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# On the Continuity Properties of the Set of Trajectories of the Control System with Limited Control Resources

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## Abstract

In this paper the control system with integral constraint on the control functions is studied where the behavior of the system by the Urysohn type integral equation is described. The admissible control functions are chosen from the closed ball of the space $L_p([a,b];R^m)$ $(p>1)$ centered at the origin with radius $r$. Dependence of the set of trajectories on $r$ and $p$ is investigated. It is proved that the set of trajectories is Lipschitz continuous with respect to $r$ and is continuous with respect to $p$. The robustness of the trajectory with respect to the fast consumption of the remaining control resource is established.

**Keywords:** control system, Hausdorff distance, integral constraint, robustness, Urysohn integral equation.

## 1. Introduction

The control systems arise in different areas of physics, mechanics, airspace navigation, economics, sociology, etc. and depending on character of control efforts are classified as control systems with geometric constraints, integral constraints and mixed constraints on the control functions. The theory of control systems with geometric constraints on the control functions is enough well investigated chapter of the control systems theory (see, e.g. [4], [13], [16], [20] and references therein). But integral constraints on the control functions arise in the cases when the control resource is exhausted by consumption such as energy, fuel, finance, etc. (see, e.g. [3], [6], [12], [15], [18], [21], [22], [23]). Note that integral boundedness of the control function does not imply its geometric boundedness. This situation causes additional difficulties and therefore studying the control systems with integral constraints on the control functions requires special methods.

Integral equations are very adequate tool to describe the behaviors of various processes arising in the theory and applications (see, e.g. [2], [7], [17], [19], [24]). In this paper the control system described by Uryshon type integral equation is considered. The control functions are chosen from the closed ball of the space $L_p([a,b];R^m)$ $(p>1)$ centered at the origin with radius $r$. Note that the different topological properties and approximate constructions methods of the set of trajectories of the control systems described by various type integral equations and integral constraints on the control functions are studied in papers [8-11].

The paper is organized as follows. In Section 2 the basic conditions and propositions are formulated which are used in following arguments. In Section 3 it is proved that the set of trajectories is Lipschitz continuous with respect to $r$ (Theorem 3.1). In Section 4 it is shown that the set of trajectories depends on $p$ continuously (Theorem 4.1). In Section 5 it is proved that system's trajectory is robust with respect to the fast consumption of the remaining control resource (Theorem 5.1) and it is shown that every trajectory can be approximated by trajectory obtained by the full consumption of the available control resource (Theorem 5.2).

## 2. The System's Description

Consider control system the behavior of which is described by Urysohn type integral equation

$$x(t)=f(t,x(t))\;+\;\lambda\int_a^b K\big(t,s,x(s),u(s)\big)ds \qquad \mathbf{(2.1)}$$

where $t\in[a,b]$, $x(t)\in R^n$ is the state vector, $u(s)\in R^m$ is the control vector, $\lambda\geq 0$.
For given $p>1$ and $r\geq 0$ we denote

$$U_{p,r} = \{u(\cdot) \in L_p([a,b];R^m): \|u(\cdot)\|_p \le r\}$$

which is called the set of admissible control functions and every $u(\cdot) \in U_{p,r}$ is said to be an admissible control function, where $L_p([a,b];R^m)$ is the space of Lebesgue measurable functions $u(\cdot):[a,b] \to R^m$ such that $\|u(\cdot)\|_p < \infty$, $\|u(\cdot)\|_p = \left(\int_a^b \|u(s)\|^p \, ds\right)^{\frac{1}{p}}$, $\|\cdot\|$ denotes the Euclidean norm.

It is obvious that the set of admissible control functions $U_{p,r}$ is the closed ball with radius $r$ and centered at the origin in the space $L_p([a,b];R^m)$.

It is assumed that the functions and a number λ given in system (2.1) satisfy the following conditions:

**2.A.** the functions $f(\cdot\,,\,\cdot):[a,b]\times R^n \to R^n$ and $K(\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot):[a,b]\times[a,b]\times R^n \times R^m \to R^n$ are continuous;

**2.B.** there exist $l_0 \in [0,1)$, $l_1 \ge 0, \gamma_1 \ge 0,\ l_2 \ge 0,$ $\gamma_2 \ge 0,\ l_3 \ge 0, \gamma_3 \ge 0$ such that

$$\|f(t,x_1) - f(t,x_2)\| \le l_0\|x_1 - x_2\|$$

for every $(t,x_1) \in [a,b]\times R^n$, $(t,x_2) \in [a,b]\times R^n$ and

$$\begin{aligned}\|K(t_1,s,x_1,u_1) - K(t_2,s,x_2,u_2)\| \\ \le [l_1 + \gamma_1(\| u_1 \| + \| u_2 \|)]\|t_1 - t_2\| \\ + [l_2 + \gamma_2(\| u_1 \| + \| u_2 \|)]\|x_1 - x_2\| \\ + [l_3 + \gamma_3(\| x_1 \| + \| x_2 \|)]\|u_1 - u_2\|\end{aligned}$$

for every $(t_1,s,x_1,u_1) \in [a,b]\times[a,b]\times R^n \times R^m$, $(t_2,s,x_2,u_2) \in [a,b]\times[a,b]\times R^n \times R^m$;

**2.C.** there exist $p_* > 1$ and $r_* > 0$ such that the inequality

$$\lambda\left(l_2(b-a) + 2\gamma_* r_*(b-a)^{\frac{p_*-1}{p_*}}\right) < 1 - l_0$$

is satisfied where $\gamma_* = \max\{\gamma_1,\ \gamma_2,\ \gamma_3\}$.

If the function $(t,s,x,u) \to K(t,s,x,u)$, $(t,s,x,u) \in [a,b]\times[a,b]\times R^n \times R^m$, is Lipschitz continuous with respect to $(t,x,u)$, then it satisfies the condition 2.B.

We set

$$L(\lambda;p,r) = l_0 + \lambda\left(l_2(b-a) + 2\gamma_* r(b-a)^{\frac{p-1}{p}}\right) \quad \textbf{(2.2)}$$

From Condition 2.C it follows that

$$0 \le L(\lambda;p_*,r_*) < 1\,. \quad \textbf{(2.3)}$$

Then there exist $\beta_1 > 0,\ \beta_2 > 0$ such that $L(\lambda;p,r) < 1$ for every $p \in [p_* - \beta_1, p_* + \beta_1]$ and $r \in [0, r_* + \beta_2]$.

Denote

$$L_*(\lambda) = \max\{L(\lambda;p,r) : p \in [p_* - \beta_1, p_* + \beta_1],\ r \in [0, r_* + \beta_2]\}. \quad \textbf{(2.4)}$$

From (2.2), (2.3) and (2.4) it follows that

$$0 \le L_*(\lambda) < 1\,,\quad L_*(\lambda) - l_0 \ge 0 \quad \textbf{(2.5)}$$

From now on, it will be assumed that $p \in [p_* - \beta_1, p_* + \beta_1]$ and $r \in [0, r_* + \beta_2]$.

Now, let us define a trajectory of the system (2.1) generated by given admissible control function $u(\cdot) \in U_{p,r}$. A continuous function $x(\cdot):[a,b] \to R^n$ satisfying the integral equation (2.1) for every $t \in [a,b]$, is said to be a trajectory of the system (2.1) generated by the admissible control function $u(\cdot) \in U_{p,r}$. The set of trajectories of the system (2.1) generated by all admissible control functions $u(\cdot) \in U_{p,r}$ is denoted by symbol $X_{p,r}$ and is called the set of trajectories of the system (2.1).

The conditions 2.A-2.C guarantee that every admissible control function $u(\cdot) \in U_{p,r}$ generates a unique trajectory $x(\cdot) \in C([a,b];R^n)$ of the system (2.1) (see, Theorem 3.1 of [8]), where $C([a,b];R^n)$ is the space of continuous functions $x(\cdot):[a,b] \to R^n$ with norm $\| x(\cdot) \|_C = \max\{\| x(t) \| : t \in [a,b]\}$. Analogously to the Theorem 4.1 of [8] it is possible to show that there exists $\beta_* > 0$ such that

$$\| x(\cdot) \|_C \le \beta_* \quad \textbf{(2.6)}$$

for every $x(\cdot) \in X_{p,r}$, $p \in [p_* - \beta_1, p_* + \beta_1]$ and $r \in [0, r_* + \beta_2]$. Moreover, by virtue of Theorem 5.1 of [8] we have that the set of trajectories $X_{p,r}$ is a precompact subset of the space $C([a,b];R^n)$.

Let us give an auxiliary proposition which will be used in following arguments.

**Proposition 2.1.** Let $u_1(\cdot) \in U_{p,r_1}$, $u_2(\cdot) \in U_{p,r_2}$ where $p \in [p_* - \beta_1, p_* + \beta_1]$, and $r_1 \in [0, r_* + \beta_2]$, $r_2 \in [0, r_* + \beta_2]$. Then

$$\lambda \int_a^b (l_2 + \gamma_2[\| u_1(s) \| + \| u_2(s) \|])ds \le L_*(\lambda) - l_0$$

where $L_*(\lambda)$ is defined by (2.4).

The proof of the proposition follows from Hölder's inequality.

For given metric space $(Z, d_Z(\cdot\,,\,\cdot))$ the Hausdorff distance between the sets $F \subset Z$ and $E \subset Z$ is denoted by $h_Z(F,E)$ and defined as

$$h_Z(F,E) = \max\left\{ sup_{x\in F}\, d_Z(x,E)\,, sup_{y\in E}\, d_Z(y,F)\right\}$$

where $d_Z(x,E) = \inf\,\{d_Z(x,y): y \in E\}$.

Now let $(Z, d_Z(\cdot\,,\,\cdot))$ be a metric space and $b(Z)$ be a family of all nonempty bounded subsets of $Z$. Then $(b(Z), h_Z(\cdot\,,\,\cdot))$ is a pseudometric space where $h_Z(\cdot\,,\,\cdot)$ stands for Hausdorff distance between subsets of the space $(Z, d_Z(\cdot\,,\,\cdot))$ (see, e.g. [1], [14]).

Let $(Y, d_Y(\cdot\,,\,\cdot))$ and $(Z, d_Z(\cdot\,,\,\cdot))$ be metric spaces, $\Phi(\cdot) : Y \to b(Z)$ be a given set valued map, and $y_* \in Y$. If $h_Z(\Phi(y), \Phi(y_*)) \to 0$ as $y \to y_*$, then the map $\Phi(\cdot)$ is called continuous at $y_*$ .

If there exists $M_0 > 0$ such that

$$h_Z(\Phi(y_1), \Phi(y_2)) \leq M_0 \cdot d_Y(y_1\,, y_2)$$

for every $y_1 \in Y$ and $y_2 \in Y$, then the map $\Phi(\cdot)$ is called Lipschitz continuous with Lipschitz constant $M_0$.

The Hausdorff distance between the sets $U \subset R^n$ and $V \subset R^n$ is denoted by $h_n(U,V)$ and the Hausdorff distance between the sets $G \subset C([a,b]; R^n)$ and $W \subset C([a,b]; R^n)$ is denoted by $h_C(G,W)$.

For $t \in [a,b]$ we set

$$X_{p,r}(t) = \{x(t) \in R^n : x(\cdot) \in X_{p,r}\}. \quad \textbf{(2.7)}$$

The set $X_{p,r}(t)$ is close to the attainable set notion used in control and dynamical systems theory and consists of points to which arrive the trajectories of the system at the instant of $t$ (see, e.g. [4], [5], [6]).

It follows from Proposition 5.2 of [8] that $h_n(X_{p,r}(t), X_{p,r}(t_*)) \to 0$ as $t \to t_*$ for every fixed $t_* \in [a,b]$.

## 3. Lipschitz Continuity of the Set of Trajectories with Respect to r

In this section for each fixed $p \in [p_* - \beta_1, p_* + \beta_1]$ the Lipschitz continuity of the set valued map $r \to X_{p,r}$ , $r \in [0, r_* + \beta_2]$, is proved. Denote

$$B_C(1) = \{x(\cdot) \in C([a,b]; R^n) : \| x(\cdot) \|_C \leq 1\} \quad \textbf{(3.1)}$$

$$l_* = \max\,\{(b-a)^{\frac{p-1}{p}} : p \in [p_* - \beta_1, p_* + \beta_1]\} \quad \textbf{(3.2)}$$

$$R_* = \frac{\lambda(l_3 + 2\beta_*\gamma_3)l_*}{1 - L_*(\lambda)} \quad \textbf{(3.3)}$$

where $L_*(\lambda)$ is defined by (2.4), $\beta_*$ is given in (2.6).

**Theorem 3.1.** Let $p \in [p_* - \beta_1, p_* + \beta_1]$ be fixed. Then

$$h_C(X_{p,r_1}\,, X_{p,r_2}) \leq R_* |r_1 - r_2|$$

for every $r_1 \in [0, r_* + \beta_2]$ , $r_2 \in [0, r_* + \beta_2]$ where $R_*$ is defined by (3.3).

*Proof.* Let $r_1 < r_2$ and $x_*(\cdot) \in X_{p,r_2}$ be an arbitrarily chosen trajectory generated by the control function $u_*(\cdot) \in U_{p,r_2}$. Define a control function $\tilde{u}(\cdot): [a,b] \to R^m$, setting

$$\tilde{u}(t) = \frac{r_1}{r_2} u_*(t), \qquad t \in [a,b]. \quad \textbf{(3.4)}$$

Since $u_*(\cdot) \in U_{p,r_2}$, then from (3.4) it follows that $\tilde{u}(\cdot) \in U_{p,r_1}$. Let $\tilde{x}(\cdot): [a,b] \to R^n$ be the trajectory of the system (2.1) generated by the control function $\tilde{u}(\cdot) \in U_{p,r_1}$. Then $\tilde{x}(\cdot) \in X_{p,r_1}$ and from condition 2.B and (2.1) we obtain

$$\begin{aligned}\| \tilde{x}(t) - x_*(t) \| &\leq l_0 \| \tilde{x}(t) - x_*(t) \| \\ &+\lambda \int_a^b [l_2 + \gamma_2(\| \tilde{u}(s) \| + \| u_*(s) \|)] \\ &\cdot \| \tilde{x}(s) - x_*(s) \|\, ds \\ &+\lambda \int_a^b [l_3 + \gamma_3(\| \tilde{x}(s) \| + \| x_*(s) \|)] \\ &\cdot \| \tilde{u}(s) - u_*(s) \|\, ds\,. \end{aligned} \quad \textbf{(3.5)}$$

From (2.6), (3.2), (3.4), Proposition 2.1 and Hölder's inequality it follows

$$\begin{aligned}&\lambda \int_a^b [l_2 + \gamma_2(\| \tilde{u}(s) \| + \| u_*(s) \|)] \cdot \| \tilde{x}(s) - x_*(s) \|\, ds \\ &\leq \lambda \int_a^b [l_2 + \gamma_2(\| \tilde{u}(s) \| + \| u_*(s) \|)] ds \cdot \| \tilde{x}(\cdot) - x_*(\cdot) \|_C \\ &\leq [L_*(\lambda) - l_0] \cdot \| \tilde{x}(\cdot) - x_*(\cdot) \|_C\,, \end{aligned} \quad \textbf{(3.6)}$$

$$\lambda \int_a^b [l_3 + \gamma_3(\| \tilde{x}(s) \| + \| x_*(s) \|)] \cdot \| \tilde{u}(s) - u_*(s) \|\, ds$$

$$\leq \lambda[l_3+2\beta_*\gamma_3]\int_a^b \left\|\frac{r_1}{r_2}u_*(s)-u_*(s)\right\| ds$$

$$\leq \lambda[l_3+2\beta_*\gamma_3]\,\frac{|r_1-r_2|}{r_2}r_2\,(b-a)^{\frac{p-1}{p}}$$

$$\leq \lambda[l_3+2\beta_*\gamma_3]\,l_*|r_1-r_2|. \quad \textbf{(3.7)}$$

(3.5), (3.6) and (3.7) imply that

$$\| \tilde{x}(t)-x_*(t) \| \leq l_0 \| \tilde{x}(\cdot)-x_*(\cdot) \|_C$$

$$+[L_*(\lambda)-l_0]\cdot\| \tilde{x}(\cdot)-x_*(\cdot) \|_C$$

$$+\lambda[l_3+2\beta_*\gamma_3]\,l_*|r_1-r_2|$$

for every $t\in[a,b]$. The last inequality, (2.5) and (3.3) yield

$$\| \tilde{x}(\cdot)-x_*(\cdot) \|_C \leq \frac{\lambda[l_3+2\beta_*\gamma_3]\,l_*}{1-L_*(\lambda)}|r_1-r_2|$$

$$=R_*|r_1-r_2|. \quad \textbf{(3.8)}$$

So, by virtue of the inequality (3.8), for each $x_*(\cdot)\in X_{p,r_2}$ there exists $\tilde{x}(\cdot)\in X_{p,r_1}$ such that that the inequality

$$\| \tilde{x}(\cdot)-x_*(\cdot) \|_C \leq R_*|r_1-r_2|$$

is satisfied. This means that

$$X_{p,r_2}\subset X_{p,r_1}+R_*|r_1-r_2|\cdot B_C(1) \quad \textbf{(3.9)}$$

where $B_C(1)$ is defined by (3.1). Keeping in mind that $X_{p,r_1}\subset X_{p,r_2}$, we have from (3.9) the proof of the theorem.

From Theorem 3.1 it follows that for each fixed $p\in[p_*-\beta_1,p_*+\beta_1]$ the set valued map $r\to X_{p,r}$, $r\in[0,r_*+\beta_2]$, is Lipschitz continuous with Lipschitz constant $R_*$.

From Theorem 3.1 we also obtain the validity of the following corollary.

**Corollary 3.1.** Let $p\in[p_*-\beta_1,p_*+\beta_1]$ be fixed. Then

$$h_C(X_{p,r_1}(t)\,,X_{p,r_2}(t))\leq R_*|r_1-r_2|$$

for every $r_1\in[0,r_*+\beta_2]$, $r_2\in[0,r_*+\beta_2]$ and $t\in[a,b]$ where $R_*$ is defined by (3.3), $X_{p,r}(t)\subset R^n$ is defined by (2.7).

## 4. Continuity of the Set of Trajectories with Respect to p

In this section the dependence of the set of trajectories on $p$ will be investigated. At first let us define a distance between the subsets of the spaces $L_{p_1}([a,b];R^m)$ and $L_{p_2}([a,b];R^m)$ where $p_1\in[1,+\infty)$, $p_2\in[1,+\infty)$.

The Hausdorff distance between the sets $Q\subset L_{p_1}([a,b];R^m)$ and $D\subset L_{p_2}([a,b];R^m)$ where $1\leq p_1<+\infty$, $1\leq p_2<+\infty$ is denoted by $H_1(Q,D)$ and is defined by

$$H_1(Q,D)=max\,\{sup_{x(\cdot)\in Q}\,d_{L_1}(x(\cdot),D)\,,$$

$$sup_{y(\cdot)\in D}\,d_{L_1}(y(\cdot),Q)\}.$$

Here

$$d_{L_1}(x(\cdot),D)=\inf\{\| x(\cdot)-y(\cdot) \|_1 : y(\cdot)\in D\},$$

$$\| x(\cdot)-y(\cdot) \|_1=\int_a^b \| x(s)-y(s) \| ds.$$

We denote

$$q_*=\frac{\lambda(l_3+2\beta_*\gamma_3)}{1-L_*(\lambda)} \quad \textbf{(4.1)}$$

where $L_*(\lambda)$ is defined by (2.4), $\beta_*$ is given in (2.6).

**Theorem 4.1.** Let $r\in[0,r_*+\beta_2]$ and $p_0\in(p_*-\beta_1,p_*+\beta_1)$ be fixed. Then for every $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,p_0,r)>0$ such that for every $p\in(p_0-\delta,p_0+\delta)$ the inequality

$$h_C(X_{p,r}\,,X_{p_0,r})\leq\varepsilon$$

holds.

*Proof.* By virtue of Theorem 3.6 from [5] we have that for fixed $r\in[0,r_*+\beta_2]$, $p_0\in(p_*-\beta_1,p_*+\beta_1)$ and for given $\frac{\varepsilon}{q_*}$ there exists $\delta=\delta(\varepsilon,p_0,r)\in(0,p_0-1)$ such that

$$H_1\left(U_{p,r},U_{p_0,r}\right)\leq\frac{\varepsilon}{q_*} \quad \textbf{(4.2)}$$

for every $p\in(p_0-\delta,p_0+\delta)$. Without loss of generality let us assume that

$$\delta=\delta(\varepsilon,p_0,r)<\min\{p_0-p_*+\beta_1,p_*-p_0+\beta_1\}$$

which implies that

$$(p_0-\delta\,,p_0+\delta)\subset(p_*-\beta_1,p_*+\beta_1). \quad \textbf{(4.3)}$$

Now, let us choose arbitrary $p\in(p_0-\delta\,,p_0+\delta)$ and $x(\cdot)\in X_{p,r}$, generated by admissible control function $u(\cdot)\in U_{p,r}$. According to (4.2) we have that for $u(\cdot)\in U_{p,r}$ there exists $v(\cdot)\in U_{p_0,r}$ such that

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$$\| u(\cdot) - v(\cdot) \|_1 \leq \frac{\varepsilon}{q_*} \qquad (4.4)$$

where $q_* > 0$ is defined by (4.1). Let $z(\cdot): [a,b] \to R^n$ be the trajectory of the system (2.1) generated by the admissible control function $v(\cdot) \in U_{p_0,r}$ . Then $z(\cdot) \in X_{p_0,r}$ and from condition 2.B it follows that

$$\| x(t) - z(t) \| \leq l_0 \| x(t) - z(t) \| + \lambda \int_a^b [l_2 + \gamma_2(\| u(s) \| + \| v(s) \|)] \cdot \| x(s) - z(s) \| ds + \lambda \int_a^b [l_3 + \gamma_3(\| x(s) \| + \| z(s) \|)] \cdot \| u(s) - v(s) \| ds \qquad (4.5)$$

From (2.6), (4.4) and Proposition 2.1 we have

$$\lambda \int_a^b [l_2 + \gamma_2(\| u(s) \| + \| v(s) \|)] \cdot \| x(s) - z(s) \| ds \leq \lambda \int_a^b [l_2 + \gamma_2(\| u(s) \| + \| v(s) \|)] ds \cdot \| x(\cdot) - z(\cdot) \|_C \leq [L_*(\lambda) - l_0] \cdot \| x(\cdot) - z(\cdot) \|_C , \qquad (4.6)$$

$$\lambda \int_a^b [l_3 + \gamma_3(\| x(s) \| + \| z(s) \|)] \cdot \| u(s) - v(s) \| ds \leq \lambda [l_3 + 2\beta_* \gamma_3] \int_a^b \| u(s) - v(s) \| ds \leq \lambda [l_3 + 2\beta_* \gamma_3] \cdot \frac{\varepsilon}{q_*} \qquad (4.7)$$

From (4.5), (4.6) and (4.7) we obtain that

$$\| x(t) - z(t) \| \leq l_0 \| x(\cdot) - z(\cdot) \|_C + [L_*(\lambda) - l_0] \cdot \| x(\cdot) - z(\cdot) \|_C + \lambda [l_3 + 2\beta_* \gamma_3] \cdot \frac{\varepsilon}{q_*} = L_*(\lambda) \cdot \| x(\cdot) - z(\cdot) \|_C + \lambda [l_3 + 2\beta_* \gamma_3] \cdot \frac{\varepsilon}{q_*}$$

for every $t \in [a,b]$. The last inequality, (2.5) and (4.1) imply that

$$\| x(\cdot) - z(\cdot) \|_C \leq \frac{\lambda (l_3 + 2\beta_* \gamma_3)}{1 - L_*(\lambda)} \cdot \frac{\varepsilon}{q_*} = \varepsilon.$$

Thus, we get that for each $x(\cdot) \in X_{p,r}$ there exists $z(\cdot) \in X_{p_0,r}$ such that the inequality

$$\| x(\cdot) - z(\cdot) \|_C \leq \varepsilon$$

holds. This yields that

$$X_{p,r} \subset X_{p_0,r} + \varepsilon \cdot B_C(1) \qquad (4.8)$$

Analogously, it is possible to show that

$$X_{p_0,r} \subset X_{p,r} + \varepsilon \cdot B_C(1) \qquad (4.9)$$

(4.8) and (4.9) complete the proof.

From Theorem 4.1 it follows that for each fixed $r \in [0, r_* + \beta_2]$ the set valued map $p \to X_{p,r}$ , $p \in (p_* - \beta_1, p_* + \beta_1)$, is continuous.

Theorem 4.1 implies the validity of the following corollary.

**Corollary 4.1.** Let $r \in [0, r_* + \beta_2]$ and $p_0 \in (p_* - \beta_1, p_* + \beta_1)$ be fixed. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, p_0, r) > 0$ such that for every $p \in (p_0 - \delta, p_0 + \delta)$ the inequality

$$h_C(X_{p,r}(t) , X_{p_0,r}(t)) \leq \varepsilon$$

is verified for every $t \in [a,b]$ where the set $X_{p,r}(t)$ is defined by (2.7).

## 5. Robustness of the Trajectories with Respect to the Remaining Control Resource

In this section we study the robustness of a trajectory with respect to the remaining control resource consumption.

**Theorem 5.1** Let $\varepsilon > 0$ be a given number, $x(\cdot) \in X_{p,r}$ be a trajectory of the system (2.1) generated by the control function $u(\cdot) \in U_{p,r}$ such that $\|u(\cdot)\|_p = r_0 < r$, $\Omega_* \subset [a,b]$ be a Lebesgue measurable set, the control function $u_0(\cdot): [a,b] \to R^m$ be defined

$$u_0(s) = \begin{cases} u(s) & if \ s \in [a,b] \setminus \Omega_*, \\ v(s) & if \ s \in \Omega_* \end{cases} \qquad (5.1)$$

such that $\|u_0(\cdot)\|_p = r$ and let $x_0(\cdot) \in X_{p,r}$ be a trajectory of the system (2.1) generated by the control function $u_0(\cdot) \in U_{p,r}$ . If

$$\mu(\Omega_*) \leq \left[\frac{1-L_*(\lambda)}{2\lambda r(l_3+2\beta_*\gamma_3)}\cdot\varepsilon\right]^{\frac{p}{p-1}}, \qquad (5.2)$$

then the inequality

$$\|x(\cdot)-x_0(\cdot)\|_C \leq \varepsilon$$

is held where $L_*(\lambda)$ and $\beta_*>0$ are defined by (2.4) and (2.6) respectively, $\mu(\Omega_*)$ denotes the Lebesgue measure of the set $\Omega_*$.

*Proof.* (2.1), (2.4), (2.6), (5.1), Proposition 2.1, condition 2.B, the inclusions $u(\cdot)\in U_{p,r}$, $u_0(\cdot)\in U_{p,r}$ and Hölder's inequality imply that

$$\begin{aligned}
\| x(t)-x_0(t) \| &\leq l_0 \| x(t)-x_0(t) \| \\
&+\lambda\int_a^b [l_2+\gamma_2(\| u(s)\| + \| u_0(s)\|)] \\
&\cdot \| x(s)-x_0(s)\| ds \\
&+\lambda\int_a^b [l_3+\gamma_3(\| x(s)\| + \| x_0(s)\|)] \\
&\cdot \| u(s)-u_0(s)\| ds \\
&\leq l_0 \| x(\cdot)-x_0(\cdot)\|_C \\
&+\lambda\int_a^b [l_2+\gamma_2(\| u(s)\| + \| u_0(s)\|)] ds \\
&\cdot \| x(\cdot)-x_0(\cdot)\|_C \\
&+\lambda(l_3+2\beta_*\gamma_3)\int_{\Omega_*} \| u(s)-u_0(s)\| ds \\
&\leq l_0\|x(\cdot)-x_0(\cdot)\|_C+(L_*(\lambda)-l_0)\|x(\cdot)-x_0(\cdot)\|_C \\
&+2\lambda r(l_3+2\beta_*\gamma_3)\,[\mu(\Omega_*)]^{\frac{p-1}{p}} \\
&= L_*(\lambda)\cdot\|x(\cdot)-x_0(\cdot)\|_C \\
&+2\lambda r(l_3+2\beta_*\gamma_3)[\mu(\Omega_*)]^{\frac{p-1}{p}}
\end{aligned}$$

for every $t\in[a,b]$. The last inequality, (2.5) and (5.2) imply

$$\|x(\cdot)-x_0(\cdot)\|_C \leq \frac{2\lambda r(l_3+2\beta_*\gamma_3)}{1-L_*(\lambda)}[\mu(\Omega_*)]^{\frac{p-1}{p}} \leq \varepsilon\,.$$

The proof of the theorem is completed.

Theorem 5.1 shows that full consumption of the remaining control resource on the domain with sufficiently small measure causes small deviation of the trajectory.

Denote

$$U_{p,r}^* = \left\{u(\cdot)\in L_p([a,b];R^m): \|u(\cdot)\|_p = r\right\}$$

and let $X_{p,r}^*$ be the set of trajectories of the system (2.1) generated by the control functions $u(\cdot)\in U_{p,r}^*$.

For fixed $t\in[a,b]$ we set

$$X_{p,r}^*(t) = \left\{x(t)\in R^n : x(\cdot)\in X_{p,r}^*\right\}. \qquad (5.3)$$

**Theorem 5.2.** The equality $cl(X_{p,r}) = cl\left(X_{p,r}^*\right)$ is satisfied where $cl$ denotes the closure of a set.

*Proof.* Since $X_{p,r}^*\subset X_{p,r}$, then we have

$$cl\,(X_{p,r}^*) \subset cl\,(X_{p,r}). \qquad (5.4)$$

Let $x_*(\cdot)\in X_{p,r}$ be an arbitrarily chosen trajectory of the system (2.1) generated by the control function $u_*(\cdot)\in U_{p,r}$ where $\|u_*(\cdot)\|_p = r_* < r$. Now we choose an arbitrary $\delta>0$ and the Lebesgue measurable set $V_*\subset[a,b]$ such that

$$\mu(V_*) \leq \left[\frac{1-L_*(\lambda)}{2\lambda r(l_3+2\beta_*\gamma_3)}\delta\right]^{\frac{p}{p-1}} \qquad (5.5)$$

where $L_*(\lambda)$ is defined by (2.4), $\beta_*$ is defined by (2.6). Assume that $\int_{[a,b]\setminus V_*}\|u_*(s)\|^p ds = r_1^p$. Define new control function $w_*(\cdot):[a,b]\to R^m$ setting

$$w_*(s) = \begin{cases} u_*(s) & if\ \ s\in[a,b]\setminus V_* \\ \left[\frac{r^p-r_1^p}{\mu(V_*)}\right]^{\frac{1}{p}}\cdot b_* & if \quad s\in V_* \end{cases}$$

where $b_*\in R^m$ is an arbitrary vector such that $\|b_*\|=1$. It is obvious that $\|w_*(\cdot)\|_p = r$, i.e. $w_*(\cdot)\in U_{p,r}^*$. Let $y_*(\cdot)$ be the trajectory of the system (2.1) generated by the control function $w_*(\cdot)$. Then $y_*(\cdot)\in X_{p,r}^*$, and keeping in mind (5.5) we obtain from Theorem 5.1 that $\|x_*(\cdot)-y_*(\cdot)\|_C\leq\delta$. Since $\delta>0$ is arbitrarily chosen, we have that $x_*(\cdot)\in cl\left(X_{p,r}^*\right)$ which implies that $X_{p,r}\subset cl\left(X_{p,r}^*\right)$, and hence

$$cl\,(X_{p,r}) \subset cl\left(X_{p,r}^*\right) \qquad (5.6)$$

From (5.4) and (5.6) we obtain the proof of the theorem.

The Theorem 5.1 means that every trajectory $x(\cdot)\in X_{p,r}$ can be approximated by the trajectory obtained by full consumption of the control resource.

From Theorem 5.2 it follows the validity of the following corollary.

**Corollary 5.1.** The equality

$$cl\left(X_{p,r}(t)\right) = cl\left(X^*_{p,r}(t)\right)$$

is satisfied for every $t \in [a,b]$ where the set $X_{p,r}(t)$ is defined by (2.7), the set $X^*_{p,r}(t)$ is defined by (5.3).

### Author's Contributions

**Anar Huseyin:** Drafted and wrote the manuscript, performed the proofs and result analysis.

### Ethics

There are no ethical issues after the publication of this manuscript.

### References

**[1].** Aubin, J-P, Frankowska, H. Set Valued Analysis. Birkhauser: Boston, USA, 1990, pp 461.

**[2].** Brauer, F. 1975. On a nonlinear integral equation for population growth problems. *SIAM J. Math. Anal.*; 69: 312-317.

**[3].** Conti, R. Problemi di Controllo e di Controllo Ottimale. UTET: Torino, Italy, 1974, pp 239.

**[4].** Deimling, K. Multivalued Differential Equations. Walter de Gruyter: Berlin, Germany, 1992, pp 260.

**[5].** Guseinov, KG, Nazlipinar AS. 2007. On the continuity property of $L_p$ balls and an application. *J. Math. Anal. Appl.*; 335: 1347-1359.

**[6].** Gusev, MI, Zykov, IV. 2017. On extremal properties of the boundary points of reachable sets for control systems with integral constraints. *Tr. Inst. Math. Mekh. UrO RAN*; 23: 103-115.

**[7].** Heisenberg, W. Physics and Philosophy. The Revolution in Modern Science. George Allen & Unwin: London, Great Britain, 1958, pp 176.

**[8].** Huseyin, A. 2017. On the existence of $\varepsilon$-optimal trajectories of the control systems with constrained control resources. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.;* 66: 75-84.

**[9].** Huseyin, N, Guseinov, KG, Ushakov, VN. 2015. Approximate construction of the set of trajectories of the control system described by a Volterra integral equation. *Math. Nachr*.; 288(16): 1891-1899.

**[10].** Huseyin, N, Huseyin, A, Guseinov KG. 2018. Approxmation of the set of trajectories of the nonlinear control system with limited control resources. *Math. Model. Anal.;* 23(1): 152-166.

**[11].** Huseyin, N. 2020. On the properties of the set of p-integrable trajectories of the control system with limited control resources. *Internat. J. Control*; 93(8): 1810-1816.

**[12].** Ibragimov, G, Alias, IA, Waziri, U, Jafaaru, AB. 2019. Differential game of optimal pursuit for an infinite system of differential equations. *Bull. Malaysian Math. Sci. Soc.;* 42(1): 391-403.

**[13].** Kalman, RE. 1963. Mathematical description of linear dynamical systems. *J. SIAM Control Ser. A*; 1: 152-192.

**[14].** Kelley, JL. General Topology. Springer: New York, USA, 1975, pp 298.

**[15].** Krasovskii, NN. Theory of Control of Motion. Linear Systems. Nauka: Moscow, USSR, 1968, pp 475.

**[16].** Krasovskii, NN, Subbotin, AI. Game-Theoretical Control Problems. Springer-Verlag: New York, USA,1988, pp 517.

**[17].** Krasnoselskii, MA, Krein, SG. 1955. On the principle of averaging in nonlinear mechanics. *Uspekhi Mat. Nauk*; 10: 147-153.

**[18].** Kostousova, EK. 2020. On the polyhedral estimation of reachable sets in the "extended" space for multistage systems with uncertain matrices and integral constraints. *Tr. Inst. Mat. Mekh*; 26(1), 141-155.

**[19].** Polyanin, AD, Manzhirov, AV. Handbook of Integral Equation. CRC Press: Boca Raton, FL, USA, 1998, pp 1108.

**[20].** Pontryagin, LS, Boltyanskii, VG, Gamkrelidze, RV, Mishchenko, EF. The Mathematical Theory of Optimal Processes. John Wiley & Sons: New York, USA, 1962, pp 360.

**[21].** Subbotin, AI, Ushakov, VN. 1975. Alternative for an encounter-evasion differential game with integral constraints on the players' controls. *J. Appl. Math. Mech.;* 39(3): 367-375.

**[22].** Subbotina, NN, Subbotin, AI. 1975. Alternative for the encounter-evasion differential game with constraints on the momenta of the players controls. *J. Appl. Math. Mech*.; 39(3): 376-385.

**[23].** Ukhobotov, VI, Izmestev, IV. 2018. Impulse differential game with a mixed constraint on the choice of the control of the first player. *Tr. Inst. Math. Mekh. UrO RAN*; 24(1): 209-222.

**[24].** Urysohn, PS. 1923. On a type of nonlinear integral equation. *Mat. Sb*; 31: 236-255.