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RESEARCH ARTICLE

THE NOWICKI CONJECTURE FOR BICOMMUTATIVE ALGEBRAS

Şehmus FINDIK *

Department of Mathematics, Faculty of Science and Letters, Çukurova University, Adana, Türkiye

ABSTRACT

Let K be a field of characteristic zero, and $K[X_n, Y_n]$ be the commutative associative unitary polynomial algebra of rank $2n$ generated by the set $X_n \cup Y_n = \{x_1, \dots, x_n, y_1, \dots, y_n\}$. It is well known that the algebra $K[X_n, Y_n]^\delta$ of constants of the locally nilpotent linear derivation δ of $K[X_n, Y_n]$ sending y_i to x_i, and x_i to 0, is generated by $x_1, \dots, x_n$ and the determinants of the form $x_i y_j - x_j y_i$; that was first conjectured by Nowicki in 1994, and later proved by several authors. Bicommutative algebras are nonassociative noncommutative algebras satisfying the identities $(xy)z = (xz)y$ and $x(yz) = y(xz)$. In this study, we work in the $2n$ generated free bicommutative algebra as a noncommutative nonassociative analogue of the Nowicki conjecture, and find the generators of the algebra of constants in this algebra.

Keywords: Algebra of constants, Bicommutative algebra, The Nowicki conjecture

1. INTRODUCTION

Roots of the Nowicki conjecture dates back to 1900, when the famous German mathematician David Hilbert posed 23 unsolved major questions at the Paris International Congress of Mathematicians [1]. In the fourteenth problem, he asked the finite generation of the algebra $K[X_n]^G$ of invariants of any subgroup G of the general linear group consisting of $n \times n$ invertible matrices with entries from a field K of characteristic zero, where $K[X_n]$ is the commutative associative unitary polynomial algebra of rank n.

The negative answer to the fourteenth problem was given by Nagata [2] in 1959, while many partially affirmative cases were considered by several authors. One may count the work by Noether [3] who showed that $K[X_n]^G$ finitely generated for every finite group G. Another remarkable approach was given by Weitzenböck [4] who considered algebras constants of linear nilpotent derivations δ of $K[X_n]$. He showed that the algebra $K[X_n]^\delta$ is finitely generated that is equal to the algebra $K[X_n]^{\langle \exp \delta \rangle}$ of invariants. However, no information about the explicit forms of generators were provided. Many years later in 1994, Nowicki [5] conjectured an explicit generating set for the algebra $K[X_n, Y_n]^\delta$ of constants of the Weitzenböck derivation δ sending y_i to x_i, and x_i to 0, where $K[X_n, Y_n]$ is the polynomial algebra of rank $2n$ generated by the set $X_n \cup Y_n = \{x_1, \dots, x_n, y_1, \dots, y_n\}$. He proposed that $K[X_n, Y_n]^\delta$ is generated by $x_1, \dots, x_n$ and the elements of the form $x_i y_j - x_j y_i$, where $1 \leq i < j \leq n$. Then, the conjecture was verified by many mathematicians [6, 7, 8, 9].

Noncommutative nonassociative analogues of the Nowicki conjecture have been studied, recently. See e.g. [10], in which the authors consider the free metabelian Lie algebra F_{2n} of rank $2n$ generated by $X_n \cup Y_n$. They gave a finite generating set for the algebra $(F'_{2n})^\delta$ included in the commutator ideal F'_{2n} of F_{2n} as a $K[X_n, Y_n]^\delta$-module. As a continuation of this work a finite generation set for the algebra of constants in the commutator ideal of the free metabelian associative algebra generated by $X_n \cup Y_n$ as a $K[X_n, Y_n]^\delta$-bimodule was given in [11]. In the same work, a set of finite generators was obtained for

*Corresponding Author: sfindik@cu.edu.tr
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the free algebra in the variety of infinite dimensional Grassmann algebras. There is also the free metabelian Possion algebra analogue of the Nowicki conjecture [12].

In the current study, we consider the free algebra of rank $2n$ in the variety of bicommutative algebras and determine the generators of the algebra of constants of Weitzenböck derivation that was stated in the Nowicki conjecture.

2. PRELIMINARIES

We assume that $\boldsymbol{K}$ is a field of characteristic zero throughout the paper. Let $\boldsymbol{K}[\boldsymbol{X_n}]$, $\boldsymbol{K}[\boldsymbol{Y_n}]$, and $\boldsymbol{K}[\boldsymbol{X_n}, \boldsymbol{Y_n}]$ be the polynomial algebras generated by sets $\boldsymbol{X_n} = \{\boldsymbol{x_1}, \dots, \boldsymbol{x_n}\}, \boldsymbol{Y_n} = \{\boldsymbol{y_1}, \dots, \boldsymbol{y_n}\}$, and $\boldsymbol{X_n} \cup \boldsymbol{Y_n}$, respectively. We also fix notations $\boldsymbol{\omega}(\boldsymbol{K}[\boldsymbol{X_n}])$ and $\boldsymbol{\omega}(\boldsymbol{K}[\boldsymbol{X_n}])$ for augmentation ideals of $\boldsymbol{K}[\boldsymbol{X_n}]$ and $\boldsymbol{K}[\boldsymbol{Y_n}]$, respectively, consisting of the polynomials without constant terms.

We call a noncommutative nonassociative algebra over $\boldsymbol{K}$ *right symmetric* and *left symmetric* if it satisfies the identity $(\boldsymbol{xy})\boldsymbol{z} = (\boldsymbol{xz})\boldsymbol{y}$ and $\boldsymbol{x}(\boldsymbol{yz}) = \boldsymbol{y}(\boldsymbol{xz})$, respectively. An algebra over $\boldsymbol{K}$ is called *bicommutative* if it is left and right symmetric.

Let $\boldsymbol{F_{2n}}$ be the free algebra of rank $\boldsymbol{2n}$ generated by $\boldsymbol{X_n} \cup \boldsymbol{Y_n}$ in the variety of bicommutative algebras over the field $\boldsymbol{K}$, and let $\boldsymbol{a} = \boldsymbol{a_1 a_2}$, $\boldsymbol{b} = \boldsymbol{b_1 b_2}$, $\boldsymbol{c} \in \boldsymbol{F_{2n}^2}$ for some $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{b_1}, \boldsymbol{b_2} \in \boldsymbol{F_{2n}}$. Then the following straightforward computations show that the ideal $\boldsymbol{F_{2n}^2} = \boldsymbol{F_{2n}F_{2n}}$ of $\boldsymbol{F_{2n}}$ is commutative and associative.

$$\begin{aligned}\boldsymbol{ab} &= (\boldsymbol{a_1a_2})(\boldsymbol{b_1b_2}) = \big(\boldsymbol{a_1}(\boldsymbol{b_1b_2})\big)\boldsymbol{a_2} = \big(\boldsymbol{b_1}(\boldsymbol{a_1b_2})\big)\boldsymbol{a_2} = (\boldsymbol{b_1a_2})(\boldsymbol{a_1b_2}) = \boldsymbol{a_1}\big((\boldsymbol{b_1a_2})\boldsymbol{b_2}\big)\\ &= \boldsymbol{a_1}\big((\boldsymbol{b_1b_2})\boldsymbol{a_2}\big) = (\boldsymbol{b_1b_2})(\boldsymbol{a_1a_2}) = \boldsymbol{ba},\end{aligned}$$

and

$$(\boldsymbol{ab})\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{ab}) = \boldsymbol{a}(\boldsymbol{cb}) = \boldsymbol{a}(\boldsymbol{bc}).$$

Therefore, $\boldsymbol{F_{2n}}$ can be considered as a direct sum of the vector space $\boldsymbol{K}(\boldsymbol{X_n} \cup \boldsymbol{Y_n}) = \mathbf{Span}\{\boldsymbol{X_n} \cup \boldsymbol{Y_n}\}$ and $\boldsymbol{\omega}(\boldsymbol{K}[\boldsymbol{A_n}, \boldsymbol{B_n}])\boldsymbol{\omega}(\boldsymbol{K}[\boldsymbol{C_n}, \boldsymbol{D_n}])$, where

$$\boldsymbol{A_n} = \{\boldsymbol{a_1}, \dots, \boldsymbol{a_n}\}, \boldsymbol{B_n} = \{\boldsymbol{b_1}, \dots, \boldsymbol{b_n}\}, \boldsymbol{C_n} = \{\boldsymbol{c_1}, \dots, \boldsymbol{c_n}\}, \boldsymbol{D_n} = \{\boldsymbol{d_1}, \dots, \boldsymbol{d_n}\}$$

such that

$$\begin{aligned}\boldsymbol{x_i x_j} &= \boldsymbol{a_i c_j},\\ \boldsymbol{y_i y_j} &= \boldsymbol{b_i d_j},\\ \boldsymbol{x_i y_j} &= \boldsymbol{a_i d_j},\\ \boldsymbol{y_i x_j} &= \boldsymbol{b_i c_j}.\end{aligned}$$

Note that $\boldsymbol{F_{2n}^2} \cong \boldsymbol{\omega}(\boldsymbol{K}[\boldsymbol{A_n}, \boldsymbol{B_n}])\boldsymbol{\omega}(\boldsymbol{K}[\boldsymbol{C_n}, \boldsymbol{D_n}])$ contains elements as linear combinations of the form

$$\boldsymbol{a_1^{\alpha_1} \cdots a_n^{\alpha_n} b_1^{\beta_1} \cdots b_n^{\beta_n} c_1^{\gamma_1} \cdots c_n^{\gamma_n} d_1^{\varepsilon_1} \cdots d_n^{\varepsilon_n}},$$

where $\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n > 0$, $\gamma_1 + \cdots + \gamma_n + \varepsilon_1 + \cdots + \varepsilon_n > 0$. We refer to the paper [13] for more details.

Now let $\delta: F_{2n} \to F_{2n}$ be the locally nilpotent derivation of F_{2n} acting linearly on the vector space spanned on $X_n \cup Y_n$ such that $\delta(y_i) = x_i$, $\delta(x_i) = 0$ for each $i = 1, \dots, n$. Our main result concerns with the generators of the subalgebra

$$F_{2n}^{\delta} = \{f \in F_{2n} : \delta(f) = 0\}$$

of constants of the derivation δ in the free bicommmutative algebra F_{2n}. For this purpose, we will work in the algebra

$$F_{2n} = K(X_n \cup Y_n) \oplus F_{2n}^2 \cong K(X_n \cup Y_n) \oplus \omega(K[A_n, B_n])\omega(K[C_n, D_n]).$$

An easy observation gives that

$$\begin{aligned} F_{2n}^{\delta} &\cong K(X_n \cup Y_n)^{\delta} \oplus \big(\omega(K[A_n, B_n])\omega(K[C_n, D_n])\big)^{\delta} \\ &= KX_n \oplus \big(\omega(K[A_n, B_n])\omega(K[C_n, D_n])\big)^{\delta}. \end{aligned}$$

Here, we assume that δ acts on $K(A_n \cup B_n)$ and $K(C_n \cup D_n)$ same as on $K(X_n \cup Y_n)$; i.e.,

$$\delta(b_i) = a_i \,,\ \delta(a_i) = 0$$
$$\delta(d_i) = c_i \,,\ \delta(c_i) = 0$$

for each $i = 1, \dots, n$. Hence, it is sufficient to determine constants of δ in the algebra

$$(F_{2n}^2)^{\delta} = \big(\omega(K[A_n, B_n])\omega(K[C_n, D_n])\big)^{\delta}.$$

In the next section, we determine the elements of $(F_{2n}^2)^{\delta}$, and consequently describe the algebra F_{2n}^{δ}.

3. MAIN RESULTS

The following theorem and corrollary are our main results.

Theorem 1. The algebra $\big(\omega(K[A_n, B_n])\omega(K[C_n, D_n])\big)^{\delta}$ is generated by determinants

$$\begin{vmatrix} a_i & c_j \\ b_i & d_j \end{vmatrix} = a_i d_j - b_i c_j \,, \quad 1 \le i, j \le n,$$

and it is a $K\big[A_n, C_n, a_i b_j - b_i a_j, c_i d_j - d_i c_j, a_k d_l - b_k c_l : 1 \le i < j \le n, 1 \le k, l \le n\big]^{\delta}$-module.

Proof. Clearly, $\omega(K[A_n, B_n])\omega(K[C_n, D_n]) \subset K[A_n, B_n, C_n, D_n]$ is a $K[A_n, B_n, C_n, D_n]$-module, and $\big(\omega(K[A_n, B_n])\omega(K[C_n, D_n])\big)^{\delta}$ is a $K[A_n, B_n, C_n, D_n]^{\delta}$-module. It is well known, see e.g. [7], that $K[A_n, B_n, C_n, D_n]^{\delta}$ is generated by $a_1, \dots, a_n, c_1, \dots, c_n$ together with

$$\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} = a_i b_j - b_i a_j \,, \qquad \begin{vmatrix} c_i & c_j \\ d_i & d_j \end{vmatrix} = c_i d_j - d_i c_j \,, \qquad 1 \le i < j \le n,$$

$$\begin{vmatrix} a_i & c_j \\ b_i & d_j \end{vmatrix} = a_i d_j - b_i c_j \,, \quad 1 \le i, j \le n.$$

It is straightforward to see that a polynomial $p(A_n, B_n, C_n, D_n) \in K[A_n, B_n, C_n, D_n]$ belongs to $\omega(K[A_n, B_n])\omega(K[C_n, D_n])$ if and only if

$$p(A_n, B_n, C_n, D_n) \not\equiv 0 \ (\mathrm{mod}\ K[A_n, B_n] \oplus K[C_n, D_n]).$$

Since,

$$a_1, \ldots, a_n \equiv 0 \ (\text{mod}\ K[A_n, B_n] \oplus K[C_n, D_n])$$
$$c_1, \ldots, c_n \equiv 0 \ (\text{mod}\ K[A_n, B_n] \oplus K[C_n, D_n])$$
$$a_i b_j - b_i a_j \equiv 0 \ (\text{mod}\ K[A_n, B_n] \oplus K[C_n, D_n])$$
$$c_i d_j - d_i c_j \equiv 0 \ (\text{mod}\ K[A_n, B_n] \oplus K[C_n, D_n])$$
$$a_i d_j - b_i c_j \not\equiv 0 \ (\text{mod}\ K[A_n, B_n] \oplus K[C_n, D_n])$$

we obtain that $\left(\omega(K[A_n, B_n])\omega(K[C_n, D_n])\right)^\delta$ is generated by the elements of the form $a_i d_j - b_i c_j$, $1 \leq i, j \leq n$, and it is a

$$K[A_n, B_n, C_n, D_n]^\delta = K\left[A_n, C_n, a_i b_j - b_i a_j, c_i d_j - d_i c_j, a_k d_l - b_k c_l \colon 1 \leq i < j \leq n, 1 \leq k, l \leq n\right]^\delta$$

-module.

Corollary 2. F_{2n}^δ is generated by $x_1, \ldots, x_n$ together with elements of the form

$$x_i y_j - y_i x_j \ , \ \ 1 \leq i, j \leq n.$$

Example 3. $(\boldsymbol{i})$ Let $n = 1$, and the free bicommutative algebra F_2 be generated by $x_1 = x$ and $y_1 = y$. Then the algebra F_2^δ is generated by $\{x, xy - yx\}$.
$(\boldsymbol{ii})$ Let $n = 2$, and the free bicommutative algebra F_4 be generated by $x_1 = x, y_1 = y, x_2 = z, y_2 = t$. Then the algebra F_4^δ is generated by $\{x, z, xy - yx, zt - tz, xt - yz\}$.

Remark 4. Note that in the case of commutativity the above example is compatible with the following well known results:
$(\boldsymbol{i})$ Let $n = 1$. Then $K[x, y]^\delta$ is generated the set $\{x\}$ in the commutative polynomial algebra generated by $x_1 = x$ and $y_1 = y$.
$(\boldsymbol{ii})$ Let $n = 2$. Then $K[x, y, z, t]^\delta$ is generated the set $\{x, z, xt - yz\}$ in the commutative polynomial algebra generated by $x_1 = x, y_1 = y, x_2 = z, y_2 = t$.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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