Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (3) (2024), 748 – 776
DOI : 10.15672/hujms.1198408

Research Article

Time-frequency analysis associated with the generalized Stockwell transform

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Abstract

The Riemann-Liouville operator has been extensively investigated and has witnessed a remarkable development in numerous fields of harmonic analysis. In this paper, we consider the Stockwell transform associated with the Riemann-Liouville operator. Knowing the fact that the study of the time-frequency analysis are both theoretically interesting and practically useful, we investigated several problems for this subject on the setting of this generalized Stockwell transform. Firstly, we study the boundedness and compactness of localization operators associated with the generalized Stockwell transform. Next, we explore the Shapiro uncertainty principle for the previous transform. Finally, the scalogram for the generalized Stockwell transform are introduced and studied at the end.

Mathematics Subject Classification (2020). 47G10, 42B10, 47G30, 47B10, 42C25, 42C40

Keywords. Riemann-Liouville operator, localization operator, generalized Stockwell transform, Shapiro's theorem, scalogram

1. Introduction

The spherical mean operators constitute a vital class of operators in harmonic analysis in the sense that all the harmonic functions are characterized by the fact that they coincide with their spherical mean values. These operators can also be viewed as the generalized Radon transform that is self dual in the context of Helgason's double fibration. In the classical work of John [19], the spherical means have been successfully applied to diverse problems in the theory of partial differential equations. Subsequently, they paved the way into the Fourier analysis with the celebrated theorem of Stein on spherical analogue of the Lebesgue differentiation theorem. A recent addition to the theory of spherical mean operators on $\mathbb{R}^2$ appeared with the work of Trimèche [36], wherein the author generalized the spherical mean operators on $\mathbb{R}^2$ by introducing the permutation operator which commutes with some partial differential operators. Besides, Trimèche also studied the

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Received: 02.11.2022; Accepted: 04.08.2023

harmonic analysis associated with this permutation operator, which is being widely employed in literature under the name Riemann-Liouville operator [3–8, 18, 22–25]. As of now, these operators have found numerous applications in image processing of synthetic aperture, radar data and acoustics [15, 17].

One of the aims of the Fourier transform, is the study of the time-frequency analysis. In the sixties the time-frequency analysis has emerged with the works of Gabor [16] who provided an interesting way to study the local frequency spectrum of signals by introducing many time-frequency representations, as, for instance, the short-time Fourier transform (STFT), the continuous wavelet transform or also the Wigner distribution where all of these representations have a same common point, that is the simultaneous representation of the space and the frequency variables in a same set called the time-frequency plane.

The major drawback of the short-time Fourier transform is the fixed width of the analysing window. Indeed, in many applications, the high frequency content of a signal is more time/space-localized than the low-frequency one. Removing of the rigidity of the window function is one of the motivations for continuous wavelet transform. Although, the wavelet transform captures more information than the short-time Fourier transform (STFT), however, it suffers from two apparent limitations: first, the detail measured by the wavelet transform is not directly analogs to the frequency, because the wavelet transform is essentially a time-scale transform with the inverse scale being interpreted as frequency; second, the phase-information is completely lost in the case of wavelet transform, because each wavelet component acts a local filter and the translation of the mother wavelet destroys the phase information with respect to the origin [37, 38]. To circumvent these limitations, Stockwell et al. [33] introduced the notion of Stockwell transform as a bridge between the STFT and the wavelet transform. By adopting the progressive resolution of wavelets, the Stockwell transform is able to resolve a wider range of frequencies than the ordinary STFT, and by using a Fourier-like basis and maintaining a phase of zero about the time $t = 0$, Fourier based analysis could be performed locally. This unique feature of the Stockwell transform makes it a highly valuable tool for signal processing and is one of the hottest research areas of the contemporary era. Indeed, the Stockwell transform has been successfully used to analyse signals in numerous applications, such as seismic recordings, ground vibrations, geophysics, medical imaging, hydrology, gravitational waves, power system analysis and many other areas. Finally, we note that many extensions of the Stockwell transform have been proposed in recent years. See, for example, [10–14, 26, 28, 29, 31, 32] and others.

As the harmonic analysis associated to the Riemann-Liouville operator has been extensively investigated and has witnessed a remarkable development, it is natural to study several aspects of the time-frequency analysis in the Riemann-Liouville operator setting.

In this paper, we continue the study of some problems of harmonic analysis associated with the generalized Stockwell transform started in [9, 27]. Particularly, motivated by Wong's approaches, the aim of the first part of this paper is to study the boundedness and compactness of localization operators associated with generalized Stockwell transforms. Our second endeavour is to study the spectral analysis associated with the generalized concentration operator. In particular, we introduce and we study the scalogram associated with the generalized Stockwell transforms. We note that the scalogram has many applications, for example in [2], the authors used Morlet wavelet scalogram to detected a previously unknown coordinated contractility behavior of the atrium during ventricular fibrillation, a phenomenon which is not captured in a normal electrocardiogram. Other applications can also be found in [35], where the authors applied the scalogram to biomedical signals to detect their short-lived temporal interactions.

The remaining part of the paper is organized as follows. Section 2 deals with preliminaries including the fundamental results about the harmonic analysis associated with the Riemann-Liouville operator and basic theory on the generalized Stockwell transform. In Section 3, we study the localization operators theory in the setting of the generalized Stockwell transform. In particular the boundedness and compactness of proposed operators are investigated in the Schatten classes. Section 4, is devoted to investigate the Shapiro uncertainty principles associated with the generalized Stockwell transform. Finally, in the last section, we study the spectral analysis associated with the time-frequency concentration operators to describe functions that have time-frequency content in a subset of finite measure. Moreover, we introduce and we study the scalogram associated with the generalized Stockwell transform.

2. Preliminaries

The aim of this section is to present a healthy overview of the prerequisites circumscribing the Riemann-Liouville operator, Schatten-von Neumann classes, and the generalized Stockwell transform. For a detailed perspective regarding the content of the section, we refer to [4, 9, 27, 36, 38]. For the sake of distinction, we sub-divide the section into three subsections.

2.1. Harmonic analysis associated with the Riemann-Liouville operator

Prior to starting the formal aspects of this subsection, we fix some notations as under:

- $C_*(\mathbb{R}^2)$ denotes the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.
- $C_{*,c}(\mathbb{R}^2)$ denotes the subspace of $C_*(\mathbb{R}^2)$ formed by functions with compact support.
- $\mathcal{E}_*(\mathbb{R}^2)$ is the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable.
- $\mathcal{S}_*(\mathbb{R}^2)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$, even with respect to the first variable.
- S^1 is the unit sphere in $\mathbb{R}^2$,
$$S^1 = \Big\{(\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 = 1\Big\}.$$
- $\mathbb{R}^2_+ = \Big\{(r, x) \in \mathbb{R}^2 : \ r \geq 0\Big\}.$

Note that, for all $(\mu, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) &= -i\lambda u(r,x), \\ \Delta_2 u(r,x) &= -\mu^2 u(r,x) \\ u(0,0) &= 1, \quad \frac{\partial u}{\partial r}(0,x) = 0, \ \forall\, x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by [36]

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2 + \lambda^2})e^{-i\lambda x},$$

where Δ_1 and Δ_2 denote the singular partial differential operators, given by

$$\begin{aligned} \Delta_1 &= \frac{\partial}{\partial x}, \\ \Delta_2 &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \qquad (r,x) \in (0,\infty) \times \mathbb{R}, \qquad \alpha \geqslant 0, \end{aligned}$$

and j_α is the normalized Bessel function defined as

$$\forall\, z \in \mathbb{C}, \quad j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\alpha)}(z/2)^{2k}.$$

Definition 2.1. *For any $(r,x)\in\mathbb{R}^2_+$, the Riemann-Liouville operator on $C_*(\mathbb{R}^2)$ is defined by:*

$$\mathcal{R}_\alpha f(r,x)=\begin{cases}\frac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}f(rs\sqrt{1-t^2},x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds & \text{if}\quad \alpha>0\\ \frac{1}{\pi}\displaystyle\int_{-1}^{1}f(r\sqrt{1-t^2},x+rt)(1-t^2)^{-\frac{1}{2}}dt & \text{if}\quad \alpha=0.\end{cases}$$

Remark 2.1. (i) The function $\varphi_{\mu,\lambda}$, $(\mu,\lambda)\in\mathbb{C}^2$, can be expressed as

$$\forall (r,x)\in\mathbb{R}^2_+,\quad \varphi_{\mu,\lambda}(r,x)=\mathcal{R}_\alpha(\cos(\mu.)e^{-i\lambda.})(r,x).$$

(ii) For all $\nu\in\mathbb{N}^2, (r,x)\in\mathbb{R}^2_+$ and $z=(\mu,\lambda)\in\mathbb{C}^2$, we have

$$|D_z^\nu\varphi_{\mu,\lambda}(r,x)|\le ||(r,x)||^{|\nu|}\exp(2||(r,x)||\,||\mathrm{Im}z||),\tag{2.1}$$

where

$$D_z^\nu=\frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}\quad\text{and}\quad |\nu|=\nu_1+\nu_2.$$

In particular, for all $\nu\in\mathbb{N}^2, (r,x)\in\mathbb{R}^2_+$ and $z=(\mu,\lambda)\in\mathbb{C}^2$:

$$|\varphi_{\mu,\lambda}(r,x)|\le 1.\tag{2.2}$$

Next, consider the set Γ defined as

$$\Gamma=\mathbb{R}^2\cup\Big\{(it,x):(t,x)\in\mathbb{R}^2, |t|\le|x|\Big\},$$

let Γ_+ denotes the subset of Γ given by

$$\Gamma_+=\mathbb{R}^2_+\cup\Big\{(it,x):(t,x)\in\mathbb{R}^2, 0\le t\le|x|\Big\}.$$

Then for all $(\mu,\lambda)\in\Gamma$, we have

$$\sup_{(r,x)\in\mathbb{R}^2}|\varphi_{\mu,\lambda}(r,x)|=1.$$

In the following, we denote by

- $d\nu_\alpha(r,x)$ the measure defined on $\mathbb{R}^2_+$ by
$$d\nu_\alpha(r,x)=k_\alpha r^{2\alpha+1}dr\otimes dx,$$
with
$$k_\alpha=\frac{1}{2^\alpha\Gamma(\alpha+1)(2\pi)^{1/2}}.$$
- For $p\in[1,\infty]$, p' denotes as in all that follows, the conjugate exponent of p.
- $L^p(d\nu_\alpha), 1\le p\le\infty$, the space of measurable functions on $\mathbb{R}^2_+$, satisfying
$$\begin{aligned}\|f\|_{L^p(d\nu_\alpha)} &:= \left(\int_{\mathbb{R}^2_+}|f(r,x)|^p d\nu_\alpha(r,x)\right)^{1/p}<\infty,\quad 1\le p<\infty,\\ \|f\|_{L^\infty(d\nu_\alpha)} &:= \underset{(r,x)\in\mathbb{R}^2_+}{ess\ sup}\ |f(r,x)|<\infty,\quad p=\infty.\end{aligned}$$
- $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by
$$\mathcal{B}_{\Gamma_+}=\Big\{\theta^{-1}(B):\quad B\in\mathcal{B}_{Bor}(\mathbb{R}^2_+)\Big\},$$
where θ defined on the set Γ_+ by
$$\theta(\mu,\lambda)=(\sqrt{\mu^2+\lambda^2},\lambda).\tag{2.3}$$
- $d\gamma_\alpha$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by
$$\forall\, A\subset\mathcal{B}_{\Gamma_+},\quad \gamma_\alpha(A)=\nu_\alpha(\theta(A)).$$

- $L^p(d\gamma_\alpha), 1 \le p \le \infty$, the space of measurable functions on Γ_+, satisfying

$$\begin{aligned} \|f\|_{L^p(d\gamma_\alpha)} &:= \left(\int_{\Gamma_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L^\infty(d\gamma_\alpha)} &:= \underset{(\mu,\lambda)\in\Gamma_+}{ess\ sup} |f(\mu,\lambda)| < \infty, \quad p = \infty. \end{aligned}$$

We have the following properties.

Proposition 2.1. *i) For every nonnegative measurable function f on Γ_+, we have*

$$\int_{\Gamma_+} f(\mu,\lambda) d\gamma_\alpha(\mu,\lambda) = k_\alpha \Big[\int_{\mathbb{R}^2_+} f(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu_\alpha d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu_\alpha d\lambda\Big].$$

ii) For every nonnegative measurable function f on $\mathbb{R}^2_+$ (resp. integrable on $\mathbb{R}^2_+$ with respect to the measure $d\nu_\alpha$), $f\circ\theta$ is a measurable nonnegative function on Γ_+, (resp. integrable on Γ_+ with respect to the measure $d\gamma_\alpha$) and we have

$$\int_{\Gamma_+} f\circ\theta(\mu,\lambda) d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x) d\nu_\alpha(r,x). \tag{2.4}$$

Remark 2.2. For all $(\mu,\lambda)\in\Gamma$, $(r,x),(s,y)\in\mathbb{R}^2_+$, the eigenfunction $\varphi_{\mu,\lambda}$, satisfies the following product formula

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi \varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta.$$

Definition 2.2. *Let f be in $L^p(d\nu_\alpha)$, $p\in[1,\infty]$, for all $(r,x)\in\mathbb{R}^2_+$, we define the translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville operator by*

$$\tau_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta, \tag{2.5}$$

for all $(s,y)\in\mathbb{R}^2_+$.

Proposition 2.2. *For every $f\in L^p(d\nu_\alpha)$, $1\leqslant p\leqslant\infty$ and $(r,x)\in\mathbb{R}^2_+$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\left\|\tau_{(r,x)}(f)\right\|_{L^p(d\nu_\alpha)} \leqslant \|f\|_{L^p(d\nu_\alpha)}. \tag{2.6}$$

Definition 2.3. *The convolution product of $f,g\in L^1(d\nu_\alpha)$ is defined by*

$$f*_\alpha g(r,x) = \int_{\mathbb{R}^2_+} \tau_{(r,x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y), \quad \text{for all} \quad (r,x)\in\mathbb{R}^2_+, \tag{2.7}$$

with $\check{f}(s,y) = f(s,-y)$.

Proposition 2.3. *Let $1\le p,q,r\le\infty$, such that $\frac{1}{p}+\frac{1}{q}-\frac{1}{r}=1$. If f is a function in $L^p(d\nu_\alpha)$ and g an element of $L^q(d\nu_\alpha)$, then $f*_\alpha g$ belongs to $L^r(d\nu_\alpha)$ and we have*

$$\|f*_\alpha g\|_{L^r(d\nu_\alpha)} \le \|f\|_{L^p(d\nu_\alpha)}\|g\|_{L^q(d\nu_\alpha)}. \tag{2.8}$$

Next, we have the notion of generalized Fourier transform $\mathcal{F}_\alpha$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$.

Definition 2.4. *The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by*

$$\forall\,(\mu,\lambda)\in\Gamma, \quad \mathcal{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x). \tag{2.9}$$

Below, we recall some fundamental properties of the generalized Fourier transform $\mathcal{F}_\alpha$.
(i) For all $f \in L^1(d\nu_\alpha)$,
$$||\mathcal{F}_\alpha(f)||_{L^\infty(d\gamma_\alpha)} \leq ||f||_{L^1(d\nu_\alpha)}. \tag{2.10}$$
(ii) For every $f \in L^1(d\nu_\alpha)$, we have
$$\mathcal{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathcal{F}}_\alpha(f) \circ \theta(\mu,\lambda), \quad (\mu,\lambda) \in \Gamma,$$
where for every $(\mu,\lambda) \in \mathbb{R}^2$,
$$\widetilde{\mathcal{F}}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x) j_\alpha(r\mu) e^{-i\lambda x} d\nu_\alpha(r,x),$$
and θ is the function defined by the relation (2.3).
(iii) For $f \in L^1(d\nu_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(d\gamma_\alpha)$, we have the inversion formula for $\mathcal{F}_\alpha$: for almost every $(r,x) \in \mathbb{R}^2_+$,
$$f(r,x) = \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu,\lambda) \overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_\alpha(\mu,\lambda). \tag{2.11}$$

Theorem 2.1. i*) (Plancherel's formula for $\mathcal{F}_\alpha$). For every f in $\mathcal{S}_*(\mathbb{R}^2)$, we have*
$$\int_{\Gamma_+} |\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} |f(r,x)|^2 d\nu_\alpha(r,x). \tag{2.12}$$
In particular, the generalized Fourier transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.
ii*) (Parseval's formula for $\mathcal{F}_\alpha$). For all f, g in $L^2(d\nu_\alpha)$ we have*
$$\int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu,\lambda) \overline{\mathcal{F}_\alpha(g)(\mu,\lambda)} d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x) \overline{g(r,x)} d\nu_\alpha(r,x). \tag{2.13}$$

2.2. Basic generalized Stockwell theory

In this subsection, we shall recall some fundamental results on the generalized Stockwell transforms due to Ben Hamadi, Ghandouri and Hafirassou (see [9]) and Mejjaoli [27].
For $(a,b) \in (0,\infty) \times \mathbb{R}^*$, the dilation operator $D_{(a,b)}$ of any measurable function h on $\mathbb{R}^2_+$ is defined by
$$\forall\, (r,x) \in \mathbb{R}^2_+,\ D_{(a,b)}h(r,x) := a^{\alpha+1}|b|^{\frac{1}{2}} h(ar,bx). \tag{2.14}$$
In the following proposition, we assemble some fundamental properties of the dilation operators.

Proposition 2.4. *(i) For all $(a,b),(c,d) \in (0,\infty) \times \mathbb{R}^*$, we have*
$$D_{(a,b)} \circ D_{(c,d)} = D_{(ac,bd)}. \tag{2.15}$$
(ii) Let $(a,b) \in (0,\infty) \times \mathbb{R}^*$. For all $h \in L^p(d\nu_\alpha)$, $p \in [1,\infty]$. The function $D_{(a,b)}h$ belongs to $L^p(d\nu_\alpha)$ and we have
$$||D_{(a,b)}h||_{L^p(d\nu_\alpha)} = a^{(2\alpha+2)(\frac{1}{2}-\frac{1}{p})}|b|^{\frac{p-2}{2p}} ||h||_{L^p(d\nu_\alpha)}. \tag{2.16}$$
In particular, $D_{(a,b)}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself whose the inverse operator is $D_{(\frac{1}{a},\frac{1}{b})}$. Moreover we have
$$\forall\, (\mu,\lambda) \in \mathbb{R}^2_+, \quad \widetilde{\mathcal{F}}_\alpha(D_{(a,b)}h)(\mu,\lambda) = \frac{1}{a^{\alpha+1}|b|^{\frac{1}{2}}} \widetilde{\mathcal{F}}_\alpha(h)(\frac{\mu}{a},\frac{\lambda}{b}). \tag{2.17}$$
(iii) Let $(a,b) \in (0,\infty) \times \mathbb{R}^*$. For all h, g in $L^2(d\nu_\alpha)$, we have
$$\langle D_{(a,b)}h, g\rangle_{L^2(d\nu_\alpha)} = \langle h, D_{(\frac{1}{a},\frac{1}{b})}g\rangle_{L^2(d\nu_\alpha)}. \tag{2.18}$$
(iv) Let $(a,b) \in (0,\infty) \times \mathbb{R}^*$ and $(r,x) \in \mathbb{R}^2_+$. We have
$$D_{(a,b)}\tau_{(r,x)} = \tau_{(\frac{r}{a},\frac{x}{b})}\, D_{(a,b)}. \tag{2.19}$$

Definition 2.5. *The modulation operator $M_{(a,b)}$ is defined for every function h in $L^2(d\nu_\alpha)$ and for all $(a,b)\in\mathbb{R}^2_+$ by*

$$M_{(a,b)}h=\widetilde{\mathcal{F}_\alpha}\Big(\sqrt{\tau_{(a,b)}|\widetilde{\mathcal{F}_\alpha}(h)|^2}\Big).$$

Proposition 2.5. *1) For every $h\in L^2(d\nu_\alpha)$, $(a,b)\in\mathbb{R}^2_+$, $M_{(a,b)}h$ belongs to $L^2(d\nu_\alpha)$ and we have*

$$\|M_{(a,b)}h\|_{L^2(d\nu_\alpha)}=\|h\|_{L^2(d\nu_\alpha)}. \tag{2.20}$$

2) For every $h\in L^2(d\nu_\alpha)$, $(a,b)\in(0,\infty)\times\mathbb{R}^*$, we have

$$M_{(a,b)}D_{(a,b)}h=D_{(a,b)}M_{(1,1)}h. \tag{2.21}$$

Definition 2.6. *Let $h\in L^2(d\nu_\alpha)$, we define the family $h^\alpha_{a,b,r,x}$, $(a,b)\in(0,\infty)\times\mathbb{R}^*$, $(r,x)\in\mathbb{R}^2_+$, as*

$$h^\alpha_{a,b,r,x}(s,y)=\tau_{(r,x)}M_{(a,b)}D_{(a,b)}h(s,y),\quad \forall (s,y)\in\mathbb{R}^2_+. \tag{2.22}$$

Definition 2.7. *We say that a function $h\in L^2(d\nu_\alpha)$ is a generalized Stockwell wavelet if*

$$0<C_h:=c_\alpha\int_0^\infty\int_0^\infty\tau_{(1,1)}(|\widetilde{\mathcal{F}_\alpha}(h)|^2)(a,b)\frac{da}{a}\frac{db}{b}<\infty,$$

where

$$c_\alpha=\frac{1}{2^\alpha\sqrt{2\pi}\Gamma(\alpha+1)}.$$

Remark 2.3. If $h\in L^2(d\nu_\alpha)$ is a generalized Stockwell wavelet in the sense of the previous Definition, then for every $(\mu,\lambda)\in(0,\infty)\times(0,\infty)$, we have

$$0<C_h=c_\alpha\int_0^\infty\int_0^\infty\tau_{(1,1)}(|\widetilde{\mathcal{F}_\alpha}(h)|^2)(\frac{\mu}{a},\frac{\lambda}{b})\frac{da}{a}\frac{db}{b}<\infty. \tag{2.23}$$

Remark 2.4. Let h be in $L^2(d\nu_\alpha)$. We have

$$\forall\,(a,b)\in(0,\infty)\times\mathbb{R}^*,\,\forall\,(r,x)\in\mathbb{R}^2_+,\quad ||h_{a,b,r,x}||_{L^2(d\nu_\alpha)}\le||h||_{L^2(d\nu_\alpha)}. \tag{2.24}$$

Notation. We denote by
$L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$, $p\in[1,\infty]$, the space of measurable functions f on $\mathbb{R}^2_+\times\mathbb{R}^2_+$ such that

$$\begin{aligned}\|f\|_{L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &:= \left(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}|f(a,b,r,x)|^p d\mu_\alpha(a,b,r,x)\right)^{\frac{1}{p}}<\infty,\ \ 1\le p<\infty,\\ \|f\|_{L^\infty_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &:= \operatorname*{ess\,sup}_{(a,b,r,x)\in\mathbb{R}^2_+\times\mathbb{R}^2_+}|f(a,b,r,x)|<\infty,\end{aligned}$$

where the measure μ_α is defined by

$$\forall\,(a,b,r,x)\in\mathbb{R}^2_+\times\mathbb{R}^2_+,\quad d\mu_\alpha(a,b,r,x)=d\nu_\alpha(a,b)d\nu_\alpha(r,x).$$

Definition 2.8. *Let h be a generalized Stockwell wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$. The generalized continuous Stockwell transform $\mathcal{S}^\alpha_h$ on $\mathbb{R}^2_+$ is defined for regular functions f on $\mathbb{R}^2_+$ by*

$$\forall\,(a,b)\in(0,\infty)\times\mathbb{R}^*,\ (r,x)\in\mathbb{R}^2_+,\quad \mathcal{S}^\alpha_h(f)(a,b,r,x)=\int_{\mathbb{R}^2_+}f(s,y)\overline{h^\alpha_{a,b,r,x}(s,y)}d\nu_\alpha(s,y), \tag{2.25}$$

where $h^\alpha_{a,b,r,x}$ is given by relation (2.22).

Definition 2.8 can be recast as

$$\mathcal{S}_h^\alpha(f)(a,b,r,x) = \breve{f} *_\alpha \overline{M_{(a,b)}D_{(a,b)}h}(r,x), \tag{2.26}$$

where $*_\alpha$ is the generalized convolution product given by (2.7).
We note that the adjoint of $\mathcal{S}_h^\alpha$ is $(\mathcal{S}_h^\alpha)^* : L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2) \to L^2(d\nu_\alpha)$ and is defined as

$$(\mathcal{S}_h^\alpha)^*(F)(s,y) = \frac{1}{C_h}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} F(a,b;r,x)h_{a,b;r,x}(s,y)d\mu_\alpha(a,b;r,x),\ (s,y)\in\mathbb{R}_+^2. \tag{2.27}$$

Theorem 2.2. *(Plancherel's formula for $\mathcal{S}_h^\alpha$). Let h be a generalized Stockwell wavelet on $\mathbb{R}_+^2$ in $L^2(d\nu_\alpha)$. For all f in $L^2(d\nu_\alpha)$ we have*

$$\int_{\mathbb{R}_+^2} |f(r,x)|^2 d\nu_\alpha(r,x) = \frac{1}{C_h}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\mathcal{S}_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x). \tag{2.28}$$

Corollary 2.1. *(Parseval's formula for $\mathcal{S}_h^\alpha$). Let h be a generalized Stockwell wavelet on $\mathbb{R}_+^2$ in $L^2(d\nu_\alpha)$ and f_1, f_2 in $L^2(d\nu_\alpha)$. Then, we have*

$$\int_{\mathbb{R}_+^2} f_1(r,x)\overline{f_2(r,x)}d\nu_\alpha(r,x) = \frac{1}{C_h}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \mathcal{S}_h^\alpha(f_1)(a,b,r,x)\overline{\mathcal{S}_h^\alpha(f_2)(a,b,r,x)}d\mu_\alpha(a,b,r,x). \tag{2.29}$$

Remark 2.5. Let h be a generalized Stockwell wavelet in $L^2(d\nu_\alpha)$. Then from the relations (2.25) and (2.24), for all f in $L^2(d\nu_\alpha)$ we have

$$\|\mathcal{S}_h^\alpha(f)\|_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \le \|f\|_{L^2(d\nu_\alpha)}\|h\|_{L^2(d\nu_\alpha)}. \tag{2.30}$$

2.3. Schatten-von Neumann classes

In this subsection, we recall the notion of Schatten-von Neumann classes. Prior to that, we set the following notation:

- $l^p(\mathbb{N})$, $1 \le p \le \infty$, the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$, such that

$$\begin{aligned} ||x||_p &:= \Big(\sum_{j=1}^\infty |x_j|^p\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \le p < \infty, \\ ||x||_\infty &:= \sup_{j\in\mathbb{N}} |x_j| < \infty. \end{aligned}$$

  For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x, y\rangle_2 := \sum_{j=1}^\infty x_j\overline{y_j}.$$

- $B(L^p(d\nu_\alpha))$, $1 \le p \le \infty$, the space of bounded operators from $L^p(d\nu_\alpha)$ into itself.

Definition 2.9. *(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2(d\nu_\alpha))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.*
(ii) For $1 \le p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \Big(\sum_{n=1}^\infty (s_n(A))^p\Big)^{\frac{1}{p}}. \tag{2.31}$$

Remark 2.6. We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.

Definition 2.10. *The trace of an operator A in S_1 is defined by*

$$tr(A) = \sum_{n=1}^{\infty} \langle Av_n, v_n \rangle_{L^2(d\nu_\alpha)} \tag{2.32}$$

where $(v_n)_n$ is any orthonormal basis of $L^2(d\nu_\alpha)$.

Remark 2.7. If A is positive, then

$$tr(A) = ||A||_{S_1}. \tag{2.33}$$

Moreover, a compact operator A on the Hilbert space $L^2(d\nu_\alpha)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2(d\nu_\alpha)} \tag{2.34}$$

for any orthonormal basis $(v_n)_n$ of $L^2(d\nu_\alpha)$.

Definition 2.11. *We define $S_\infty := B(L^2(d\nu_\alpha))$, equipped with the norm,*

$$||A||_{S_\infty} := \sup_{v \in L^2(d\nu_\alpha): ||v||_{L^2(d\nu_\alpha)}=1} ||Av||_{L^2(d\nu_\alpha)}. \tag{2.35}$$

3. Localization operators for the generalized Stockwell transform

3.1. Preliminaries

In this subsection, we define the two-wavelet localization operator associated for the generalized Stockwell transform, and we give the expression of its adjoint.

Definition 3.1. *Let h, k be measurable functions on $\mathbb{R}^2_+$ and σ be measurable function on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$, we define the two-wavelet localization operator for the generalized Stockwell transform, denoted by $\mathcal{L}_{h,k}(\sigma)$, on $L^p(d\nu_\alpha)$, $1 \le p \le \infty$, by $\forall\, (s, y) \in \mathbb{R}^2_+$,*

$$\mathcal{L}_{h,k}(\sigma)(f)(s, y) = \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \sigma(a, b, r, x) \mathcal{S}^\alpha_h(f)(a, b, r, x)\, k_{a,b,r,x}(s, y) d\mu_\alpha(a, b, r, x). \tag{3.1}$$

Often it is more convenient to interpret the definition of $\mathcal{L}_{h,k}(\sigma)$ in a weak sense, that is, for f in $L^p(d\nu_\alpha)$, $1 \le p \le \infty$, and g in $L^{p'}(d\nu_\alpha)$, where p' is the conjugate exponent of p

$$\langle \mathcal{L}_{h,k}(\sigma)(f), g \rangle_{L^2(d\nu_\alpha)} = \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \sigma(a, b, r, x) \mathcal{S}^\alpha_h(f)(a, b, r, x) \overline{\mathcal{S}^\alpha_k(g)(a, b, r, x)} d\mu_\alpha(a, b, r, x). \tag{3.2}$$

In accordance with the different choices of the symbols σ and the different continuities required, we need to impose different conditions on h and k. And then we obtain an operator on $L^p(d\nu_\alpha)$. In what follows, such operator $\mathcal{L}_{h,k}(\sigma)$ will be named localization operator for the sake of simplicity.

Proposition 3.1. *Let $p \in [1, \infty)$. The adjoint of the localization operator*

$$\mathcal{L}_{h,k}(\sigma) : L^p(d\nu_\alpha) \to L^p(d\nu_\alpha)$$

is $\mathcal{L}_{k,h}(\overline{\sigma}) : L^{p'}(d\nu_\alpha) \to L^{p'}(d\nu_\alpha)$.

Proof. For all f in $L^p(d\nu_\alpha)$ and g in $L^{p'}(d\nu_\alpha)$ it follows immediately from (3.2)

$$\begin{aligned}
\langle \mathcal{L}_{h,k}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)} &= \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \sigma(a,b,r,x) \mathcal{S}_h^\alpha(f)(a,b,r,x)\overline{\mathcal{S}_k^\alpha(g)(a,b,r,x)} d\mu_\alpha(a,b,r,x) \\
&= \overline{\frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \overline{\sigma(a,b,r,x)} \mathcal{S}_k^\alpha(g)(a,b,r,x)\overline{\mathcal{S}_h^\alpha(f)(a,b,r,x)} d\mu_\alpha(a,b,r,x)} \\
&= \overline{\langle \mathcal{L}_{k,h}(\overline{\sigma})(g), f\rangle_{L^2(d\nu_\alpha)}} = \langle f, \mathcal{L}_{k,h}(\overline{\sigma})(g)\rangle_{L^2(d\nu_\alpha)}.
\end{aligned}$$

Thus we get

$$\mathcal{L}_{h,k}^*(\sigma) = \mathcal{L}_{k,h}(\overline{\sigma}). \tag{3.3}$$

□

In the sequel of this section, h and k will be a generalized Stockwell wavelets in $L^2(d\nu_\alpha)$ such that

$$\|h\|_{L^2(d\nu_\alpha)} = \|k\|_{L^2(d\nu_\alpha)} = 1.$$

3.2. Boundedness

In this subsection, we prove that the linear operators

$$\mathcal{L}_{h,k}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$$

are bounded for all symbol $\sigma \in L^p_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$, $p \in [1,\infty]$.
We first tackle this problem for σ in $L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$ or $L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$ and we conclude by using interpolation theory.

Proposition 3.2. *Let σ be in $L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$, then the localization operator $\mathcal{L}_{h,k}(\sigma)$ is in S_∞ and we have*

$$||\mathcal{L}_{h,k}(\sigma)||_{S_\infty} \leqslant \frac{1}{\sqrt{C_h C_k}} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.$$

Proof. For every functions f and g in $L^2(d\nu_\alpha)$, we have from the relations (3.2) and (2.30),

$$\begin{aligned}
|\langle \mathcal{L}_{h,k}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| &\leqslant \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\sigma(a,b,r,x)||\mathcal{S}_h^\alpha(f)(a,b,r,x)||\overline{\mathcal{S}_k^\alpha(g)(a,b,r,x)}| d\mu_\alpha(a,b,r,x) \\
&\leqslant \frac{1}{\sqrt{C_h C_k}} \|\mathcal{S}_h^\alpha(f)\|_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \|\mathcal{S}_k^\alpha(g)\|_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \\
&\leqslant \frac{1}{\sqrt{C_h C_k}} \|f\|_{L^2(d\nu_\alpha)} \|g\|_{L^2(d\nu_\alpha)} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.
\end{aligned}$$

Thus,

$$||\mathcal{L}_{h,k}(\sigma)||_{S_\infty} \leqslant \frac{1}{\sqrt{C_h C_k}} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.$$

□

Proposition 3.3. *Let σ be in $L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$, then the localization operator $\mathcal{L}_{h,k}(\sigma)$ is in S_∞ and we have*

$$||\mathcal{L}_{h,k}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.$$

Proof. For all functions f and g in $L^2(d\nu_\alpha)$, we have from Hölder's inequality

$$\begin{aligned}
|\langle \mathcal{L}_{h,k}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| &\leqslant \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\sigma(a,b,r,x)||\mathcal{S}_h^\alpha(f)(a,b,r,x)||\overline{\mathcal{S}_k^\alpha(g)(a,b,r,x)}| d\mu_\alpha(a,b,r,x) \\
&\leqslant \frac{1}{\sqrt{C_h C_k}} \|\sigma\|_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \|\mathcal{S}_h^\alpha(f)\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \|\mathcal{S}_k^\alpha(g)\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.
\end{aligned}$$

Using Plancherel's formula for $\mathcal{S}_h^\alpha$ and $\mathcal{S}_h^\alpha$, given by the relation (2.28), we get

$$|\langle \mathcal{L}_{h,k}(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)}| \leqslant \|\sigma\|_{L^\infty_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\|f\|_{L^2(d\nu_\alpha)}\|g\|_{L^2(d\nu_\alpha)}.$$

Thus,

$$||\mathcal{L}_{h,k}(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

□

We can now associate a localization operator $\mathcal{L}_{h,k}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ to every function σ in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $1 \leq p \leq \infty$ and prove that $\mathcal{L}_{h,k}(\sigma)$ is in S_∞. The precise result is the following theorem.

Theorem 3.1. *Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator $\mathcal{L}_{h,k}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$, such that*

$$||\mathcal{L}_{h,k}(\sigma)||_{S_\infty} \leqslant (\frac{1}{\sqrt{C_hC_k}})^{\frac{1}{p}}\|\sigma\|_{L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

Proof. Let f be in $L^2(d\nu_\alpha)$. We consider the following operator

$$T : L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \cap L^\infty_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \quad \to \quad L^2(d\nu_\alpha),$$

given by

$$T(\sigma) := \mathcal{L}_{h,k}(\sigma)(f).$$

Then by Proposition 3.2 and Proposition 3.3

$$||T(\sigma)||_{L^2(d\nu_\alpha)} \leq \frac{1}{\sqrt{C_hC_k}}||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \tag{3.4}$$

and

$$||T(\sigma)||_{L^2(d\nu_\alpha)} \leq ||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^\infty_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}. \tag{3.5}$$

Therefore, by (3.4), (3.5) and the Riesz-Thorin interpolation theorem (see [34, Theorem 2], and [38, Theorem 2.11]), T may be uniquely extended to a linear transformation on $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, and we have

$$||\mathcal{L}_{h,k}(\sigma)(f)||_{L^2(d\nu_\alpha)} = ||T(\sigma)||_{L^2(d\nu_\alpha)} \leq (\frac{1}{\sqrt{C_hC_k}})^{\frac{1}{p}}||f||_{L^2(d\nu_\alpha)}\|\sigma\|_{L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}. \tag{3.6}$$

Since (3.6) is true for arbitrary functions f in $L^2(d\nu_\alpha)$, then we obtain the desired result.

□

3.3. Schatten-von Neumann properties for $\mathcal{L}_{h,k}(\sigma)$

In this subsection, we will prove that, the localization operator

$$\mathcal{L}_{h,k}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$$

is in the Schatten class S_p. The first result on the Schatten property of localization operators, is given in the following proposition.

Proposition 3.4. *Let σ be in $L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, then the localization operator*

$$\mathcal{L}_{h,k}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$$

is in S_2 and we have

$$\|\mathcal{L}_{h,k}(\sigma)\|_{S_2} \leqslant \frac{1}{\sqrt{C_hC_k}}\|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

Proof. Let $\{\phi_j, j = 1, 2...\}$ be an orthonormal basis for $L^2(d\nu_\alpha)$. Then by (3.2), Fubini's theorem, Parseval's identity and the relations (2.25) and (3.3), we have

$$\begin{aligned}\sum_{j=1}^{\infty} ||\mathcal{L}_{h,k}(\sigma)(\phi_j)||^2_{L^2(d\nu_\alpha)} &= \sum_{j=1}^{\infty} \langle \mathcal{L}_{h,k}(\sigma)(\phi_j), \mathcal{L}_{h,k}(\sigma)(\phi_j)\rangle_{L^2(d\nu_\alpha)} \\ &= \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \sigma(a,b,r,x) \langle \mathcal{L}^*_{h,k}(\sigma)\, k_{a,b,r,x}, h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)} d\mu_\alpha(a,b,r,x).\end{aligned}$$

Thus, we get

$$\begin{aligned}\sum_{j=1}^{\infty} ||\mathcal{L}_{h,k}(\sigma)(\phi_j)||^2_{L^2(d\nu_\alpha)} &\leq \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} |\sigma(a,b,r,x)|\; ||\mathcal{L}^*_{h,k}(\sigma)||_{S_\infty} d\mu_\alpha(a,b,r,x) \\ &= \frac{1}{\sqrt{C_h C_k}} ||\mathcal{L}^*_{h,k}(\sigma)||_{S_\infty} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} < \infty. \end{aligned} \tag{3.7}$$

So, by (3.7) and Proposition 2.8 in the book [38], by Wong, $\mathcal{L}_{h,k}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ is in the Hilbert-Schmidt class S_2 and hence compact. □

Proposition 3.5. *Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $1 \leqslant p < \infty$. Then, the localization operator $\mathcal{L}_{h,k}(\sigma)$ is compact.*

Proof. Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of functions in

$$L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \bigcap L^\infty_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$$

such that $\sigma_n \to \sigma$ in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ as $n \to \infty$. Then by Theorem 3.1

$$||\mathcal{L}_{h,k}(\sigma_n) - \mathcal{L}_{h,k}(\sigma)||_{S_\infty} \leq \left(\frac{1}{\sqrt{C_h C_k}}\right)^{\frac{1}{p}} ||\sigma_n - \sigma||_{L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

Hence $\mathcal{L}_{h,k}(\sigma_n) \to \mathcal{L}_{h,k}(\sigma)$ in S_∞ as $n \to \infty$. On the other hand as by Proposition 3.4, $\mathcal{L}_{h,k}(\sigma_n)$ is in S_2 hence compact, it follows that $\mathcal{L}_{h,k}(\sigma)$ is compact. □

Theorem 3.2. *Let σ be in $L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$. Then,*

$$\frac{2}{C_h + C_k} \|\widetilde{\sigma}\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \leqslant \|\mathcal{L}_{h,k}(\sigma)\|_{S_1} \leqslant \frac{1}{\sqrt{C_h C_k}} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}, \tag{3.8}$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(a,b,r,x) = \langle \mathcal{L}_{h,k}(\sigma)(\, h_{a,b,r,x}),\, k_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}, \quad (a,b,r,x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+.$$

Proof. Since σ is in $L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, by Proposition 3.4, $\mathcal{L}_{h,k}(\sigma)$ is in S_2. Using [38, Theorem 2.2], there exists an orthonormal basis $\{\phi_j,\ j = 1, 2...\}$ for the orthogonal complement of the kernel of $\mathcal{L}_{h,k}(\sigma)$, consisting of eigenvectors of $|\mathcal{L}_{h,k}(\sigma)|$ and $\{\varphi_j,\ j = 1, 2...\}$ an orthonormal set in $L^2(d\nu_\alpha)$, such that

$$\mathcal{L}_{h,k}(\sigma)(f) = \sum_{j=1}^{\infty} s_j \langle f, \phi_j\rangle_{L^2(d\nu_\alpha)} \varphi_j, \tag{3.9}$$

where $s_j,\ j = 1, 2...$ are the positive singular values of $\mathcal{L}_{h,k}(\sigma)$ corresponding to ϕ_j. Then, we get

$$\|\mathcal{L}_{h,k}(\sigma)\|_{S_1} = \sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} \langle \mathcal{L}_{h,k}(\sigma)(\phi_j), \varphi_j\rangle_{L^2(d\nu_\alpha)}.$$

Thus, by Fubini's theorem, Schwarz's inequality, Bessel's inequality, relations (2.24) and (2.25), we get

$$\begin{aligned}
\|\mathcal{L}_{h,k}(\sigma)\|_{S_1} &= \sum_{j=1}^{\infty}\langle \mathcal{L}_{h,k}(\sigma)(\phi_j), \varphi_j\rangle_{L^2(d\nu_\alpha)} \\
&= \sum_{j=1}^{\infty} \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \sigma(a,b,r,x)\mathcal{S}_h^\alpha(\phi_j)(a,b,r,x)\overline{\mathcal{S}_k^\alpha(\varphi_j)(a,b,r,x)}d\mu_\alpha(a,b,r,x) \\
&\leqslant \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\sigma(a,b,r,x)| \left(\sum_{j=1}^{\infty}|\mathcal{S}_h^\alpha(\phi_j)(a,b,r,x)|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty}|\mathcal{S}_k^\alpha(\varphi_j)(a,b,r,x)|^2\right)^{\frac{1}{2}} d\mu_\alpha(a,b,r,x) \\
&\leqslant \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\sigma(a,b,r,x)|\| h_{a,b,r,x}\|_{L^2(d\nu_\alpha)}\| k_{a,b,r,x}\|_{L^2(d\nu_\alpha)} d\mu_\alpha(a,b,r,x) \\
&\leqslant \frac{1}{\sqrt{C_h C_k}} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.
\end{aligned}$$

Thus

$$\|\mathcal{L}_{h,k}(\sigma)\|_{S_1} \le \frac{1}{\sqrt{C_h C_k}} \|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.$$

We prove now that $\mathcal{L}_{h,k}(\sigma)$ satisfies the first member of (3.8). It is easy to see that $\widetilde{\sigma}$ belongs to $L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$ and using formula (3.9), we get

$$\begin{aligned}
|\widetilde{\sigma}(a,b,r,x)| &= |\langle \mathcal{L}_{h,k}(\sigma)(h_{a,b,r,x}), k_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}| \\
&= |\sum_{j=1}^{\infty} s_j \langle h_{a,b,r,x}, \phi_j\rangle_{L^2(d\nu_\alpha)}\langle \varphi_j, k_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}| \\
&\leqslant \frac{1}{2}\sum_{j=1}^{\infty} s_j \Big(|\langle h_{a,b,r,x}, \phi_j\rangle_{L^2(d\nu_\alpha)}|^2 + |\langle k_{a,b,r,x}, \varphi_j\rangle_{L^2(d\nu_\alpha)}|^2\Big).
\end{aligned}$$

Then by Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\widetilde{\sigma}(a,b,r,x)|d\mu_\alpha(a,b,r,x) \le \frac{1}{2}\sum_{j=1}^{\infty} s_j \Big(&\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\langle h_{a,b,r,x}, \phi_j\rangle_{L^2(d\nu_\alpha)}|^2 d\mu_\alpha(a,b,r,x) \\
+ &\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\langle k_{a,b,r,x}, \varphi_j\rangle_{L^2(d\nu_\alpha)}|^2 d\mu_\alpha(a,b,r,x)\Big).
\end{aligned}$$

Thus using Plancherel's identity for $\mathcal{S}_h^\alpha$, $\mathcal{S}_h^\alpha$, we get

$$\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} |\widetilde{\sigma}(a,b,r,x)|d\mu_\alpha(a,b,r,x) \le \frac{C_h + C_k}{2}\sum_{j=1}^{\infty} s_j = \frac{C_h + C_k}{2}\|\mathcal{L}_{h,k}(\sigma)\|_{S_1}.$$

The proof is complete. □

Corollary 3.1. *For σ in $L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$, we have the following trace formula*

$$tr(\mathcal{L}_{h,k}(\sigma)) = \frac{1}{\sqrt{C_h C_k}} \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \sigma(a,b,r,x)\langle k_{a,b,r,x}, h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)} d\mu_\alpha(a,b,r,x). \tag{3.10}$$

Proof. From Theorem 3.2, the localization operator $\mathcal{L}_{h,k}(\sigma)$ belongs to S_1, then by the definition of the trace given by the relation (2.32), we have

$$
\begin{aligned}
tr(\mathcal{L}_{h,k}(\sigma)) &= \sum_{j=1}^{\infty}\langle\mathcal{L}_{h,k}(\sigma)(\phi_j),\phi_j\rangle_{L^2(d\nu_\alpha)}\\
&= \sum_{j=1}^{\infty}\frac{1}{\sqrt{C_hC_k}}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2}\sigma(a,b,r,x)\langle\phi_j,\, h_{(a,b,r,x)}\rangle_{L^2(d\nu_\alpha)}\overline{\langle\phi_j,\, k_{(a,b,r,x)}\rangle_{L^2(d\nu_\alpha)}}d\mu_\alpha(a,b,r,x)\\
&= \frac{1}{\sqrt{C_hC_k}}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2}\sigma(a,b,r,x)\sum_{j=1}^{\infty}\langle\phi_j,\, h_{(a,b,r,x)}\rangle_{L^2(d\nu_\alpha)}\overline{\langle\phi_j,\, k_{(a,b,r,x)}\rangle_{L^2(d\nu_\alpha)}}d\mu_\alpha(a,b,r,x)\\
&= \frac{1}{\sqrt{C_hC_k}}\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2}\sigma(a,b,r,x)\langle k_{(a,b,r,x)},\, h_{(a,b,r,x)}\rangle_{L^2(d\nu_\alpha)}d\mu_\alpha(a,b,r,x),
\end{aligned}
$$

and the proof is complete. $\square$

In the following we give the main result of this section.

Corollary 3.2. *Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$, $1\leqslant p\leqslant\infty$. Then, the localization operator*

$$\mathcal{L}_{h,k}(\sigma): L^2(d\nu_\alpha)\longrightarrow L^2(d\nu_\alpha)$$

is in S_p and we have

$$\|\mathcal{L}_{h,k}(\sigma)\|_{S_p}\leqslant\left(\frac{1}{\sqrt{C_hC_k}}\right)^{\frac{1}{p}}\|\sigma\|_{L^p_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.$$

Proof. The result follows from Proposition 3.3, Theorem 3.2 and by interpolation [38, Theorem 2.10 and Theorem 2.11]. relation $\square$

Remark 3.1. If $h=k$ and if σ is a real valued and nonnegative function in $L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$ then $\mathcal{L}_{h,k}(\sigma): L^2(d\nu_\alpha)\to L^2(d\nu_\alpha)$ is a positive operator. So, by (2.33) and Corollary 3.1

$$||\mathcal{L}_{h,k}(\sigma)||_{S_1}=\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2}\sigma(a,b;r,x)||h_{a,b;r,x}||^2_{L^2(d\nu_\alpha)}d\mu_\alpha(a,b;r,x). \tag{3.11}$$

Now we state a result concerning the trace of products of localization operators.

Corollary 3.3. *Let σ_1 and σ_2 be any real-valued and nonnegative functions which belong to $L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)$. We assume that $h=k$ and h is a function in $L^2(d\nu_\alpha)$ such that $||h||_{L^2(d\nu_\alpha)}=1$. Then, the localization operators $\mathcal{L}_{h,k}(\sigma_1)$, $\mathcal{L}_{h,k}(\sigma_2)$ are positive trace class operators and*

$$
\begin{aligned}
\Big|\Big|\Big(\mathcal{L}_{h,k}(\sigma_1)\,\mathcal{L}_{h,k}(\sigma_2)\Big)^n\Big|\Big|_{S_1} &= tr\Big(\mathcal{L}_{h,k}(\sigma_1)\,\mathcal{L}_{h,k}(\sigma_2)\Big)^n\\
&\leq \Big(tr(\mathcal{L}_{h,k}(\sigma_1))\Big)^n\Big(tr(\mathcal{L}_{h,k}(\sigma_2))\Big)^n\\
&= \Big|\Big|\mathcal{L}_{h,k}(\sigma_1)\Big|\Big|^n_{S_1}\Big|\Big|\mathcal{L}_{h,k}(\sigma_2)\Big|\Big|^n_{S_1},
\end{aligned}
$$

for any natural number n.

Proof. By Theorem 1 in the paper [20] by Liu we know that if A and B are in the trace class S_1 and are positive operators, then

$$\forall\, n\in\mathbb{N},\quad tr(AB)^n\leq\Big(tr(A)\Big)^n\Big(tr(B)\Big)^n.$$

So, if we take $A=\mathcal{L}_{h,k}(\sigma_1)$, $B=\mathcal{L}_{h,k}(\sigma_2)$ and we invoke the previous remark, the desired result is obtained and the proof is completed. $\square$

4. Mean dispersion theorem for the generalized Stockwell transform

In this section, we shall present some useful results regarding the concentration of $\mathcal{S}_h^\alpha(f)$ on small sets.

Proposition 4.1. *Suppose that $U \subset \mathbb{R}_+^2 \times \mathbb{R}_+^2$ satisfies*

$$\mu_\alpha(U) < \frac{C_h}{||h||^2_{L^2(d\nu_\alpha)}}, \tag{4.1}$$

then, for all f in $L^2(d\nu_\alpha)$, we have

$$\|\chi_{U^c}\mathcal{S}_h^\alpha(f)\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \geq \sqrt{C_h}\sqrt{1-\frac{||h||^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(U)}\|f\|_{L^2(d\nu_\alpha)}, \tag{4.2}$$

where χ_{U^c} denotes the characteristic function of the complementary U^c of U.

Proof. From Plancherel's Theorem 2.2, we have

$$C_h||f||^2_{L^2(d\nu_\alpha)} = \|\mathcal{S}_h^\alpha(f)\|^2_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} = \|\mathcal{S}_h^\alpha(f)\|^2_{L^2_{\mu_\alpha}(U)} + \|\mathcal{S}_h^\alpha(f)\|^2_{L^2_{\mu_\alpha}(U^c)}. \tag{4.3}$$

On the other hand from the relation (2.30), we have

$$\begin{aligned}\int_U |\mathcal{S}_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) &\leq \|\mathcal{S}_h^\alpha(f)\|^2_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}\mu_\alpha(U)\\ &\leq \mu_\alpha(U)\|f\|^2_{L^2(d\nu_\alpha)}\|h\|^2_{L^2(d\nu_\alpha)}.\end{aligned} \tag{4.4}$$

Thus, the result follows immediately from the relations (4.3) and (4.4). □

Remark 4.1. Let U be a subset of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ satisfying the relation (4.1). If $\mathcal{S}_h^\alpha(f)$ is supported in U, then $f = 0$.

Proposition 4.2. *Let h be a generalized Stockwell wavelet such that $\|h\|_{L^2(d\nu_\alpha)} = 1$. Let $s > 0$. Then the following uncertainty inequality hold.*
There exists a positive constant $C(s)$ such that, for all f in $L^2(d\nu_\alpha)$, we have

$$\Big|\Big| \, ||(a,b,r,x)||^s \mathcal{S}_h^\alpha(f)\Big|\Big|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \geq C(s)\|f\|_{L^2(d\nu_\alpha)}. \tag{4.5}$$

Proof. Let $\delta > 0$. We consider the subset V_δ of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ defined by

$$V_\delta = \Big\{(a,b,r,x) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : ||(a,b,r,x)|| < \delta\Big\},$$

and satisfying $0 < \mu_\alpha(V_\delta) < C_h$. By applying the relation (4.2) with $U = V_\delta$ we obtain

$$\begin{aligned}||f||^2_{L^2(d\nu_\alpha)} &\leq \tfrac{1}{C_h - \mu_\alpha(V_\delta)}\int_{V_\delta^c}|\mathcal{S}_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x)\\ &\leq \tfrac{1}{\delta^{2s}(C_h - \mu_\alpha(V_\delta))}\int_{||(a,b,r,x)||\geq\delta}||(a,b,r,x)||^{2s}|\mathcal{S}_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x)\\ &\leq \tfrac{1}{\delta^{2s}(C_h - \mu_\alpha(V_\delta))}\Big|\Big| \, ||(a,b,r,x)||^s\mathcal{S}_h^\alpha(f)\Big|\Big|^2_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}.\end{aligned}$$

Thus, we obtain the relation (4.5) with $C(s) := \delta^s\sqrt{C_h - \mu_\alpha(V_\delta)}$. □

Proposition 4.3. *([27]). Let h be a generalized Stockwell wavelet on $\mathbb{R}_+^2$ in $L^1(d\nu_\alpha)\bigcap L^2(d\nu_\alpha)$. Then, $\mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function*

$$\mathcal{K}_h(a',b',r',x';a,b,r,x) := \frac{1}{C_h}\int_{\mathbb{R}_+^2} h_{a',b',r',x'}(s,y)\overline{h_{a,b,r,x}(s,y)}d\nu(s,y). \tag{4.6}$$

The kernel satisfies:

$$\forall\,(a',b',r',x'),\ (a,b,r,x)\in\mathbb{R}^2_+\times\mathbb{R}^2_+,\quad |\mathcal{K}_h(a',r',x';a,b,r,x)|\leq\frac{||h||^2_{L^2(d\nu_\alpha)}}{C_h}. \tag{4.7}$$

Notation. We shall adopt the following notations:
(i) $P_h: L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)\to L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ denotes the orthogonal projection from $L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ onto $\mathcal{S}^\alpha_h(L^2(d\nu_\alpha))$.
(ii) $P_U: L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)\to L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ denotes the orthogonal projection from $L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ onto the subspace of functions of $L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ supported in a subset $U\subset\mathbb{R}^2_+\times\mathbb{R}^2_+$ satisfying

$$0<\mu_\alpha(U):=\int_U d\mu_\alpha(a,b,r,x)<\infty. \tag{4.8}$$

Next, we recall that

$$\begin{aligned} ||P_UP_h||_{HS} &:= \Big(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\times U}|\mathcal{K}_h(a',b',r',x';a,b,r,x)|^2 d\mu_\alpha(a',b',r',x')d\mu_\alpha(a,b,r,x)\Big)^{\frac{1}{2}} \\ &\leq \frac{||h||_{L^2(d\nu_\alpha)}}{\sqrt{C_h}}\sqrt{\mu_\alpha(U)}<\infty. \end{aligned} \tag{4.9}$$

That is, P_UP_h is a Hilbert-Schmidt operator and, therefore it is a compact operator.

Remark 4.2. i) The operator $P_h=\mathcal{S}^\alpha_h(\mathcal{S}^\alpha_h)^*$ can be explicitly expressed as an integral operator

$$P_hF(z)=\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}F(a,b,r,x)\mathcal{K}_h(z;a,b,r,x)d\mu_\alpha(a,b,r,x),\ z=(a',b',r',x')\in\mathbb{R}^2_+\times\mathbb{R}^2_+,$$

with integral kernel $\mathcal{K}_h$.
ii) As $\mathcal{K}_h$ is the integral kernel of an orthogonal projection, it satisfies

$$\mathcal{K}_h(z;z')=\overline{\mathcal{K}_h(z';z)},\quad \text{for all } z,z'\in\mathbb{R}^2_+\times\mathbb{R}^2_+, \tag{4.10}$$

and

$$\mathcal{K}_h(z;z')=\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\mathcal{K}_h(z;z'')\mathcal{K}_h(z'';z')d\mu_\alpha(z''),\ z,z'\in\mathbb{R}^2_+\times\mathbb{R}^2_+. \tag{4.11}$$

iii) If $\{v_n: n\in\mathbb{N}\}$ is an orthonormal basis of $\mathcal{S}^\alpha_h(L^2(d\nu_\alpha))$, $\mathcal{K}_h$ can be expanded as

$$\mathcal{K}_h(z;z')=\sum_{n=1}^{\infty}v_n(z)\overline{v_n(z')},\ z,z'\in\mathbb{R}^2_+\times\mathbb{R}^2_+. \tag{4.12}$$

Definition 4.1. *Let $0<\varepsilon<1$ and $U\subset\mathbb{R}^2_+\times\mathbb{R}^2_+$ be a measurable subset. Let h be a generalized Stockwell wavelet and let $f\in L^2(d\nu_\alpha)$ be a nonzero function.*
We say that $\mathcal{S}^\alpha_h(f)$ is ε-time-concentrated on U, if

$$\Big|\Big|\mathcal{S}^\alpha_h(f)\Big|\Big|_{L^2_{\mu_\alpha}(U^c)}\leq\varepsilon\Big|\Big|\mathcal{S}^\alpha_h(f)\Big|\Big|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

Proposition 4.4. *Let h be a generalized Stockwell wavelet and $(u_\beta)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$ and U be a measurable subset of $\mathbb{R}^2_+\times\mathbb{R}^2_+$. If $\mu_\alpha(U)<\infty$, then for every nonempty finite subset $\mathcal{K}\subset\mathbb{N}^2$, we have*

$$\sum_{\beta\in\mathcal{K}}\Big(\sqrt{C_h}-\|\chi_{U^c}\mathcal{S}^\alpha_h(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\Big)\leqslant\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{\sqrt{C_h}}\mu_\alpha(U).$$

Proof. As P_UP_h is an Hilbert-Schmidt operator then by (2.34)

$$\begin{aligned}\sum_{\beta\in\mathcal{K}}\langle P_U\mathcal{S}_h^\alpha(u_\beta),\mathcal{S}_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= \sum_{\beta\in\mathcal{K}}\langle P_hP_UP_h\mathcal{S}_h^\alpha(u_\beta),\mathcal{S}_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\\ &\le C_h tr(P_hP_UP_h)\\ &= C_h\|P_UP_h\|^2_{HS}.\end{aligned}$$

Then by (4.9) we get

$$\sum_{\beta\in\mathcal{K}}\langle P_U\mathcal{S}_h^\alpha(u_\beta),\mathcal{S}_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \leqslant \|h\|^2_{L^2(d\nu_\alpha)}\ \mu_\alpha(U). \tag{4.13}$$

Now by the Cauchy-Schwarz inequality we have for every $\beta\in\mathcal{K}$,

$$\begin{aligned}\langle P_U\mathcal{S}_h^\alpha(u_\beta),\mathcal{S}_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= C_h-\langle P_{U^c}\mathcal{S}_h^\alpha(u_\beta),\mathcal{S}_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\\ &\geqslant C_h-\sqrt{C_h}\|\chi_{U^c}\mathcal{S}_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\end{aligned}$$

in particular, using relation (4.13), we obtain

$$\sum_{\beta\in\mathcal{K}}\Big(C_h-\sqrt{C_h}\|\chi_{U^c}\mathcal{S}_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\Big)\leqslant\sum_{\beta\in\mathcal{K}}\langle P_U\mathcal{S}_h^\alpha(u_\beta),\mathcal{S}_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\leqslant\|h\|^2_{L^2(d\nu_\alpha)}\ \mu_\alpha(U),$$

and the conclusion follows. □

As a consequence of the proposition 4.4, we shall demonstrate that, if the generalized continuous Stockwell transform of an othornormal sequence are ε time-frequency concentrated in a given centered ball of $\mathbb{R}^2_+\times\mathbb{R}^2_+$, then such a sequence is necessarily finite.

Proposition 4.5. *Let ε and δ be positive real numbers such that $0<\varepsilon<1$, and h be a generalized Stockwell wavelet. Let $\mathcal{K}\subset\mathbb{N}^2$ be a nonempty subset and $(u_\beta)_{\beta\in\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. If $\mathcal{S}_h^\alpha(u_\beta)$ is ε-time-frequency concentrated in the set*

$$B_\delta:=\Big\{(a,b,r,x)\in\mathbb{R}^2_+\times\mathbb{R}^2_+ : ||(a,b,r,x)||\le\delta\Big\}$$

for every $\beta\in\mathcal{K}$, then $\mathcal{K}$ is finite and

$$\mathrm{Card}(\mathcal{K})\leqslant\frac{\delta^{4\alpha+6}}{1-\varepsilon}M(\alpha,h). \tag{4.14}$$

where $M(\alpha,h)=\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(B_1)$.

Proof. Let $\mathcal{M}\subset\mathcal{K}$ be a nonempty finite subset, then by Proposition 4.4, we deduce that

$$\sum_{\beta\in\mathcal{M}}\Big(\sqrt{C_h}-\|\chi_{U^c}\mathcal{S}_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\Big)\leqslant\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{\sqrt{C_h}}\mu_\alpha(B_\delta), \tag{4.15}$$

however for every $\beta\in\mathcal{M}$, $\|\chi_{B^c_\delta}\mathcal{S}_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\leqslant\varepsilon\sqrt{C_h}$, and

$$\mu_\alpha(B_\delta)=\mu_\alpha(B_1)\delta^{4\alpha+6}, \tag{4.16}$$

hence by combining relations (4.15) and (4.16), we deduce that

$$\mathrm{Card}(\mathcal{M})\leqslant\frac{\mu_\alpha(B_1)\|h\|^2_{L^2(d\nu_\alpha)}}{(1-\varepsilon)C_h}\delta^{4\alpha+6},$$

which means that $\mathcal{K}$ is finite and satisfies relation (4.14). □

Let p be a positive real number, h be a generalized Stockwell wavelet and $f\in L^2(d\nu_\alpha)$, we define the generalized p^{th} time-frequency dispersion of $\mathcal{S}_h^\alpha(f)$ by

$$\rho_p(\mathcal{S}_h^\alpha(f))=\left(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}||(a,b,r,x)||^p\,|\mathcal{S}_h^\alpha(f)(a,b;r,x)|^2\,d\mu_\alpha(a,b,r,x)\right)^{\frac{1}{p}}.$$

Corollary 4.1. *Let A, p be positive real numbers and $h \in L^2(d\nu_\alpha)$ be a generalized Stockwell wavelet. Let $\mathcal{K} \subset \mathbb{N}^2$ be a nonempty subset and $(u_\beta)_{\beta\in\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Assume that for every $\beta \in \mathcal{K}$,*

$$\rho_p(\mathcal{S}_h^\alpha(u_\beta)) \leqslant A,$$

then $\mathcal{K}$ is finite and

$$\mathrm{Card}(\mathcal{K}) \leqslant A^{4\alpha+6} M'(\alpha,p,h),$$

where $M'(\alpha,p,h) = 2^{1+\frac{8\alpha+12}{p}} M(\alpha,h)$.

Proof. Assume that $\rho_p(\mathcal{S}_h^\alpha(u_\beta)) \leqslant A$ for every $\beta \in \mathcal{K}$, then we have

$$\int_{B^c_{A2^{\frac{2}{p}}}} |\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \leqslant \frac{1}{\left(A2^{\frac{2}{p}}\right)^p} \rho_p^p(\mathcal{S}_h^\alpha(f)) \leqslant \frac{1}{4}. \tag{4.17}$$

Relation (4.17) means that for every $\beta \in \mathcal{K}$, u_β is $\dfrac{1}{2}$-concentrated in the set $B_{A2^{\frac{2}{p}}}$, hence according to Proposition 4.5, we deduce that $\mathcal{K}$ is finite and

$$\mathrm{Card}(\mathcal{K}) \leqslant A^{4\alpha+6} M'(\alpha,p,h).$$

□

Lemma 4.1. *Let h be a generalized Stockwell wavelet and p be a positive real number. If $(u_\beta)_{\beta\in\mathbb{N}^2}$ is an orthonormal sequence in $L^2(d\nu_\alpha)$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\forall \beta \in \mathbb{N}^2,\ \rho_p(\mathcal{S}_h^\alpha(u_\beta)) \geqslant 2^{j_0}.$$

Proof. The proof is an immediate consequence of Heisenberg-type inequality (4.5). □

Theorem 4.1 (Shapiro's Dispersion Theorem)**.** *Let h be a generalized Stockwell wavelet and $(u_\beta)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then for every positives reals numbers p and for every nonempty finite subset $\mathcal{K} \subset \mathbb{N}^2$, we have*

$$\sum_{\beta\in\mathcal{K}} (\rho_p(\mathcal{S}_h^\alpha(u_\beta)))^p \geq \frac{1}{2}\left(\frac{3}{M'(\alpha,p,h)2^{8\alpha+13}}\right)^{\frac{p}{4\alpha+6}} (\mathrm{Card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}}. \tag{4.18}$$

Proof. For every $j \in \mathbb{Z}$, let

$$P_j = \left\{\beta \in \mathbb{N}^2\ :\ \rho_p(\mathcal{S}_h^\alpha(u_\beta)) \in [2^{j-1}, 2^j)\right\},$$

then for every $\beta \in P_j$

$$\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} ||(a,b,r,x)||^p \left|\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) \leqslant 2^{pj},$$

thus, using the relation (4.17) yields

$$\int_{B^c_{2^{j+\frac{2}{p}}}} \left|\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) \leqslant \frac{1}{4}\frac{\rho_p(u_\beta)^p}{2^{jp}} \leqslant \frac{1}{4}. \tag{4.19}$$

Therefore, as a consequence of the relation (4.19), we deduce that every $\beta \in P_j$, u_β is $\frac{1}{2}$-concentrated in the ball $B_{2^{j+\frac{2}{p}}}$, In other words, the sequence $(u_\beta)_{\beta\in P_j}$ satisfies the conditions of proposition 4.5, which shows that P_j is finite and

$$\mathrm{Card}(P_j) \leqslant 2^{j(4\alpha+6)} M'(\alpha,p,h). \tag{4.20}$$

For $m\in\mathbb{Z},\ m\geqslant j_0$, we denote by $Q_m=\bigcup_{j=j_0}^{m}P_j$ then according to relation (4.20), we have

$$\mathrm{Card}\,(Q_m)=\sum_{j=j_0}^{m}\mathrm{Card}(P_j)\leqslant\frac{M'(\alpha,p,h)}{3}2^{(m+1)(4\alpha+6)}.$$

Now, if $\mathrm{Card}(\mathcal{K})>\dfrac{2M'(\alpha,p,h)}{3}2^{(j_0+1)(4\alpha+6)}$, then we can choose an integer $n>j_0$ such that

$$\frac{2M'(\alpha,p,h)}{3}2^{n(4\alpha+6)}<\mathrm{Card}(\mathcal{K})\leqslant\frac{2M'(\alpha,p,h)}{3}2^{(n+1)(4\alpha+6)}. \tag{4.21}$$

Thus, by relation (4.21) we get

$$\sum_{\beta\in\mathcal{K}}(\rho_p(\mathcal{S}_h^\alpha(v_\beta)))^p\geqslant\frac{\mathrm{Card}(\mathcal{K})}{2}2^{(n-1)p}\geqslant\frac{1}{2}\left(\mathrm{Card}(\mathcal{K})\right)^{1+\frac{p}{4\alpha+6}}\left(\frac{3}{2^{8\alpha+13}M'(\alpha,p,h)}\right)^{\frac{p}{4\alpha+6}}.$$

Finally, if $\mathrm{Card}(\mathcal{K})\leqslant\dfrac{2M'(\alpha,p,h)}{3}2^{(j_0+1)(4\alpha+6)}$, then

$$\sum_{\beta\in\mathcal{K}}(\rho_p(\mathcal{S}_h^\alpha(v_\beta)))^p\geqslant\mathrm{Card}(\mathcal{K})2^{(j_0-1)p}\geqslant\mathrm{Card}(\mathcal{K})^{1+\frac{p}{4\alpha+6}}\left(\frac{3}{M'(\alpha,p,h)2^{8\alpha+13}}\right)^{\frac{p}{4\alpha+6}}.$$

□

Corollary 4.2. *Let $p>0$, h be a generalized Stockwell wavelet and let $(u_\beta)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then for every $\mathcal{K}\subset\mathbb{N}^2$*

$$\sum_{\beta\in\mathcal{K}}\left(\Big\|\,||(a,b)||^{\frac{p}{2}}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|^2_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}+\Big\|\,||(r,x)||^{\frac{p}{2}}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|^2_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)$$
$$\geq\tfrac{1}{2}\left(\tfrac{3}{M'(\alpha,p,h)2^{12\alpha+19}}\right)^{\frac{p}{4\alpha+6}}\mathrm{Card}(\mathcal{K})^{1+\frac{p}{4\alpha+6}}.$$

Proof. The result is an immediate consequence of the previous theorem and the fact that

$$||(a,b,r,x)||^p\leq 2^p(||(a,b)||^p+||(r,x)||^p).$$

□

As a consequence of the last dispersion inequality, we infer that, there does not exist an infinite sequence $(u_\beta)_{\beta\in\mathcal{K}}$ in $L^2(d\nu_\alpha)$ such that the two sequences

$$\Big\|\,||(a,b)||^{\frac{p}{2}}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}$$

and

$$\Big\|\,||(r,x)||^{\frac{p}{2}}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}$$

are bounded.

Corollary 4.3. *Let $p>0$, h be a generalized Stockwell wavelet and let $(u_\beta)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then for every $\mathcal{K}\subset\mathbb{N}^2$*

$$\sup_{\beta\in\mathcal{K}}\left(\Big\|\,||(a,b)||^{\frac{p}{2}}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|^2_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)},\Big\|\,||(r,x)||^{\frac{p}{2}}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|^2_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)$$
$$\geq\tfrac{1}{4}\left(\tfrac{3}{M'(\alpha,p,h)2^{12\alpha+19}}\right)^{\frac{p}{4\alpha+6}}\mathrm{Card}(\mathcal{K})^{\frac{p}{4\alpha+6}}.$$

In particular

$$\sup_{\beta\in\mathbb{N}^2}\left(\Big\|\,||(a,b)||^{p}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}+\Big\|\,||(r,x)||^{p}\mathcal{S}_h^\alpha(u_\beta)(a,b,r,x)\Big\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)=\infty.$$

Theorem 4.2 (Shapiro's Umbrella Theorem). *Let h be a generalized Stockwell wavelet and $\mathcal{K} \subset \mathbb{N}^2$ be a nonempty subset and $(u_\beta)_{\beta\in\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, if there is a function $g \in L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ such that*

$$|\mathcal{S}^\alpha_h(u_\beta)(a,b,r,x)| \leqslant g(a,b,r,x),$$

for every $\beta \in \mathcal{K}$ and for almost every $(a,b,r,x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, then $\mathcal{K}$ is finite.

Proof. Following the idea of Malinnikova [21], for every positive real number $0 < \varepsilon < C_h$, there is a subset $\Delta_{g,\varepsilon} \subset \mathbb{R}^2_+ \times \mathbb{R}^2_+$, such that

$$\mu_\alpha(\Delta_{g,\varepsilon}) = \inf\left\{\mu_\alpha(U) \ : \ \int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\backslash U} |g(a,b,r,x)|^2 \, d\mu_\alpha(a,b,r,x) \leqslant \varepsilon^2\right\},$$

and

$$\int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\backslash\Delta_{g,\varepsilon}} |g(a,b,r,x)|^2 \, d\mu_\alpha(a,b,r,x) = \varepsilon^2.$$

Hence, according to the hypothesis, for every $\alpha \in \mathcal{K}$, we have

$$\int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\backslash\Delta_{g,\varepsilon}} |\mathcal{S}^\alpha_h\,(u_\beta)\,(a,b,r,x)|^2 \, d\mu_\alpha(a,b,r,x) \leqslant \varepsilon^2,$$

and by the Proposition 4.4, we get $\mathrm{Card}(\mathcal{K})(\sqrt{C_h} - \varepsilon) \leqslant \frac{\|h\|^2_{L^2(d\nu_\alpha)}}{\sqrt{C_h}} \mu_\alpha(\Delta_{g,\varepsilon})$. □

5. Spectral analysis for the generalized concentration operator

The aim of this section is to study the scalograms associated with the Riemann-Liouville Stockwell transform.

5.1. Calderón-Toeplitz operator

Definition 5.1. *Let h be a generalized Stockwell wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$. We define the Riemann-Liouville wavelet scalogram of f as*

$$\mathbf{S}^\alpha_h(f)(a,b,r,x) = C_h^{-1}|\mathcal{S}^\alpha_h f(a,b,r,x)|^2,\ (a,b,r,x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+. \tag{5.1}$$

Remark 5.1. From the Plancherel formula associated with $\mathcal{S}^\alpha_h$, we have

$$\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \mathbf{S}^\alpha_h(f)(a,b,r,x) d\mu_\alpha(a,b,r,x) = \|f\|^2_{L^2(d\nu_\alpha)}. \tag{5.2}$$

It justifies the interpretation of a scalogram as a time-frequency energy density. Also, note that (3.2)

$$\langle \mathcal{L}_{h,h}(\sigma)f,f\rangle_{L^2(d\nu_\alpha)} = \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \sigma(a,b,r,x)\mathbf{S}^\alpha_h(f)(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{5.3}$$

In this section we shall keep our focus on localization operators $\mathcal{L}_{h,h}(\sigma)$ with symbol $\sigma = \chi_U$, and h is a generalized Stockwell wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$, and U is subset of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ with finite measure $\mu_\alpha(U) < \infty$. For the sake of simplicity, such an operator will be denoted as $\mathcal{L}_h(U)$.

Definition 5.2. *We define the Calderón-Toeplitz operator*

$$T_{h,U} \ : \mathcal{S}^\alpha_h(L^2(d\nu_\alpha)) \to \mathcal{S}^\alpha_h(L^2(d\nu_\alpha))$$

by

$$T_{h,U}\, F = P_h P_U F. \tag{5.4}$$

Proposition 5.1. *The operator $T_{h,U} : \mathcal{S}_h^\alpha(L^2(d\nu_\alpha)) \to \mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$ is trace-class operator and satisfies*

$$0 \le T_{h,U} \le P_U \le I, \tag{5.5}$$

and

$$T_{h,U} = \mathcal{S}_h^\alpha \mathcal{L}_h(U)(\mathcal{S}_h^\alpha)^*. \tag{5.6}$$

Proof. For all $F \in \mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$, we have

$$\begin{aligned}\langle T_{h,U} F, F\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= \langle P_h(P_U F), F\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \\ &= \langle P_U F, F\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \\ &= \int_U |F(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x).\end{aligned} \tag{5.7}$$

Thus we deduce (5.5), and $T_{h,U}$ is bounded and positive.
Now, we want to prove (5.6). Indeed, using $\mathcal{S}_h^\alpha$ and $(\mathcal{S}_h^\alpha)^*$, the time-frequency localization operator

$$\mathcal{L}_h(U) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$$

can be expressed as

$$\mathcal{L}_h(U)(f) = (\mathcal{S}_h^\alpha)^*(P_U \mathcal{S}_h^\alpha f), \quad f \in L^2(d\nu_\alpha).$$

Therefore,

$$(\mathcal{S}_h^\alpha \mathcal{L}_h(U)(\mathcal{S}_h^\alpha)^*)F = P_h P_U F = T_{h,U} F, \quad F \in \mathcal{S}_h^\alpha(L^2(d\nu_\alpha)). \tag{5.8}$$

Therefore, the time-frequency operator $\mathcal{L}_h(U)$ and the Calderón-Toeplitz operator $T_{h,U}$ are related by

$$T_{h,U} = \mathcal{S}_h^\alpha \mathcal{L}_h(U)(\mathcal{S}_h^\alpha)^*.$$

□

Remark 5.2. From the above proposition, we deduce that $T_{h,U}$ and $\mathcal{L}_h(U)$ enjoy the same spectral properties, in particular, we have the following proposition.

Proposition 5.2. *The Calderón-Toeplitz operator $T_{h,U}$ is compact and even trace class with*

$$tr(T_{h,U}) = tr(\mathcal{L}_h(U)) = M_\alpha(h,U), \tag{5.9}$$

where

$$M_\alpha(h,U) := \frac{1}{C_h}\int_U \|h_{a,b,r,x}\|^2_{L^2(d\nu_\alpha)} d\mu_\alpha(a,b,r,x). \tag{5.10}$$

Proof. Note that the operator $T_{h,U} : \mathcal{S}_h^\alpha(L^2(d\nu_\alpha)) \to \mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$ is bounded and positive. Now, let $\{e_n\}_{n=1}^\infty$ be an arbitrary orthonormal basis for $\mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$. Then, if we denote by $v_n = \sqrt{C_h}(\mathcal{S}_h^\alpha)^*(e_n)$, then $\{v_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2(d\nu_\alpha)$.
Thus, by (3.2) and Fubini's theorem, we get

$$\begin{aligned}\sum_{n=1}^\infty \langle T_{h,U}(e_n), e_n\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= C_h \sum_{n=1}^\infty \langle \mathcal{L}_h(U)(\mathcal{S}_h^\alpha)^*(e_n), (\mathcal{S}_h^\alpha)^*(e_n)\rangle_{L^2(d\nu_\alpha)} \\ &= \tfrac{1}{C_h}\sum_{n=1}^\infty \int_U |\mathcal{S}_h^\alpha(v_n)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \\ &= \tfrac{1}{C_h}\int_U \sum_{n=1}^\infty |\mathcal{S}_h^\alpha(v_n)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \\ &= \tfrac{1}{C_h}\int_U \sum_{n=1}^\infty |\langle v_n, h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}|^2 d\mu_\alpha(a,b,r,x) \\ &= \tfrac{1}{C_h}\int_U \|h_{a,b,r,x}\|^2_{L^2(d\nu_\alpha)} d\mu_\alpha(a,b,r,x) \\ &= M_\alpha(h,U).\end{aligned}$$

Therefore, by Definition 2.10 and Remark 2.7, the operator $T_{h,U}$ is trace class with

$$\|T_{h,U}\|_{S_1} = tr(T_{h,U}) = M_\alpha(h,U).$$

□

Let $\mathbf{V}_{h,U} : L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \to L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ the operator defined by $\mathbf{V}_{h,U} = P_h P_U P_h$. The advantage of $\mathbf{V}_{h,U}$ compared to $T_{h,U}$ is that it is defined on $L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and consequently its spectral properties can be easily related to its integral kernel. Since $T_{h,U}$ is positive and trace-class, then using the decomposition

$$L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) = \mathcal{S}^\alpha_h(L^2(d\nu_\alpha)) \oplus \left(\mathcal{S}^\alpha_h(L^2(d\nu_\alpha))\right)^\perp,$$

we deduce that $\mathbf{V}_{h,U}$ is also positive and trace-class with

$$tr(\mathbf{V}_{h,U}) = tr(T_{h,U}) = M_\alpha(h,U). \tag{5.11}$$

In addition, we have the following result.

Proposition 5.3. *The trace of $T^2_{h,U}$ is given by*

$$tr(T^2_{h,U}) = \int_U \int_U |\mathcal{K}_h(a,b,r,x;a',b',r',x')|^2 d\mu_\alpha(a,b,r,x) d\mu_\alpha(a',b',r',x'). \tag{5.12}$$

Proof. Since, $\mathbf{V}_{h,U}$ is positive, then

$$tr(T^2_{h,U}) = tr(\mathbf{V}^2_{h,U}). \tag{5.13}$$

On the other hand using the fact that the space $\mathcal{S}^\alpha_h(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel $\mathcal{K}_h$, we get that for $F \in \mathcal{S}^\alpha_h(L^2(d\nu_\alpha))$

$$\begin{aligned}\mathbf{V}_{h,U}F(a,b,r,x) &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} F(a',b',r',x') \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(c,d,t,y)\mathcal{K}_h(a,b,r,x;c,d,t,y) \\ &\quad \mathcal{K}_h(c,d,t,y;a',b',r',x') d\mu_\alpha(c,d,t,y) d\mu_\alpha(a',b',r',x'). \end{aligned} \tag{5.14}$$

That is, $\mathbf{V}_{h,U}$ has integral kernel

$$\begin{aligned}&\mathbf{N}_{h,U}(a,b,r,x;a',b',r',x') = \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(c,d,t,y)\mathcal{K}_h(a,b,r,x;c,d,t,y) \\ &\mathcal{K}_h(c,d,t,y;a',b',r',x') d\mu_\alpha(c,d,t,y).\end{aligned} \tag{5.15}$$

Therefore,

$$\begin{aligned} tr(\mathbf{V}^2_{h,U}) &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} |\mathbf{N}_{h,U}(a,b,r,x;a',b',r',x')|^2 d\mu_\alpha(a,b,r,x) d\mu_\alpha(a',b',r',x') \\ &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(z_1)\chi_U(z_2)\mathbf{K}_h(z_1;z_2) d\mu_\alpha(z_1) d\mu_\alpha(z_2) \end{aligned}$$

where by using the properties of the kernel of the reproducing kernel Hilbert space

$$\begin{aligned} \mathbf{K}_h(z_1;z_2) &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \mathcal{K}_h(z_2;a,b,r,x)\mathcal{K}_h(a,b,r,x;z_1)\mathcal{K}_h(z_1;a',b',r',x') \\ &\quad \mathcal{K}_h(a',b',r',x';z_2) d\mu_\alpha(a,b,r,x) d\mu_\alpha(a',b',r',x') \\ &= \mathcal{K}_h(z_2;z_1)\mathcal{K}_h(z_1;z_2). \end{aligned}$$

Using (4.10), we get

$$\mathbf{K}_h(z_1;z_2) = |\mathcal{K}_h(z_1;z_2)|^2. \tag{5.16}$$

This follows us to conclude. □

5.2. Eigenvalues and eigenfunctions

Since the localization operator $\mathcal{L}_h(U) = (\mathcal{S}_h^\alpha)^* \chi_U \mathcal{S}_h^\alpha$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation

$$\mathcal{L}_h(U)(f) = \sum_{n=1}^{\infty} s_n(U) \left\langle f, v_n^U \right\rangle_{L^2(d\nu_\alpha)} v_n^U, \quad f \in L^2(d\nu_\alpha), \tag{5.17}$$

where $\{s_n(U)\}_{n=1}^\infty$ are the positive eigenvalues arranged in a nonincreasing manner and $\{v_n^U\}_{n=1}^\infty$ is the corresponding orthonormal set of eigenfunctions. Note that $s_n(U) \searrow 0$ and we have for all $n \geq 1$,

$$s_n(U) \leq s_1(U) \leq 1. \tag{5.18}$$

This, together with (5.6), we can deduce that the Calderón-Toeplitz operator

$$T_{h,U} : \mathcal{S}_h^\alpha(L^2(d\nu_\alpha)) \to \mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$$

can be diagonalized as

$$T_{h,U}\, F = \sum_{n=1}^{\infty} s_n(U) \left\langle F, e_n^U \right\rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} e_n^U, \quad F \in \mathcal{S}_h^\alpha(L^2(d\nu_\alpha)), \tag{5.19}$$

where $e_n^U = \frac{1}{\sqrt{C_h}} \mathcal{S}_h^\alpha(v_n^U)$.

Lemma 5.1. *For all $z = (a,b,r,x) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$, we have*

$$\Theta(z) := \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \chi_U(\omega) |\mathcal{K}_h(\omega; z)|^2 d\mu_\alpha(\omega) = \sum_{n=1}^{\infty} s_n(U) \mathbf{S}_h^\alpha(v_n^U)(z). \tag{5.20}$$

Proof. From (4.6), we have for all $z = (a,b,r,x) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2$, the function $\mathcal{K}_h(.; z)$ is in $\mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$. Therefore using the properties of the kernel of the reproducing kernel Hilbert space, we get

$$\begin{aligned}
\langle T_{h,U}\, \mathcal{K}_h(.; z), \mathcal{K}_h(.; z) \rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} &= \langle P_U \mathcal{K}_h(.; z), \mathcal{K}_h(.; z) \rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} \\
&= \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \chi_U(\omega) \mathcal{K}_h(\omega; z) \overline{\mathcal{K}_h(\omega; z)} d\mu_\alpha(\omega) \\
&= \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \chi_U(\omega) |\mathcal{K}_h(\omega; z)|^2 d\mu_\alpha(\omega).
\end{aligned}$$

Let $\{w_n^U\}_{n=1}^\infty \subset \mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$ be an orthonormal basis of $Ker(T_{h,U})$ (eventually empty). Hence, $\{e_n^U\}_{n=1}^\infty \cup \{w_n^U\}_{n=1}^\infty$ is an orthonormal basis of $\mathcal{S}_h^\alpha(L^2(d\nu_\alpha))$ and therefore the reproducing kernel $\mathcal{K}_h$ can be written as

$$\begin{aligned}
\mathcal{K}_h(a,b,r,x; a',b',r',x') &= \overline{\mathcal{K}_h(a',b',r',x'; z)} \\
&= \sum_{n=1}^{\infty} e_n^U(z) \overline{e_n^U(a',b',r',x')} + \sum_{n=1}^{\infty} w_n^U(z) \overline{w_n^U(a',b',r',x')}.
\end{aligned} \tag{5.21}$$

Using this, we compute again

$$\begin{aligned}
\langle T_{h,U}\, \mathcal{K}_h(.; z), \mathcal{K}_h(.; z) \rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} &= \left\langle T_{h,U} \sum_{n=1}^{\infty} \overline{e_n^U(z)} e_n^U, \sum_{k=1}^{\infty} \overline{\phi_k^U(z)} \phi_k^U \right\rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} \\
&= \sum_{n,k} \overline{e_n^U(z)} \phi_\alpha^U(z) \left\langle T_{h,U}\, e_n^U, \phi_k^U \right\rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} \\
&= \sum_{n=1}^{\infty} s_n(U) |e_n^U(z)|^2,
\end{aligned}$$

and the conclusion follows. □

Let $\varepsilon \in (0,1)$ and define the quantity

$$n(\varepsilon,U) := card\Big\{j : s_j(U) \geq 1-\varepsilon\Big\}.$$

Then an easy adaptation of the proof of Lemma 3.3 in [1], we obtain the following estimate for the eigenvalue distribution.

Proposition 5.4. *Let $\varepsilon \in (0,1)$. We have*

$$\begin{aligned} |n(\varepsilon,U) - M_\alpha(h,U)| \quad &\leq \quad max\{\tfrac{1}{\varepsilon}, \tfrac{1}{1-\varepsilon}\}\Big|\tfrac{1}{C_h}\int_U\int_U |\mathcal{K}_h(a',b',r',x';a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \\ &\qquad d\mu_\alpha(a',b',r',x') - M_\alpha(h,U)\Big|. \end{aligned}$$

5.3. Scalogram of a subspace

Given an N-dimentional subspace V of $L^2(d\nu_\alpha)$, P_V the orthogonal projection onto V with projection kernel k_V, is defined as

$$P_V f(.) = \int_{\mathbb{R}^2_+} k_V(.;t,s) f(t,s) d\nu_\alpha(t,s). \tag{5.22}$$

Recall that if $\{v_n\}_{n=1}^N$ is an orthonormal basis of V, then

$$k_V(r,x;,t,s) = \sum_{n=1}^{N} v_n(r,x)\overline{v_n(t,s)}. \tag{5.23}$$

The kernel k_V is independent of the choice of orthonormal basis for V.

Definition 5.3. *The scalogram of the space V with generalized Stockwell wavelet h is defined*

$$\mathbf{SCAL}_h^\alpha V(a,b,r,x) := \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} k_V(t,s;b,y)\overline{h_{a,b,r,x}(t,s)} h_{a,b,r,x}(b,y) d\nu_\alpha(t,s) d\nu_\alpha(b,y). \tag{5.24}$$

Then, we have the following result.

Lemma 5.2. *The scalogram $\mathbf{SCAL}_h^\alpha V$ is given by*

$$\mathbf{SCAL}_h^\alpha V = C_h \sum_{n=1}^{N} \mathbf{S}_h^\alpha(v_n). \tag{5.25}$$

Proof. We have

$$\begin{aligned} \mathbf{SCAL}_h^\alpha V(a,b,r,x) \quad &= \quad \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} \sum_{n=1}^{N} v_n(t,s)\overline{v_n(b,y) h_{a,b,r,x}(t,s)} h_{a,b,r,x}(b,y) d\nu_\alpha(t,s) d\nu_\alpha(b,y) \\ &= \quad \sum_{n=1}^{N} \langle v_n, h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)} \overline{\langle v_n, h_{a,b,r,x}\rangle}_{L^2(d\nu_\alpha)} \\ &= \quad \sum_{n=1}^{N} \mathcal{S}_h^\alpha(v_n)(a,b,r,x)\overline{\mathcal{S}_h^\alpha(v_n)(a,b,r,x)} \\ &= \quad \sum_{n=1}^{N} |\mathcal{S}_h^\alpha(v_n)(a,b,r,x)|^2. \end{aligned}$$

This completes the proof. □

Definition 5.4. *We define the time-frequency concentration of a subspace V in U as:*

$$\xi_{U,h}(V) := \frac{1}{N}\int_U \mathbf{SCAL}_h^\alpha V(a,b,r,x) d\mu_\alpha(a,b,r,x). \tag{5.26}$$

Then, using Lemma 5.2, we get the desired result:

$$\xi_{U,h}(V) := \frac{C_h}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^\alpha(v_n)(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{5.27}$$

Theorem 5.1. *The N-dimentional signal space $V_N = span\{v_n^U\}_{n=1}^N$ consisting of the first N eigenfunctions of $\mathcal{L}_h(U)$ corresponding to the N largest eigenvalues $\{s_n(U)\}_{n=1}^N$ maximize the regional concentration $\xi_{U,h}(V)$ and*

$$\xi_{U,h}(V_N) := \frac{C_h}{N}\sum_{n=1}^{N} s_n(U). \tag{5.28}$$

Proof. We have

$$\xi_{U,h}(V_N) := \frac{C_h}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^\alpha(v_n^U)(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{5.29}$$

Moreover, the min-max lemma for self-adjoint operators states that (see e. g. Sec.95 in [30])

$$\begin{aligned} s_n(U) &= \int_U \mathbf{S}_h^\alpha(v_n^U)(a,b,r,x)d\mu_\alpha(a,b,r,x) \\ &= \max\Big\{\langle \mathcal{L}_h(U)(f),f\rangle_{L^2(d\nu_\alpha)} : \|f\|_{L^2(d\nu_\alpha)}=1, f\perp v_1^U,...,v_{n-1}^U\Big\}. \end{aligned}$$

So, the eigenvalues of $\mathcal{L}_h(U)$ determine the number of orthogonal functions that have a well-concentrated scalogram in U. Thus,

$$\xi_{U,h}(V_N) = \frac{C_h}{N}\sum_{n=1}^{N} s_n(U). \tag{5.30}$$

The min-max characterization of the eigenvalues of compact operators implies that the first N eigenfunctions of the time-frequency operator $\mathcal{L}_h(U)$ have optimal cumulative time-frequency concentration inside U, in the sense,

$$\sum_{n=1}^{N}\Big\langle \mathcal{L}_h(U)(v_n^U),v_n^U\Big\rangle_{L^2(d\nu_\alpha)} = \max\Big\{\sum_{n=1}^{N}\langle \mathcal{L}_h(U)v_n,v_n\rangle_{L^2(d\nu_\alpha)} : \{v_n\}_{n=1}^N \text{ orthonormal}\Big\}. \tag{5.31}$$

Therefore any N-dimensional subset V of $L^2(d\nu_\alpha)$ cannot to be better concentrated in U than V_N, i.e

$$\xi_{U,h}(V) \le \xi_{U,h}(V_N). \tag{5.32}$$

The proof is complete. □

Remark 5.3. The time-frequency concentration of a subspace V_N in U satisfies,

$$s_N(U) \le \frac{1}{C_h}\xi_{U,h}(V_N) \le s_1(U) \le 1. \tag{5.33}$$

5.4. Accumulated scalogram

Let $\rho_{(h,U)} := \mathbf{SCAL}_h^\alpha V_{N_\alpha(h,U)}$, the $\rho_{(h,U)}$ is called the accumulated scalogram, provided that $N_\alpha(h,U) = [M_\alpha(h,U)]$ is the smallest integer greater than or equal to $M_\alpha(h,U)$ and

$$V_{N_\alpha(h,U)} = span\{v_n^U\}_{n=1}^{N_\alpha(h,U)}.$$

Observe that,

$$\rho_{(h,U)}(a,b,r,x) = \sum_{n=1}^{N_\alpha(h,U)} |\mathcal{S}_h^\alpha(v_n^U)(a,b,r,x)|^2. \tag{5.34}$$

Also,

$$\|\rho_{(h,U)}\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} = C_h N_\alpha(h,U) = C_h M_\alpha(h,U) + O(1).$$

Moreover, since

$$\sum_{n=1}^{N_\alpha(h,U)} s_n(U) \leq tr(\mathcal{L}_h(U)) = M_\alpha(h,U)$$

then we can define the quantity

$$E(h,U) := 1 - \frac{\sum_{n=1}^{N_\alpha(h,U)} s_n(U)}{M_\alpha(h,U)}. \tag{5.35}$$

which satisfies,

$$0 \leq E(h,U) \leq 1. \tag{5.36}$$

More precisely, we have the following result.

Lemma 5.3. *Let $\varepsilon \in (0,1)$. We have*

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\min(1, \frac{n(\varepsilon,U)}{M_\alpha(h,U)}). \tag{5.37}$$

Proof. Let $\varepsilon \in (0,1)$ and define $l_\alpha(\varepsilon,U) = \min(N_\alpha(h,U), n(\varepsilon,U))$. It follows that

$$s_n(U) \geq 1-\varepsilon, \quad 1 \leq n \leq l_\alpha(\varepsilon,U). \tag{5.38}$$

As $N_\alpha(h,U) \geq l_\alpha(h,U)$, we get

$$\sum_{n=1}^{N_\alpha(h,U)} s_n(U) \geq \sum_{n=1}^{l_\alpha(\varepsilon,U)} s_n(U) \geq (1-\varepsilon) l_\alpha(\varepsilon,U). \tag{5.39}$$

Therefore

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\frac{l_\alpha(\varepsilon,U)}{M_\alpha(h,U)}. \tag{5.40}$$

As $N_\alpha(\varepsilon,U) \geq M_\alpha(h,U)$, we obtain the desired result. $\square$

Consequently when the eigenvalues $\{s_n(U)\}_{n=0}^{n(\varepsilon,U)}$ are close to 1, then $E(h,U) \to 0$. Moreover, we have the following result bounding the error between $\rho_{(h,U)}$ and $C_h\Theta$.

Proposition 5.5. *We have*

$$\frac{1}{M_\alpha(h,U)}\|\rho_{(h,U)} - C_h\Theta\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \leq \frac{C_h}{M_\alpha(h,U)} + 2C_h E(h,U). \tag{5.41}$$

Proof. From Lemma 5.1, we have, for all $z = (a,b,r,x) \in U$

$$\rho_{(h,U)}(z) - C_h\Theta(z) = \sum_{n=1}^{\infty}(t_n - s_n(U))|\mathcal{S}_h^\alpha(v_n^U)(z)|^2, \tag{5.42}$$

where $t_n = 1$ if $n \leq N_\alpha(h,U)$ and 0 otherwise. Now since

$$\||\mathcal{S}_h^\alpha(v_n^U)|^2\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} = C_h$$

and

$$\sum_{n=1}^{\infty} s_n(U) = M_\alpha(h,U),$$

we obtain

$$\begin{aligned}
\|\rho_{(h,U)} - C_h\Theta\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &\leq C_h \sum_{n=1}^{\infty} |t_n - s_n(U)| \\
&= C_h \sum_{n=1}^{N_\alpha(h,U)} (1 - s_n(U)) + C_h \sum_{n>N_\alpha(h,U)} s_n(U) \\
&= C_h N_\alpha(h,U) + C_h \sum_{n=1}^{\infty} s_n(U) - 2C_h \sum_{n=1}^{N_\alpha(h,U)} s_n(U) \\
&= C_h N_\alpha(h,U) + C_h M_\alpha(h,U) - 2C_h \sum_{n=1}^{N_\alpha(h,U)} s_n(U) \\
&= C_h\Big(N_\alpha(h,U) - M_\alpha(h,U)\Big) + 2C_h\Big(M_\alpha(h,U) - \sum_{n=1}^{N_\alpha(h,U)} s_n(U)\Big) \\
&\leq C_h + 2C_h\Big(M_\alpha(h,U) - \sum_{n=1}^{N_\alpha(h,U)} s_n(U)\Big),
\end{aligned}$$

and the estimate (5.41) follows. □

Acknowledgment. The authors are deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article.

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