathbf{k}+q_u]\\
&\quad\quad+q_u(QLe_n-QLe_{n+r})+(QLe_n-QLe_{n-r})q_u\\
&\quad=2(-1)^{n+1}[(-1)^{r+1}(Le_{r-1}+1)][QLe_{r-1}-3(2F_{r+1}-F_r)\mathbf{k}+q_u]\\
&\quad\quad+q_u(QLe_n-QLe_{n+r})+(QLe_n-QLe_{n-r})q_u
\end{aligned}$$

□

Corollary 3.11 (d'Ocagne's identity)**.** *For $m, n \in \mathbb{Z}$,*

$$\begin{aligned} &QLe_{n+1}QLe_m - QLe_nQLe_{m+1} \\ &\quad = 4(-1)^{n+1}[QLe_{m-n-1} - 3(2F_{m-n+1} - F_{m-n})\mathbf{k} + q_u] \\ &\qquad + q_u(QLe_{m+1} - QLe_m) + (QLe_n - QLe_{n+1})q_u. \end{aligned} \tag{3.21}$$

Proof. Taking $r = m - n$ and $k = 1$ in (3.19) leads to (3.21), since

$$\begin{aligned} &QLe_{n+1}QLe_m - QLe_nQLe_{m+1} \\ &\quad = 2(-1)^{n+1}[Le_0 + 1][QLe_{m-n-1} - 3(2F_{m-n+1} - F_{m-n})\mathbf{k} + q_u] \\ &\quad +q_u(QLe_{m+1} - QLe_m) + (QLe_n - QLe_{n+1})q_u. \end{aligned}$$

□

Corollary 3.12 (Cassini's identity)**.** *For $n \in \mathbb{Z}$,*

$$QLe_{n-1}QLe_{n+1} - QLe_n^2 = 4(-1)^{n+1}(QLe_0 - 3\mathbf{k} + q_u) - q_uQLe_{n-1} + QLe_{n-2}q_u, \tag{3.22}$$

Proof. With $r = -k = 1$ in (3.19), we get

$$\begin{aligned} &QLe_{n-1}QLe_{n+1} - QLe_n^2 \\ &\quad = 2(-1)^{n+1}(Le_{-2} + 1)[QLe_0 - 3(2F_2 - F_1)\mathbf{k} + q_u] \\ &\qquad +q_u(QLe_n - QLe_{n+1}) + (QLe_n - QLe_{n-1})q_u \end{aligned}$$

Hence, applying (3.7) of Proposition 3.4, (3.22) holds. □

Partially following the ideas of Daşdemir in [24], where he presented Gelin-Cesàro's identity for Fibonacci quaternions and Gelin-Cesàro's identity for Lucas quaternions, Gelin-Cesàro's identity for Leonardo quaternions is deduced in the following theorem which is preceded by auxiliary results.

Lemma 3.13. *Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. For $s, n \in \mathbb{Z}$, the following equalities hold:*

(a)

$$\begin{aligned} \widetilde{QLe_s} \times \widetilde{QLe_{n+2}} \;=\; &(Le_{s+2}Le_{n+5} - Le_{s+3}Le_{n+4})\mathbf{i} \\ &-(Le_{s+1}Le_{n+5} - Le_{s+3}Le_{n+3})\mathbf{j} \\ &+(Le_{s+1}Le_{n+4} - Le_{s+2}Le_{n+3})\mathbf{k}; \end{aligned}$$

(b)

$$(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u}) \times \widetilde{QLe_{n+2}} = (2Le_{n+5} - 5Le_{n+4})\mathbf{i} + 5Le_{n+3}\mathbf{j} - 2Le_{n+3}\mathbf{k};$$

(c)

$$\widetilde{q_u} \times \widetilde{QLe_s} = (Le_{s+1} + 1)\mathbf{i} - (Le_{s+2} + 1)\mathbf{j} + (Le_s + 1)\mathbf{k};$$

(d)

$$\begin{aligned} &[(Le_{n+1} + 1)\mathbf{i} - (Le_{n+2} + 1)\mathbf{j} + (Le_n + 1)\mathbf{k}] \times \widetilde{QLe_{n+2}} \\ &\quad = [-(Le_{n+2} + 1)Le_{n+5} - (Le_n + 1)Le_{n+4}]\mathbf{i} \\ &\qquad -[(Le_{n+1} + 1)Le_{n+5} - (Le_n + 1)Le_{n+3}]\mathbf{j} \\ &\qquad +[(Le_{n+1} + 1)Le_{n+4} + (Le_{n+2} + 1)Le_{n+3}]\mathbf{k}. \end{aligned}$$

Proof. The equalities in $(a), (b)$ and (d) come straightforward from the vector cross product in each statement. For (b), observe also that

$$\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u} = 2\mathbf{j} + 5\mathbf{k}.$$

As far as (c), notice that the associated vector cross product gives $(Le_{s+3} - Le_{s+2})\mathbf{i} - (Le_{s+3} - Le_{s+1})\mathbf{j} + (Le_{s+2} - Le_{s+1})\mathbf{k}$. □

Lemma 3.14. *For $k \in \mathbb{Z}$, let $x_k = Le_k + 1$. For $n \in \mathbb{Z}$, the following equalities hold:*

(a)

$$Le_{n+2}Le_{n+5} - Le_{n+3}Le_{n+4} \;=\; -x_{n+2} + 4(-1)^n;$$

(b)
$$Le_{n+1}Le_{n+5} - Le_{n+3}^2 \quad = \quad x_{n+2} - x_{n+4} + 4(-1)^{n+1};$$

(c)
$$Le_{n+1}Le_{n+4} - Le_{n+2}Le_{n+3} \quad = \quad -x_{n+1} + 4(-1)^{n+3};$$

(d)
$$2Le_{n+5} - 5Le_{n+4} \quad = \quad -3(x_{n+4} - 1) + 2x_{n+3};$$

(e)
$$Le_{n-1}Le_{n+5} - Le_nLe_{n+4} \quad = \quad -3x_{n+1} + x_n + 20(-1)^{n+1};$$

(f)
$$Le_{n-2}Le_{n+5} - Le_nLe_{n+3} \quad = \quad -2x_{n+2} - x_n + 20(-1)^n;$$

(g)
$$Le_{n-2}Le_{n+4} - Le_{n-1}Le_{n+3} \quad = \quad -x_{n+2} + x_{n-3} + 20(-1)^n.$$

Proof. Applying Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = 1$ and $r = 2$, leads to

$$\begin{aligned}
&Le_{n+2}Le_{n+5} - Le_{n+3}Le_{n+4} \\
&= (-1)^{n+4}(Le_0 + 1)\left[\frac{1}{2}(-1)^{n+3}(Le_{n+3} - Le_{n+1}) + Le_1 + 1\right] \\
&\quad -\frac{1}{2}(Le_1 - 1)(Le_{n+4} - Le_{n+2}) \\
&= (-1)^{2n+7}(Le_{n+2} + 1) + 4(-1)^{n+4},
\end{aligned}$$

and, as easily seen, the latter expression is equal to the right-hand side of the equality in (a).

Using Catalan's identity (2.30) in Corollary 2.22, with $r = 2$, allows us to arrive at (b):

$$\begin{aligned}
Le_{n+1}Le_{n+5} - Le_{n+3}^2 &= -Le_{n+1} - Le_{n+5} + 2Le_{n+3} + (-1)^{n+1}(Le_1 + 1)^2 \\
&= -Le_{n+1} - Le_{n+4} + Le_{n+3} - 1 + 4(-1)^{n+1} \\
&= Le_{n+2} + 1 - (Le_{n+4} + 1) + 4(-1)^{n+1} \\
&= x_{n+2} - x_{n+4} + 4(-1)^{n+1}.
\end{aligned}$$

As far as (c), once again invoking Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = 1$ and $r = 2$, observe that

$$\begin{aligned}
&Le_{n+1}Le_{n+4} - Le_{n+2}Le_{n+3} \\
&= -(Le_{n+2}Le_{n+3} - Le_{n+1}Le_{n+4}) \\
&= (-1)^{n+3}(Le_0 + 1)\left[\frac{1}{2}(-1)^{n+2}(Le_{n+2} - Le_n) + Le_1 + 1\right] \\
&\quad -\frac{1}{2}(Le_1 - 1)(Le_{n+3} - Le_{n+1}) \\
&= (-1)^{2n+5}(Le_{n+2} - Le_n) + 4(-1)^{n+3} \\
&= (-1)^{2n+5}(Le_{n+1} + 1) + 4(-1)^{n+3}.
\end{aligned}$$

For (d), notice that

$$\begin{aligned}
2Le_{n+5} - 5Le_{n+4} &= -3Le_{n+4} + 2(Le_{n+3} + 1) \\
&= -3(x_{n+4} - 1) + 2x_{n+3}.
\end{aligned}$$

Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k = -1$ and $r = 5$, allows to write

$$\begin{aligned}
&Le_{n-1}Le_{n+5} - Le_nLe_{n+4} \\
&= (-1)^{n+1}(Le_{-2} + 1)\left[\frac{1}{2}(-1)^{n+1}(Le_{n+4} - Le_{n-1}) + Le_4 + 1\right] \\
&\quad +\frac{1}{2}(Le_{-1} - 1)(Le_{n+5} - Le_n) \\
&= Le_{n+4} - Le_{n-1} + 20(-1)^{n+1} + Le_n - Le_{n+5} \\
&= -3Le_{n+1} + Le_n - 2 + 20(-1)^{n+1},
\end{aligned}$$

and (e) follows.

In what concerns (f), Tagiuri-Vajda's identity (2.29) in Theorem 2.21, with $k=5$ and $r=2$, implies

$$\begin{aligned}
&Le_{n-2}Le_{n+5}-Le_nLe_{n+3}\\
&\quad=-(Le_{n+3}Le_n-Le_{n-2}Le_{n+5})\\
&\quad=-\Big\{(-1)^{n-1}(Le_4+1)\left[\frac{1}{2}(-1)^{n-1}(Le_{n-1}-Le_{n-3}+Le_1+1)\right]\\
&\qquad+\frac{1}{2}(Le_5-1)(Le_n-Le_{n-2})\Big\}\\
&\quad=-5Le_{n-1}+5Le_{n-3}+20(-1)^n-7Le_n+7Le_{n-2}\\
&\quad=2Le_{n-2}-7Le_n-5+20(-1)^n\\
&\quad=-2Le_{n+1}-3Le_n-5+20(-1)^n\\
&\quad=-2Le_{n+2}-Le_n-3+20(-1)^n,
\end{aligned}$$

leading to the result.

Finally, as follows, d'Ocagne's identiy (2.31) in Corollary 2.23 is applied in order to arrive at (g).

$$\begin{aligned}
Le_{n-2}Le_{n+4}-Le_{n-1}Le_{n+3} &= -(Le_{n-1}Le_{n+3}-Le_{n-2}Le_{n+4})\\
&= -[2(-1)^{n-1}(Le_4+1)+Le_{n+2}-Le_{n-3}]\\
&= 20(-1)^n-Le_{n+2}+Le_{n-3}
\end{aligned}$$

□

Theorem 3.15 (Gelin-Cesàro's identity)**.** *Let* $a=2+2\mathbf{j}+5\mathbf{k}$ *and* $b=-2+2\mathbf{i}-7\mathbf{k}$. *For* $k\in\mathbb{Z}$, *let* $x_k=Le_k+1$. *For* $n\in\mathbb{Z}$, *let*

$$A_n=2[(Le_{n-1}+1)\mathbf{i}-(Le_n+1)\mathbf{j}+(Le_{n-2}+1)\mathbf{k}]-QLe_{n-3}q_u-q_u^2,$$
$$B_n=2[(Le_n+1)\mathbf{i}-(Le_{n+1}+1)\mathbf{j}+(Le_{n-1}+1)\mathbf{k}]-QLe_nq_u-q_u^2.$$

For $n\in\mathbb{Z}$,

$$QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2}-QLe_n^4=C_n-D_n, \tag{3.23}$$

where

$$C_n=(4(-1)^{n-1}a+A_n)(QLe_n^2+B_n)+QLe_n^2(4(-1)^{n-1}b+B_n)+(4(-1)^{n-1}A_n+16a)b$$

and

$$\begin{aligned}
D_n \;=\; &\{4(x_n-1)[(-x_{n+2}+4(-1)^n)\mathbf{i}-(x_{n+2}-x_{n+4}+4(-1)^{n+1})\mathbf{j}\\
&+(-x_{n+1}+4(-1)^{n+3})\mathbf{k}]\\
&+8(-1)^{n-1}[(-3(x_{n+4}-1)+2x_{n+3})\mathbf{i}+5(x_{n+3}-1)\mathbf{j}-2(x_{n+3}-1)\mathbf{k}]\\
&-2[(-3x_{n+1}+x_n+20(-1)^{n+1})\mathbf{i}\\
&-(-2x_{n+2}-x_n+20(-1)^n)\mathbf{j}+(-x_{n+2}+x_{n-3}+20(-1)^n)\mathbf{k}]\\
&-2(x_{n-3}+1)[x_{n+3}\mathbf{i}-x_{n+4}\mathbf{j}+x_{n+2}\mathbf{k}]\\
&+2[(-x_{n+2}(x_{n+5}-1)-x_n(x_{n+4}-1))\mathbf{i}-(x_{n+1}(x_{n+5}-1)-x_n(x_{n+3}-1))\mathbf{j}\\
&+(x_{n+1}(x_{n+4}-1)+x_{n+2}(x_{n+3}-1))\mathbf{k}]\}QLe_{n-2}.
\end{aligned}$$

Proof. In order to arrive at (3.23), we develop $QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2}$ in two different ways. On the one hand, Catalan's identity (3.20) for Leonardo quaternions with $r=-1$ and $r=-2$ leads, respectively, to

$$\begin{aligned}
&QLe_{n+1}QLe_{n-1}\\
&\quad=QLe_n^2+2(-1)^{n-1}(Le_{-2}+1)(QLe_{-2}-3(2F_0-F_{-1})\mathbf{k}+q_u)\\
&\qquad+q_u(QLe_n-QLe_{n-1})+(QLe_n-QLe_{n+1})q_u\\
&\quad=QLe_n^2+4(-1)^{n-1}(QLe_{-2}+3\mathbf{k}+q_u)+q_uQLe_{n-2}-QLe_{n-1}q_u
\end{aligned}$$

and

$$\begin{aligned}
&QLe_{n+2}QLe_{n-2}\\
&\quad=QLe_n^2+2(-1)^{n-2}(Le_{-3}+1)(QLe_{-3}-3(2F_{-1}-F_{-2})\mathbf{k}+q_u)\\
&\qquad+q_u(QLe_n-QLe_{n-2})+(QLe_n-QLe_{n+2})q_u\\
&\quad=QLe_n^2+4(-1)^{n-1}(QLe_{-3}-9\mathbf{k}+q_u)+q_uQLe_{n-1}-QLe_{n+1}q_u.
\end{aligned}$$

From these equalities, also applying (3.10) in Proposition 3.4,

$$\begin{aligned}
&QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2}\\
&\quad= QLe_n^4 + 4(-1)^{n-1}[(QLe_{-2} + 3\mathbf{k} + q_u)(QLe_n^2 + q_uQLe_{n-1} - QLe_{n+1}q_u)\\
&\qquad+(QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)(QLe_{-3} - 9\mathbf{k} + q_u)]\\
&\qquad+QLe_n^2(q_uQLe_{n-1} - QLe_{n+1}q_u) + (q_uQLe_{n-2} - QLe_{n-1}q_u)QLe_n^2\\
&\qquad+(QLe_{n-1}q_u - q_uQLe_{n-2})(QLe_{n+1}q_u - q_uQLe_{n-1})\\
&\qquad+16(QLe_{-2} + 3\mathbf{k} + q_u)(QLe_{-3} - 9\mathbf{k} + q_u)\\
&\quad= QLe_n^4 + 4(-1)^{n-1}[a(QLe_n^2 + q_uQLe_{n-1} - QLe_{n+1}q_u)\\
&\qquad+(QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)b]\\
&\qquad+QLe_n^2(q_uQLe_{n-1} - QLe_{n+1}q_u) + (q_uQLe_{n-2} - QLe_{n-1}q_u)QLe_n^2\\
&\qquad+(QLe_{n-1}q_u - q_uQLe_{n-2})(QLe_{n+1}q_u - q_uQLe_{n-1}) + 16ab\\
&\quad= QLe_n^4 + 4(-1)^{n-1}[a(QLe_n^2 + q_uQLe_{n-1} - QLe_{n+1}q_u)\\
&\qquad+(QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)b]\\
&\qquad+(QLe_n^2 + q_uQLe_{n-2} - QLe_{n-1}q_u)(q_uQLe_{n-1} - QLe_{n+1}q_u)\\
&\qquad+(q_uQLe_{n-2} - QLe_{n-1}q_u)QLe_n^2 + 16ab\\
&\quad= QLe_n^4 + (4(-1)^{n-1}a + A_n)(QLe_n^2 + B_n) + QLe_n^2(4(-1)^{n-1}b + B_n)\\
&\qquad+(4(-1)^{n-1}A_n + 16a)b\\
&\quad= QLe_n^4 + C_n.
\end{aligned}$$

where $a = QLe_{-2} + 3\mathbf{k} + q_u = 2 + 2\mathbf{j} + 5\mathbf{k}$, $b = QLe_{-3} - 9\mathbf{k} + q_u = -2 + 2\mathbf{i} - 7\mathbf{k}$ and, by (3.10) in Proposition 3.4,

$$\begin{aligned}
A_n &= q_uQLe_{n-2} - QLe_{n-1}q_u\\
&= q_uQLe_{n-2} - QLe_{n-2}q_u - QLe_{n-3}q_u - q_u^2\\
&= 2[(Le_{n-1} + 1)\mathbf{i} - (Le_n + 1)\mathbf{j} + (Le_{n-2} + 1)\mathbf{k}] - QLe_{n-3}q_u - q_u^2
\end{aligned}$$

and

$$\begin{aligned}
B_n &= q_uQLe_{n-1} - QLe_{n+1}q_u\\
&= q_uQLe_{n-1} - QLe_nq_u - QLe_{n-1}q_u - q_u^2\\
&= 2[(Le_n + 1)\mathbf{i} - (Le_{n+1} + 1)\mathbf{j} + (Le_{n-1} + 1)\mathbf{k}] - QLe_nq_u - q_u^2.
\end{aligned}$$

On the other hand, invoking (2.4) in Proposition 2.2,

$$\begin{aligned}
QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2} &= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2}\\
&\quad+2(\widetilde{QLe_{n+1}QLe_{n-1}} \times \widetilde{QLe_{n+2}})QLe_{n-2}.
\end{aligned}$$

By Theorem 2.1,

$$\begin{aligned}
&\widetilde{QLe_{n+1}QLe_{n-1}}\\
&\quad= 2Le_n\widetilde{QLe_n} + 4(-1)^{n-1}(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u}) + \widetilde{q_uQLe_{n-2}} - \widetilde{QLe_{n-1}q_u}\\
&\quad= 2Le_n\widetilde{QLe_n} + 4(-1)^{n-1}(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u})\\
&\qquad+\widetilde{QLe_{n-2}} + Le_{n-2}\widetilde{q_u} + \widetilde{q_u} \times \widetilde{QLe_{n-2}} - Le_{n-1}\widetilde{q_u} - \widetilde{QLe_{n-1}} - \widetilde{QLe_{n-1}} \times \widetilde{q_u}\\
&\quad= 2Le_n\widetilde{QLe_n} + 4(-1)^{n-1}(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u})\\
&\qquad-(\widetilde{QLe_{n-3} + q_u}) - (Le_{n-3} + 1)\widetilde{q_u} + \widetilde{q_u} \times (\widetilde{QLe_n} - \widetilde{q_u})\\
&\quad= 2Le_n\widetilde{QLe_n} + 4(-1)^{n-1}(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u}) - \widetilde{QLe_{n-3}} - (Le_{n-3} + 2)\widetilde{q_u}\\
&\qquad+\widetilde{q_u} \times \widetilde{QLe_n}\\
&\quad= 2Le_n\widetilde{QLe_n} + 4(-1)^{n-1}(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u}) - \widetilde{QLe_{n-3}} - (Le_{n-3} + 2)\widetilde{q_u}\\
&\qquad+(Le_{n+1} + 1)\mathbf{i} - (Le_{n+2} + 1)\mathbf{j} + (Le_n + 1)\mathbf{k}.
\end{aligned}$$

From this equality, invoking Lemma 3.13 and Lemma 3.14, and denoting $Le_k + 1$, for $k \in \mathbb{Z}$, by x_k,

$$\begin{aligned}
&QLe_{n+1}QLe_{n-1}QLe_{n+2}QLe_{n-2}\\
&= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} + \Big\{4Le_n(\widetilde{QLe_n} \times \widetilde{QLe_{n+2}})\\
&\quad +8(-1)^{n-1}[(\widetilde{QLe_{-2}} + 3\mathbf{k} + \widetilde{q_u})\times \widetilde{QLe_{n+2}}]\\
&\quad -2(\widetilde{QLe_{n-3}} \times \widetilde{QLe_{n+2}}) - 2(Le_{n-3}+2)(\widetilde{q_u} \times \widetilde{QLe_{n+2}})\\
&\quad +2[((Le_{n+1}+1)\mathbf{i} - (Le_{n+2}+1)\mathbf{j} + (Le_n+1)\mathbf{k})\times \widetilde{QLe_{n+2}}]\Big\}QLe_{n-2}\\
&= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2}\\
&\quad +\Big\{4Le_n[(Le_{n+2}Le_{n+5} - Le_{n+3}Le_{n+4})\mathbf{i} - (Le_{n+1}Le_{n+5} - Le^2_{n+3})\mathbf{j}\\
&\quad +(Le_{n+1}Le_{n+4} - Le_{n+2}Le_{n+3})\mathbf{k}]\\
&\quad +8(-1)^{n-1}[(2Le_{n+5} - 5Le_{n+4})\mathbf{i} + 5Le_{n+3}\mathbf{j} - 2Le_{n+3}\mathbf{k}]\\
&\quad -2[(Le_{n-1}Le_{n+5} - Le_nLe_{n+4})\mathbf{i} - (Le_{n-2}Le_{n+5} - Le_nLe_{n+3})\mathbf{j}\\
&\quad +(Le_{n-2}Le_{n+4} - Le_{n-1}Le_{n+3})\mathbf{k}]\\
&\quad -2(Le_{n-3}+2)[(Le_{n+3}+1)\mathbf{i} - (Le_{n+4}+1)\mathbf{j} + (Le_{n+2}+1)\mathbf{k}]\\
&\quad +2[(-(Le_{n+2}+1)Le_{n+5} - (Le_n+1)Le_{n+4})\mathbf{i}\\
&\quad -((Le_{n+1}+1)Le_{n+5} - (Le_n+1)Le_{n+3})\mathbf{j}\\
&\quad +((Le_{n+1}+1)Le_{n+4} + (Le_{n+2}+1)Le_{n+3})\mathbf{k}]\Big\}QLe_{n-2}\\
&= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2}\\
&\quad +\{4(x_n-1)[(-x_{n+2}+4(-1)^n)\mathbf{i} - (x_{n+2} - x_{n+4} + 4(-1)^{n+1})\mathbf{j}\\
&\quad +(-x_{n+1} + 4(-1)^{n+3})\mathbf{k}]\\
&\quad +8(-1)^{n-1}[(-3(x_{n+4}-1) + 2x_{n+3})\mathbf{i} + 5(x_{n+3}-1)\mathbf{j} - 2(x_{n+3}-1)\mathbf{k}]\\
&\quad -2[(-3x_{n+1} + x_n + 20(-1)^{n+1})\mathbf{i} - (-2x_{n+2} - x_n + 20(-1)^n)\mathbf{j}\\
&\quad +(-x_{n+2} + x_{n-3} + 20(-1)^n)\mathbf{k}]\\
&\quad -2(x_{n-3}+1)[x_{n+3}\mathbf{i} - x_{n+4}\mathbf{j} + x_{n+2}\mathbf{k}]\\
&\quad +2[(-x_{n+2}(x_{n+5}-1) - x_n(x_{n+4}-1))\mathbf{i}\\
&\quad -(x_{n+1}(x_{n+5}-1) - x_n(x_{n+3}-1))\mathbf{j}\\
&\quad +(x_{n+1}(x_{n+4}-1) + x_{n+2}(x_{n+3}-1))\mathbf{k}]\}QLe_{n-2}\\
&= QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} + D_n.
\end{aligned}$$

Therefore, $QLe_{n+2}QLe_{n+1}QLe_{n-1}QLe_{n-2} - QLe_n^4 = C_n - D_n$. □

3.4. Generating functions

In the first result, the generating function of the Leonardo quaternions sequence $\{QLe_n\}_{n=0}^{\infty}$ is presented.

Theorem 3.16. *The generating function of $\{QLe_n\}_{n=0}^{\infty}$ is given by*

$$G(t) = \frac{1 + \mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + (-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})t + (1 - \mathbf{i} - \mathbf{j} - 3\mathbf{k})t^2}{1 - 2t + t^3}. \tag{3.24}$$

Proof. Let $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$. As $G(t) - G(t)t - G(t)t^2 = QLe_0(1-t) + QLe_1 t + q_u \sum_{n=0}^{\infty} t^n - q_u - q_u t$, where $\sum_{n=0}^{\infty} t^n$ is a formal power series in t whose closed form is $\frac{1}{1-t}$, then

$$G(t) = \frac{QLe_0(1-t) + QLe_1 t + q_u \dfrac{t^2}{1-t}}{1 - t - t^2}.$$

The development of the right hand side of this equality leads to (3.24). □

In the subsequent result, for each $m \in \mathbb{N}$, the generating function of the sequence $\{QLe_{m+n}\}_{n=0}^{\infty}$ is obtained.

Theorem 3.17. *Let $m \in \mathbb{N}$. The generating function of $\{QLe_{m+n}\}_{n=0}^{\infty}$ is given by*

$$\sum_{n=0}^{\infty} QLe_{m+n}t^n = \frac{QLe_m + (QLe_{m-1} - QLe_m + q_u)t - QLe_{m-1}t^2}{1 - 2t + t^3}, \tag{3.25}$$

where $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. Applying Binet's formula (3.4) for the Leonardo quaternions in Theorem 3.2, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} QLe_{m+n}t^n &= \sum_{n=0}^{\infty} \left(2\frac{\phi^{m+n+1}\widehat{\phi} - \psi^{m+n+1}\widehat{\psi}}{\phi - \psi} - q_u \right) t^n \\
&= \frac{2}{\phi - \psi} \sum_{n=0}^{\infty} \left(\phi^{m+1}\phi^n t^n \widehat{\phi} - \psi^{m+1}\psi^n t^n \widehat{\psi} \right) - q_u \sum_{n=0}^{\infty} t^n \\
&= \frac{2}{\phi - \psi} \left(\phi^{m+1}\frac{1}{1-\phi t}\widehat{\phi} - \psi^{m+1}\frac{1}{1-\psi t}\widehat{\psi} \right) - q_u\frac{1}{1-t} \\
&= \frac{2}{\phi - \psi} \frac{\phi^{m+1}\widehat{\phi} - \psi^{m+1}\widehat{\psi} + (\phi\psi\psi^m\widehat{\psi} - \phi^m\widehat{\phi}\phi\psi)t}{(1-\phi t)(1-\psi t)} - q_u\frac{1}{1-t} \\
&= \frac{2\frac{\phi^{m+1}\widehat{\phi} - \psi^{m+1}\widehat{\psi}}{\phi - \psi} - q_u + \left(2\frac{\phi^{m}\widehat{\phi} - \psi^{m}\widehat{\psi}}{\phi - \psi} - q_u \right) t}{1 - t - t^2} \\
&\quad + q_u\frac{1+t}{1-t-t^2} - q_u\frac{1}{1-t} \\
&= \frac{QLe_m + QLe_{m-1}t}{1-t-t^2} + q_u\frac{1+t}{1-t-t^2} - q_u\frac{1}{1-t}
\end{aligned}$$

which allows us to arrive at (3.25). □

The exponential generating function that generates the numbers $\frac{QLe_n}{n!}$ is presented in the last theorem of the section. As a consequence, the corresponding Poisson generating function is obtained.

Theorem 3.18. *The exponential generating function of $\{QLe_n\}_{n=0}^{\infty}$ is given by*

$$G_E(t) = 2\frac{\phi\widehat{\phi}e^{\phi t} - \psi\widehat{\psi}e^{\psi t}}{\phi - \psi} - q_u e^t, \tag{3.26}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, $\widehat{\phi} = 1 + \phi\mathbf{i} + (1+\phi)\,\mathbf{j} + (1+2\phi)\,\mathbf{k}$, $\widehat{\psi} = 1 + \psi\mathbf{i} + (1+\psi)\,\mathbf{j} + (1+2\psi)\,\mathbf{k}$ and $q_u = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then

$$\begin{aligned}
G_E(t) &= \sum_{n=0}^{\infty} QLe_n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(2\frac{\phi^{n+1}\widehat{\phi} - \psi^{n+1}\widehat{\psi}}{\phi - \psi} - q_u \right) \frac{t^n}{n!} \\
&= \frac{2}{\phi - \psi} \left(\phi\widehat{\phi} \sum_{n=0}^{\infty} \frac{(\phi t)^n}{n!} - \psi\widehat{\psi} \sum_{n=0}^{\infty} \frac{(\psi t)^n}{n!} \right) - q_u \sum_{n=0}^{\infty} \frac{t^n}{n!} \\
&= \frac{2}{\phi - \psi} \left(\phi\widehat{\phi}e^{\phi t} - \psi\widehat{\psi}e^{\psi t} \right) - q_u e^t,
\end{aligned}$$

which leads to (3.26). □

Corollary 3.19. *The Poisson generating function of* $\{QLe_n\}_{n=0}^{\infty}$ *is given by*

$$G_P(t) = 2\frac{\phi\widehat{\phi}\mathrm{e}^{\phi t} - \psi\widehat{\psi}\mathrm{e}^{\psi t}}{(\phi - \psi)\mathrm{e}^{t}} - q_u, \tag{3.27}$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$, $\psi = \dfrac{1-\sqrt{5}}{2}$, $\widehat{\phi} = 1+\phi\mathbf{i}+(1+\phi)\,\mathbf{j}+(1+2\phi)\,\mathbf{k}$, $\widehat{\psi} = 1+\psi\mathbf{i}+(1+\psi)\,\mathbf{j}+(1+2\psi)\,\mathbf{k}$ *and* $q_u = 1+\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Proof. As $G_P(t) = \mathrm{e}^{-t}G_E(t)$, then, invoking (3.26), (3.27) follows. □

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