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Regularity properties of integral problems for wave equations and applications

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Abstract

In this paper, the integral problem for linear and nonlinear wave equations are studied.The equation involves elliptic operator L and abstract operator A in Hilbert space H. Here, assuming enough smoothness on the initial data in terms of interpolation spaces, integral condition, the assumptions on operators A, L the existence, uniqueness of local and global solution, and L^p-regularity properties to solutions are established. By choosing the space H and operators L, A the regularity properties to solutions of different classes of wave equations in the field of physics are obtained.

Keywords: Abstract differential equations wave equations, operator theory L^p-regularity property of solutions Fourier multipliers.
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1. Introduction, Definitions and Background

The aim here, is to study the existence, uniqueness, regularity properties to solutions of the integral problem (IP) for the following abstract wave equatıon (WE)

$$u_{tt} - Lu + Au = f\left(u\right),\ \left(x,\ t\right) \in \mathbb{R}_T^n = \mathbb{R}^n \times \left(0, T\right), \tag{1.1}$$

$$u\left(x, 0\right) = \varphi\left(x\right) + \int_0^T \eta\left(\sigma\right) u\left(x, \sigma\right) d\sigma, \tag{1.2}$$

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$$u_t(x,0) = \psi(x) + \int_0^T \beta(\sigma)\, u_t(x,\sigma)\, d\sigma,$$

where A is a linear and $f(u)$ is a nonlinear operators in a Hilbert space H, $\eta(\sigma)$, $\beta(\sigma)$ are measurable functions on $(0,T)$ and $T \in (0, \infty]$. Here, L denotes the elliptic operator with constant coefficients a_{ij} defined by

$$Lu = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

$\varphi(x)$ and $\psi(x)$ are given H-valued initial functions.

Wave type equations occur in a wide variety of physical systems, such as in the propagation of deformation waves, hydro-dynamical process in plasma, in materials science and in the absence of mechanical stresses (see [1-14]). Note that, the existence and uniqueness of solutions and regularity properties of a wide class of wave equations were considered e.g. in [15-22]. Unlike to these studies here, we consider the abstract wave equation with operator coefficients. The abstract equations were studied e.g. in [23-32]. This paper generalises the results obtained in [26] and [32] because the general elliptic operator is involved instead of the Laplace operator in the leading part of the equation. The L^p well-posedness of the integral problem $(1.1)-(1.2)$ depends crucially on interpolation spaces that data functions belong, the presence of the linear operators L, A and nonlinear operator $f(u)$. We find the class of operators L and A such that provide the existence, uniqueness, L^p-regularity properties to solution $(1.1)-(1.2)$ in terms of fractional powers of A. By choosing the space H, operators L and A in $(1.1)-(1.2)$, we obtain the wide classes of wave equations which occur in application. Let we put $H = L^2(0,1)$ and consider the operator $A = A_1$ defined by

$$D(A_1) = W^{[2],2}(0,1,L_k),\ A_1 u = b_1 u^{[2]} + b_0 u, \tag{1.3}$$
$$L_k u = \alpha_k u^{[m_k]}(0) + \beta_k u^{[m_k]}(1) = 0,\ k = 1,2,$$

where $b_1(.)$, $b_0(.)$ are complex-valued functions, $D_y^{[j]} u = u^{[j]} = \left(y^{\varkappa} \frac{d}{dy}\right)^j u$, $m_k \in \{0,1\}$, α_k, β_k are complex numbers.

Here, $W^{[2],2}(0,1)$ is a weighted Sobolev space defined by

$$W^{[2],2}(0,1) = \{u \ \ u \in L^2(0,1),\ u^{[2]} \in L^2(0,1),$$

$$\|u\|_{W^{[2],2}(0,1)} = \|u\|_{L^2(0,1)} + \left\|u^{[2]}\right\|_{L^2(0,1)} < \infty.$$

Moreover,

$$W^{[2],2}(0,1,L_k) = \left\{u \ u \in W^{[2],2}(0,1),\ L_k u = 0\right\}.$$

Consider the integral nonlocal mixed problem for degenerate WE

$$\frac{\partial^2 u}{\partial t^2} - Lu + \left(A_1^2 + \omega\right) u = f(u),\ t \in (0,T),\ x \in \mathbb{R}^n, \tag{1.4}$$

$$u(x,y,0) = \varphi(x,y) + \int_0^T \eta(\sigma)\, u(x,y,\sigma)\, d\sigma,$$

$$u_t(x,y,0) = \psi(x,y) + \int_0^T \beta(\sigma)\, u_t(x,y,\sigma)\, d\sigma,$$

$$\alpha_k u^{[m_k]}(x,0,t)+\beta_k u^{[m_k]}(x,1,t)=0,\ k=1,2,$$

where ω is a positive number. From our results, we obtain the existence, uniqueness and regularity properties to solutions of (1.4) in $L^{\mathbf{p}}\ (\mathbb{R}^n\times(0,1))$ with terms of fractional powers of the operator A_1 and interpolation of spaces $L^2(0,1)$ and $W^{[2],2}(0,1)$. Let $\mathbf{p}=(2,p,p)$ and $L^{\mathbf{p}}\ (\mathbb{R}^n\times(0,1))$ denotes the space of all $\mathbf{p}$-summable complex-valued measurable functions f defined on Ω with the mixed norm

$$\|f\|_{L^{\mathbf{p}}(\not\leq)}=\left(\int\limits_{\mathbb{R}^n}\int\limits_0^T\left(\int\limits_0^1|f(x,y,t)|^{p_1}dy\right)^{\frac{2}{p_1}}dxdt\right)^{\frac{1}{2}}<\infty.$$

Let us put $H=L^2\left(\mathbb{R}^d\right)$ and choose A_2 as a convolution operator defined by

$$D(A_2)=W^{2l,p_1}\left(\mathbb{R}^d\right),\ Au=\sum_{|\alpha|\leq 2l}a_\alpha * D^\alpha u, \tag{1.5}$$

here, l is a positive integer, $a_\alpha=a_\alpha(y)$ are complex-valued function and $(a_\alpha * D^\alpha u)(y)$ denote the convolution of a_α and $D^\alpha u$. From the our Theorem 3.1, we get the $L^{\mathbf{p}}\left(\mathbb{R}^{n+d}\right)$-well-posedeness of integral problem for the following convolution wave equation

$$\frac{\partial^2 u}{\partial t^2}-Lu+\left(A_2^2+\omega\right)u=f(u),\ t\in(0,T),\ x\in\mathbb{R}^n, \tag{1.6}$$

$$u(x,y,0)=\varphi(x,y)+\int\limits_0^T\eta(\sigma)u(x,y,\sigma)d\sigma,$$

$$u_t(x,y,0)=\psi(x,y)+\int\limits_0^T\beta(\sigma)u_t(x,y,\sigma)d\sigma.$$

Let E be a Banach space. $L^p(\Omega;E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega\subset\mathbb{R}^n$ with the norm

$$\|f\|_p=\|f\|_{L^p(\Omega;E)}=\left(\int\limits_\Omega\|f(x)\|_E^p dx\right)^{\frac{1}{p}},\ 1\leq p<\infty,$$

$$\|f\|_{L^\infty(\Omega;E)}=ess\sup_{x\in\Omega}\|f(x)\|_E.$$

Let E_1 and E_2 be two Banach spaces. $(E_1,E_2)_{\theta,p}$ for $\theta\in(0,1)$, $p\in[1,\infty]$ denotes the real interpolation spaces defined by K-method [33, §1.3.2]. Let E_1 and E_2 be two Banach spaces. $B(E_1,E_2)$ will denote the space of all bounded linear operators from E_1 to E_2. For $E_1=E_2=E$ it will be denoted by $B(E)$.

Here,

$$S_\phi=\{\lambda\in\mathbb{C},\ \lambda\neq 0,\ |\arg\lambda|\leq\phi,\ 0\leq\phi<\pi\}.$$

A closed linear operator A is said to be ϕ-sectorial (or sectorial) in a Banach space E with bound $M>0$ if $D(A)$ and $R(A)$ are dense on E, $N(A)=\{0\}$ and

$$\left\|(A+\lambda I)^{-1}\right\|_{B(E)}\leq M|\lambda|^{-1}$$

for any $\lambda \in S_{\phi}$, $0 \leq \phi < \pi$, where I is the identity operator in E, $D(A)$ and $R(A)$ denote domain and range of the operator A, respectively. It is known that (see e.g.[33, §1.15.1]) there exist fractional powers A^{θ} of a sectorial operator A. Let $E\left(A^{\theta}\right)$ denote the space $D\left(A^{\theta}\right)$ with the graphical norm

$$\|u\|_{E(A^{\theta})} = \left(\|u\|^{p} + \left\|A^{\theta}u\right\|^{p}\right)^{\frac{1}{p}},\ 1 \leq p < \infty,\ 0 < \theta < \infty.$$

A sectorial operator $A(\xi)$ is said to be uniformly sectorial in E for $\xi \in \mathbb{R}^{n}$ if $D(A(\xi))$ is independent of ξ and the following uniform estimate

$$\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M\,|\lambda|^{-1}$$

holds for any $\lambda \in S_{\phi}$.

A function $\Psi \in L^{\infty}(\mathbb{R}^{n})$ is called a Fourier multiplier from $L^{p}(\mathbb{R}^{n};E)$ to $L^{q}(\mathbb{R}^{n};E)$ if the map P: $u \to \mathbb{F}^{-1}\Psi(\xi)\mathbb{F}u$ is well defined for $u \in S(\mathbb{R}^{n};E)$ and extends to a bounded linear operator

Let E be a Banach space. $S = S(\mathbb{R}^{n};E)$ denotes E-valued Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^{n}$ equipped with its usual topology generated by seminorms. $S(\mathbb{R}^{n};\mathbb{C})$ denoted by S. Let $S'(\mathbb{R}^{n};E)$ denote the space of all continuous linear functions from S into E, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^{n};E)$ is norm dense in $L^{p}(\mathbb{R}^{n};E)$ when $1 \leq p < \infty$. Let m be a positive integer. $W^{m,p}(\Omega;E)$ denotes an E-valued Sobolev space of all functions $u \in L^{p}(\Omega;E)$ that have the generalized derivatives $\frac{\partial^{m}u}{\partial x_{k}^{m}} \in L^{p}(\Omega;E)$ with the norm

$$\|u\|_{W^{m,p}(\Omega;E)} = \|u\|_{L^{p}(\Omega;E)} + \sum_{k=1}^{n}\left\|\frac{\partial^{m}u}{\partial x_{k}^{m}}\right\|_{L^{p}(\Omega;E)} < \infty.$$

Let $W^{s,p}(\mathbb{R}^{n};E)$ denotes the fractional Sobolev space of order $s \in \mathbb{R}$, that is defined as:

$$H^{s,p}(E) = H^{s,p}(\mathbb{R}^{n};E) = \{u \in S'(\mathbb{R}^{n};E),$$

$$\left.\|u\|_{H^{s,p}(E)} = \left\|\mathbb{F}^{-1}\left(I + |\xi|^{2}\right)^{\frac{s}{2}}\hat{u}\right\|_{L^{p}(\mathbb{R}^{n};E)} < \infty\right\}.$$

It clear that $H^{0,p}(\mathbb{R}^{n};E) = L^{p}(\mathbb{R}^{n};E)$. Let E_0, E be two Banach spaces and E_0 is continuously and densely embedded into E. Here, $H^{s,p}(\mathbb{R}^{n};E_0,E)$ denote the Sobolev-Lions type space i.e.,

$$H^{s,p}(\mathbb{R}^{n};E_0,E) = \{u \in H^{s,p}(\mathbb{R}^{n};E) \cap L^{p}(\mathbb{R}^{n};E_0),$$

$$\left.\|u\|_{H^{s,p}(\mathbb{R}^{n};E_0,E)} = \|u\|_{L^{p}(\mathbb{R}^{n};E_0)} + \|u\|_{H^{s,p}(\mathbb{R}^{n};E)} < \infty\right\}.$$

In a similar way, we define the following Sobolev-Lions type space:

$$W^{2,s,p}(\mathbb{R}_{T}^{n};E_0,E) = \{u \in L^{p}(\mathbb{R}_{T}^{n};E_0),\ \partial_{t}^{2}u \in L^{p}(\mathbb{R}_{T}^{n};E),$$

$$\mathbb{F}_{x}^{-1}\left(I + |\xi|^{2}\right)^{\frac{s}{2}}\hat{u} \in L^{p}(\mathbb{R}_{T}^{n};E),\ \|u\|_{W^{2,s,p}\left(\mathbb{R}_{T}^{n};E_0,E\right)} =$$

$$\left.\|u\|_{L^{p}\left(\mathbb{R}_{T}^{n};E_0\right)} + \left\|\partial_{t}^{2}u\right\|_{L^{p}\left(\mathbb{R}_{T}^{n};E\right)} + \left\|\mathbb{F}_{x}^{-1}\left(I + |\xi|^{2}\right)^{\frac{s}{2}}\hat{u}\right\|_{L^{p}\left(\mathbb{R}_{T}^{n};E\right)} < \infty\right\}.$$

Let $L_{q}^{*}(E)$ denote the space of all E-valued function space such that

$$\|u\|_{L_{q}^{*}(E)} = \left(\int_{0}^{\infty}\|u(t)\|_{E}^{q}\frac{dt}{t}\right)^{\frac{1}{q}} < \infty,\ 1 \leq q < \infty,\ \|u\|_{L_{\infty}^{*}(E)} = \sup_{0<t<\infty}\|u(t)\|_{E}.$$

Let $s>0$. Fourier-analytic representation of E-valued Besov space on $\mathbb{R}^n$ are defined as:

$$B_{p,q}^{s}\left(\mathbb{R}^n;E\right)=\Big\{u\in S^{'}\left(\mathbb{R}^n;E\right),$$

$$\|u\|_{B_{p,q}^{s}(\mathbb{R}^n;E)}=\left\|\mathbb{F}^{-1}\sum_{k=1}^{n}t^{\varkappa-s}\left(1+|\xi|^2\right)^{\frac{\varkappa}{2}}e^{-t|\xi|^2}\mathbb{F}u\right\|_{L_q^*(L^p(\mathbb{R}^n;E))},$$

$$p\in(1,\infty)\,,\;q\in[1,\infty]\,,\;\varkappa>s\}\,.$$

It should be note that, the norm of Besov space does not depends on $\varkappa$, (see e.g. [33, § 2.3] for $E=\mathbb{C}$).

Let A be a sectorial operator in Hilbert space H. Here,

$$X_p=L^p\left(\mathbb{R}^n;H\right),\;X_p\left(A^\gamma\right)=L^p\left(\mathbb{R}^n;H\left(A^\gamma\right)\right),\;1\leq p,\;q\leq\infty,$$

$$Y^{\;s,p}=Y^{\;s,p}\left(H\right)=H^{s,p}\left(\mathbb{R}^n;H\right),\;Y_q^{s,p}\left(H\right)=Y^{\;s,p}\left(H\right)\cap X_q,$$

$$\|u\|_{Y_q^{s,p}}=\|u\|_{H^{s,p}(\mathbb{R}^n;H)}+\|u\|_{X_q}<\infty,$$

$$H^{s,p}\left(A^\gamma\right)=H^{s,p}\left(\mathbb{R}^n;H\left(A^\gamma\right)\right),\;0<\gamma\leq1,$$

$$Y^{s,p}=Y^{s,p}\left(A,H\right)=H^{s,p}\left(\mathbb{R}^n;H\left(A\right),H\right),\;Y^{2,s,p}=Y^{2,s,p}\left(A,H\right)=$$

$$W^{2,s,p}\left(\mathbb{R}_T^n;H\left(A\right),H\right),\;Y_q^{s,p}\left(A;H\right)=Y^{s,p}\left(H\right)\cap X_q\left(A\right),$$

$$\|u\|_{Y_q^{s,p}(A,H)}=\|u\|_{Y^{s,p}(H)}+\|u\|_{X_q(A)}<\infty,$$

$$\mathbb{H}_{0p}=\left(Y^{s,p}\left(A,H\right),X_p\right)_{\frac{1}{2p},p},\;\mathbb{H}_{1p}=\left(Y^{s,p}\left(A,H\right),X_p\right)_{\frac{1+p}{2p},p},$$

where $\left(Y^{s,p},X_p\right)_{\theta,p}$ denotes the real interpolation space between $Y^{s,p}$ and X_p for $\theta\in(0,1)$, $p\in[1,\infty]$ (see e.g. [33, §1.3]).

Remark 1.1. By Fubini's theorem we get

$$L^p\left(\mathbb{R}_T^n;H\right)=L^p\left(0,T;X_p\right)\;\text{ for }X_p=L^{p}\left(\mathbb{R}^n;H\right).$$

Then by definition of spaces $Y^{2,s,p}$, $Y^{s,p}=H^{s,p}\left(\mathbb{R}^n;H\left(A\right),H\right)$ and X_p we have

$$Y^{2,s,p}=\big\{u\colon\;u\in W^{2,p}\left(0,T;Y^{s,p},X_p\right),\;\;\|u\|_{W^{2,p}(0,T;Y^{s,p},X_p)}=$$

$$\|u\|_{L^p(0,T;Y^{s,p})}+\left\|u^{(2)}\right\|_{L^p(0,T;X_p)}\Big\}.$$

By J. lions-J. Peetre result (see e.g. [33, §1.8.2]) the trace operator $u\rightarrow\frac{\partial^i u}{\partial t^i}\left(.,t_0\right)$ is bounded from $Y^{2,s,p}$ into

$$\left(Y^{s,p},X_p\right)_{\theta_j,p},\;\theta_j=\frac{1+jp}{2p},\;j=0,1.$$

Moreover, if $u\left(x,.\right)\in\left(Y^{s,p},X_p\right)_{\theta_j,p}$, then under some assumptions that will be stated in Section 3, $f\left(u\right)\in H$ for all $x,\;t\in\mathbb{R}_T^n$ and the map $u\rightarrow f\left(u\right)$ is bounded from $\left(Y^{s,p},X_p,\right)_{\frac{1}{2p},p}$ into H. Hence, the nonlinear equation (1.1) is satisfied in the Hılbert space H. Here, $H\left(A\right)$ denotes a domain of A ın H equipped with graphical norm.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α. Moreover, for u, $\upsilon>0$ the relations $u\lesssim\upsilon$, $u\approx\upsilon$ means that there exist positive constants C, C_1, C_2 independent on u and υ such that, respectively

$$u\leq C\upsilon,\;C_1\upsilon\leq u\leq C_2\upsilon.$$

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a priory estimates for solution of the linearized problem $(1.1)-(1.2)$. In Section 3, we show the existence and uniqueness of local and global strong solution of the problem $(1.1)-(1.2)$. In the Section 4, we show some applications of our general results in abstract spaces $L^p(\mathbb{R}^n;H)$.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say h, we write C_h.

2. Estimates for linearized equation

In this section, we make the necessary estimates to solutions of the integral problem for linear WE

$$u_{tt}-Lu+Au=g(x,t),\ x\in\mathbb{R}^n,\ t\in(0,T),\ T\in(0,\ \infty], \tag{2.1}$$

$$u(x,0)=\varphi(x)+\int\limits_0^T\eta(\sigma)u(x,\sigma)\,d\sigma, \tag{2.2}$$

$$u_t(x,0)=\psi(x)+\int\limits_0^T\beta(\sigma)u_t(x,\sigma)\,d\sigma,$$

where A is a linear operator in a Hilbert space H and $\eta(s)$, $\beta(s)$ are measurable functions on $(0,T)$.

Condition 2.0. Let $a_{ij}\in\mathbb{C}$. Suppose

$$L(\xi)=\sum_{i,j=1}^n a_{ij}\xi_i\xi_j\neq 0,$$

$L(\xi)\in S(\phi_1)$ for $\phi_1\in[0,\pi\ )$, $\xi\in\mathbb{R}^n$ and

$$|L(\xi)|\geq C|\xi|^2$$

for a positive constant C.

Remark 2.1. By properties of real interpolation of Banach spaces and interpolation of the intersection of the spaces (see e.g. [33, §1.3]) we obtain

$$\mathbb{H}_{0p}=\left(Y^{s,p}(A,H)\cap X_p,X_p\right)_{\frac{1}{2p},p}=$$

$$\left(Y^{s,p}(H),X_p\right)_{\frac{1}{2p},p}\cap\left(X_p(A),X_p\right)_{\frac{1}{2p},p}=$$

$$H^{s\left(1-\frac{1}{2p}\right),p}(\mathbb{R}^n;H)\cap L^p\left(\mathbb{R}^n;(H(A),H)_{\frac{1}{2p},p}\right)=$$

$$H^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n;(H(A),H)_{\frac{1}{2p},p},H\right).$$

In a similar way, we have

$$\mathbb{H}_{1p}=\left(Y^{s,p}(A,H)\cap X_p,X_p\right)_{\frac{1+p}{2p},p}=H^{\frac{s(p-1)}{2p},p}\left(\mathbb{R}^n;(H(A),H)_{\frac{1+p}{2p},p},H\right).$$

Remark 2.2. Let A be a sectorial operator in a Banach space E. In view of interpolation of sectorial operators (see e.g.[33, §1.8.2]) we have the following relation

$$E\left(A^{1-\theta+\varepsilon}\right)\subset(E(A),E)_{\theta,p}\subset E\left(A^{1-\theta-\varepsilon}\right)$$

for $0<\theta<1$ and $0<\varepsilon<1-\theta$.

We assume that A is a sectorial operator in a Hilbert space H. Let A be a generator of a strongly continuous cosine operator function in a Banach space H defined by formula

$$C(t)=C_A(t)=\frac{1}{2}\left(e^{itA^{\frac{1}{2}}}+e^{-itA^{\frac{1}{2}}}\right)$$

(see e.g. [23, §11] or [24, § 3]). Then, from the definition of sine operator-function $S(t)$ we have

$$S(t)=S_A(t)=\int_0^t C(\sigma)\,d\sigma,\text{ i.e. } S(t)=\frac{1}{2i}A^{-\frac{1}{2}}\left(e^{itA^{\frac{1}{2}}}-e^{-itA^{\frac{1}{2}}}\right).$$

Condition 2.1. Assume:(1)

$$\left|1+\int_0^T\eta(\sigma)\beta(\sigma)\,d\sigma\right|>\int_0^T(|\eta(\sigma)|+|\beta(\sigma)|)\,d\sigma; \tag{2.3}$$

(2) A is a ϕ-sectorial operator in the Hilbert space H for $0\leq\phi<\pi$ and A is a generator of a cosine function; (3) $\|A^\nu u\|\lesssim\left\|A^\theta u\right\|$ for $0\leq\nu\leq\theta$ and $u\in D\left(A^\theta\right)$; (4) Condition 2.0 holds; (5) $\varphi\in\mathbb{H}_{0p}$ and $\psi\in\mathbb{H}_{1p}$.

Definition 1.1. Let $T>0$, $\varphi\in\mathbb{H}_{0p}$ and $\psi\in\mathbb{H}_{1p}$. The function $u\in C\left([0,T];Y_1^{s,p}(A)\right)$ satisfies of the problem $(1.1)-(1.2)$ is called the continuous solution or the strong solution of $(1.1)-(1.2)$. If $T<\infty$, then $u(x,t)$ is called the local strong solution of $(1.1)-(1.2)$. If $T=\infty$, then $u(x,t)$ is called the global strong solution of $(1.1)-(1.2)$.

First we need the following lemmas:

Lemma 2.1. Let the Condition 2.1 holds. Then, problem $(2.1)-(2.2)$ has a solution.

Proof. By using of the Fourier transform, we get from $(2.1)-(2.2)$:

$$\hat{u}_{tt}(\xi,t)+A_L\hat{u}(\xi,t)=\hat{g}(\xi,t), \tag{2.4}$$

$$\hat{u}(\xi,0)=\hat{\varphi}(\xi)+\int_0^T\eta(\sigma)\,\hat{u}(\xi,\sigma)\,d\sigma, \tag{2.5}$$

$$\hat{u}_t(\xi,0)=\hat{\psi}(\xi)+\int_0^T\beta(\sigma)\,\hat{u}_t(\xi,\sigma)\,d\sigma,$$

where $\hat{u}(\xi,t)$ is a Fourier transform of $u(x,t)$ in x, $\hat{\varphi}(\xi)$, $\hat{\psi}(\xi)$ are Fourier transform of φ and ψ, respectively and

$$A_L=A_L(\xi)=[A+L(\xi)]^{\frac{1}{2}},\ \xi\in\mathbb{R}^n.$$

Consider first, the Cauchy problem

$$\hat{u}_{tt}(\xi,t)+A_L^2\hat{u}(\xi,t)=\hat{g}(\xi,t), \tag{2.6}$$

$$\hat{u}(\xi,0)=u_0(\xi),\ \hat{u}_t(\xi,0)=u_1(\xi),\ \xi\in\mathbb{R}^n,\ t\in[0,T],$$

where $u_0(\xi)$, $u_1(\xi)\in D(A)$ for $\xi\in\mathbb{R}^n$.

By Condition 2.0, in view of the assumptions (2), (3) and by [27, Lemma 2.3], we get the following estimate

$$\left\|[A+L((\xi))+\lambda]^{-1}\right\|_{B(H)}\lesssim|L(\xi)+\lambda|^{-1}\lesssim\left(|\xi|^2+|\lambda|\right)^{-1}\lesssim|\lambda|^{-1},$$

uniformly in $\xi \in \mathbb{R}^n$ for $\lambda \in S(\phi_2)$ and $\xi \in \mathbb{R}^n$ with $0 < \phi_1 + \phi_2 \leq \phi < \pi$ for $\phi_2 > \frac{\pi}{2}$, i.e. $A + L(\xi)$ is uniform sectorial operator. Hence, in view [33, §1.15] the operator $A_L(\xi)$ is sectorial in H. Then, by virtue of [23, §11.2, 11.4] from here, we obtain that A_L is a generator of a strongly continuous cosine operator function and the Cauchy problem (2.6) has a unique solution for all $\xi \in \mathbb{R}^n$. Moreover, the solution of (2.6) can be expressed as

$$\hat{u}(\xi, t) = C(t) u_0(\xi) + S(t) u_1(\xi) + \int_0^t S(\xi, t-\tau, A) \hat{g}(\xi, \tau) d\tau, \ t \in (0, T), \tag{2.7}$$

where $C(t)$ is a cosine and $S(t)$ is a sine operator-functions generated by A_L, i.e.

$$C(t) = C(\xi, t, A) = \frac{1}{2}\left(e^{itA_L} + e^{-itA_L}\right),$$

$$S(t) = S(\xi, t, A) = \frac{1}{2i} A_L^{-1}\left(e^{itA_L} - e^{-itA_L}\right).$$

Using the formula (2.7) and the first integral condition in (2.5), we have

$$u_0(\xi) = \hat{\varphi}(\xi) + \int_0^T \eta(\sigma)\left[C(\sigma) u_0(\xi) + S(\sigma) u_1(\xi)\right] d\sigma +$$

$$\int_0^T \int_0^T \eta(\sigma) S(\xi, \sigma-\tau, A) \hat{g}(\xi, \sigma) d\tau d\sigma, \ \tau \in (0, T),$$

i.e. we obtain the first equation with respect to $u_0(\xi)$, $u_1(\xi)$:

$$b_{10}(\xi) u_0(\xi) + b_{11}(\xi) u_1(\xi) = g_{10}(\xi), \tag{2.8}$$

where

$$b_{10}(\xi) = \left[1 - \int_0^T \eta(\sigma) C(\sigma) d\sigma\right], \ b_{11}(\xi) = \int_0^T \eta(\sigma) S(\sigma) d\sigma,$$

$$g_{10}(\xi) = \hat{\varphi}(\xi) + \int_0^T \int_0^T \eta(\sigma) S(\xi, \sigma-\tau, A) \hat{g}(\sigma, \xi) d\tau d\sigma.$$

Differentiating both sides of formula (2.7) and using the seconf integral condition (2.5), we have

$$u_1(\xi) = \hat{\psi}(\xi) - \int_0^T \beta(\sigma)\left[A_L^2 S(\sigma) u_0(\xi) + C(\sigma) u_1(\xi)\right] +$$

$$\int_0^T \int_0^T \beta(\sigma) C(\xi, \sigma-\tau, A) \hat{g}(\xi, \sigma) d\tau d\sigma,$$

i.e. we get the second equation with respect to $u_0(\xi)$, $u_1(\xi)$:

$$b_{20}(\xi) u_0(\xi) + b_{21}(\xi) u_1(\xi) = g_{20}(\xi), \tag{2.9}$$

where

$$b_{20}(\xi) = -\int_0^T \beta(\sigma) A_L^2 S(\sigma)\, d\sigma,\ b_{21}(\xi) = \int_0^T \beta(\sigma)\left[1 - C(\sigma)\right] d\sigma,$$

$$g_{20}(\xi) = \hat{\psi}(\xi) + \int_0^T \int_0^T \beta(\sigma) C(\xi, \sigma - \tau, A)\, \hat{g}(\xi, \sigma)\, d\tau d\sigma.$$

Now, we consider the system of equations (2.8)-(2.9) in $u_0(\xi)$ and $u_1(\xi)$. By assumption (2.3) and due to uniformly boundedness of A_L^{-1}, we get that the operator function

$$D(\xi) = \begin{vmatrix} b_{10}(\xi) & b_{11}(\xi) \\ b_{20}(\xi) & b_{21}(\xi) \end{vmatrix} = \left[I - \int_0^T \eta(\sigma) C(\sigma)\, d\sigma\right] \left[\int_0^T \beta(\sigma)\left[1 - C(\sigma)\right] d\sigma\right] -$$

$$\left[\int_0^T \eta(\sigma) S(\sigma)\, d\sigma\right] \left[-\int_0^T \beta(\sigma) A_L^2 S(\sigma)\, d\sigma\right]$$

has a bounded inverse $D^{-1}(\xi)$ for all $\xi \in \mathbb{R}^n$. By solving the system (2.8)-(2.9), we have

$$u_0(\xi) = D_1(\xi) D^{-1}(\xi),\ u_1(\xi) = D_2(\xi) D^{-1}(\xi), \tag{2.10}$$

$$D_1(\xi) = b_{21}(\xi) g_{10}(\xi) - b_{11}(\xi) g_{20}(\xi),$$

$$D_2(\xi) = b_{10}(\xi) g_{20}(\xi) - b_{20}(\xi) g_{10}(\xi).$$

By substituting the values $u_0(\xi)$ and $u_1(\xi)$ in (2.7), we obtain

$$\hat{u}(\xi, t) = C(\xi, t) D_1(\xi) D^{-1}(\xi) + S(\xi, t) D_2(\xi) D^{-1}(\xi) + \int_0^t S(\xi, t - \tau)\, \hat{g}(\xi, \tau)\, d\tau, \tag{2.11}$$

i.e. problem $(2.1) - (2.2)$ has a solution

$$u(x, t) = C_1(t)\varphi + S_1(t)\psi + Qg, \tag{2.12}$$

where $C_1(t)$, $S_1(t)$, Q are linear operator functions defined by

$$C_1(t)\varphi = \mathbb{F}^{-1}\left[C(\xi, t) D_1(\xi) D^{-1}(\xi)\right] \hat{\varphi}(\xi),$$

$$S_1(t)\psi = \mathbb{F}^{-1}\left[S(\xi, t) D_2(\xi) D^{-1}(\xi)\right] \hat{\psi}(\xi),$$

$$Qg = \mathbb{F}^{-1}\tilde{Q}(\xi, t),\ \tilde{Q}(\xi, t) = \int_0^t \mathbb{F}^{-1}\left[S(\xi, t - \tau)\, \hat{g}(\xi, \tau)\right] d\tau.$$

Theorem 2.1. Assume the Condition 2.1 holds and

$$s > \frac{2pn}{(2p-1)q} \tag{2.13}$$

for $p \in [1, \infty]$ and for a $q \in [1, 2]$. Let $0 \leq \alpha < 1 - \frac{1}{2p}$. Then for $\varphi \in \mathbb{H}_{0p} \cap X_1(A^\alpha)$, $\psi \in \mathbb{H}_{1p} \cap X_1\left(A^{\alpha - \frac{1}{2}}\right)$, $g(., t) \in Y_1^{s,p}$ and $g(x, .) \in L^1(0, T; Y_1^{s,p})$ for $x \in \mathbb{R}^n$ with $t \in [0, T]$ problem $(2.1) - (2.2)$ has a unique solution $u(x,t) \in C([0, T]; X_\infty(A))$ and the following estimate holds

$$\|A^\alpha u\|_{X_\infty} \leq C_0 \left[\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha \varphi\|_{X_1} + \right. \tag{2.14.1}$$

$$\left\|\psi\right\|_{\mathbb{H}_{1p}}+\left\|A^{\alpha-\frac{1}{2}}\psi\right\|_{X_1}+\int\limits_0^t\left(\left\|g\left(.,\tau\right)\right\|_{Y_1^{s,p}}+\left\|g\left(.,\tau\right)\right\|_{X_1}\right)d\tau\right],$$

uniformly in $t\in[0,T]$, where the constant $C_0>0$ depends only on A, the space H and initial data.

Moreover, for $\varphi\in\mathbb{H}_{0p}\cap X_1\left(A^{\alpha+\frac{1}{2}}\right)$, $\psi\in\mathbb{H}_{1p}\cap X_1\left(A^{\alpha}\right)$, $g\left(.,t\right)\in Y_1^{s,p}\left(A^{\frac{1}{2}}\right)$ and $g\left(x,.\right)\in L^1\left(0,T;Y_1^{s,p}\left(A^{\frac{1}{2}}\right)\right)$ for $x\in\mathbb{R}^n$ the following estimate holds

$$\left\|A^{\alpha}u_t\right\|_{X_\infty}\leq C_0\left[\left\|\varphi\right\|_{\mathbb{H}_{0p}}+\left\|A^{\alpha+\frac{1}{2}}\varphi\right\|_{X_1}+\right. \tag{2.14.2}$$

$$\left.\left\|\psi\right\|_{\mathbb{H}_{1p}}+\left\|A^{\alpha}\psi\right\|_{X_1}+\int\limits_0^t\left(\left\|A^{\frac{1}{2}}g\left(.,\tau\right)\right\|_{Y_1^{s,p}}+\left\|A^{\frac{1}{2}}g\left(.,\tau\right)\right\|_{X_1}\right)d\tau\right].$$

Proof. By Lemma 2.1, the problem $(2.1)-(2.2)$ has a solution for $\varphi\in\mathbb{H}_{0p}$, $\psi\in\mathbb{H}_{1p}$ and $g\left(.,t\right)\in Y_1^{s,p}$. Let $N\in\mathbb{N}$ and

$$\Pi_N=\left\{\xi:\xi\in\mathbb{R}^n,\ |\xi|\leq N\right\},\ \Pi_N'=\left\{\xi:\xi\in\mathbb{R}^n,\ |\xi|\geq N\right\}.$$

From (2.12) we deduced that

$$\left\|A^{\alpha}u\right\|_{X_\infty}\lesssim\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^{\alpha}D_1\left(\xi\right)D^{-1}\left(\xi\right)\hat{\varphi}\left(\xi\right)\right\|_{L^\infty\left(\Pi_N\right)}+ \tag{2.15}$$

$$\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^{\alpha}D_2\left(\xi\right)D^{-1}\left(\xi\right)\hat{\psi}\left(\xi\right)\right\|_{L^\infty\left(\Pi_N\right)}+\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^{\alpha}D_1\left(\xi\right)D^{-1}\left(\xi\right)\hat{\varphi}\left(\xi\right)\right\|_{L^\infty\left(\Pi_N'\right)}+$$

$$\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^{\alpha}D_2\left(\xi\right)D^{-1}\left(\xi\right)\hat{\psi}\left(\xi\right)\right\|_{L^\infty\left(\Pi_N'\right)}+\left\|\mathbb{F}^{-1}A^{\alpha}\int\limits_0^t\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)d\tau\right\|_{L^\infty\left(\Pi_N\right)}+$$

$$\left\|\mathbb{F}^{-1}A^{\alpha}\int\limits_0^t\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)d\tau\right\|_{L^\infty\left(\Pi_N'\right)}.$$

By virtue of Remakes 2.1, 2.2 and the properties of sectorial operators, we have

$$\left\|\mathbb{F}^{-1}A^{\alpha}\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)\right\|_{L^\infty\left(\Pi_N\right)}\leq C\left\|g\right\|_{X_1}.$$

Hence, due to uniform boundedness of operator functions $C\left(\xi,t\right)$, $S\left(\xi,t\right)$ by (2.3), in view of $(2.8)-(2.10)$ and by Minkowski's inequality for integrals, we get the following uniform estimate

$$\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^{\alpha}D_1\left(\xi\right)D^{-1}\left(\xi\right)\hat{\varphi}\left(\xi\right)\right\|_{L^\infty\left(\Pi_N\right)}+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^{\alpha}D_2\left(\xi\right)D^{-1}\left(\xi\right)\hat{\psi}\left(\xi\right)\right\|_{L^\infty\left(\Pi_N\right)}\lesssim$$

$$\left[\left\|A^{\alpha}\varphi\right\|_{X_1}+\left\|A^{\alpha}\psi\right\|_{X_1}+\left\|g\right\|_{X_1}\right]. \tag{2.16}$$

Let

$$\Phi_0\left(\xi\right)=\left[A^{1-\frac{1}{2p}+\varepsilon_0}+\left(1+|\xi|^2\right)^{\frac{s}{2}\left(1-\frac{1}{2p}\right)}\right]^{-1},\ 0<\varepsilon_0<\frac{1}{2p}, \tag{2.17}$$

$$\Phi_1\left(\xi\right)=\left[A^{\frac{1}{2}-\frac{1}{2p}+\varepsilon_1}+\left(1+|\xi|^2\right)^{\frac{s}{2}\left(\frac{1}{2}-\frac{1}{2p}\right)}\right]^{-1},\ 0<\varepsilon_1<\frac{1}{2}+\frac{1}{2p}.$$

By using the resolvent properties of sectorial operators, for for $0 \leq \alpha < 1 - \frac{1}{2p}$ and in view of the assumption (3), we have

$$\left\| A^{\alpha} C(\xi, t) \Phi_0(\xi) \right\|_{B(H)} \lesssim \left\| A^{\alpha} A^{-\left(1 - \frac{1}{2p}\right)} A^{1 - \frac{1}{2p}} \Phi_0(\xi) \right\|_{B(E)} \lesssim \tag{2.18}$$

$$\left\| A^{\left(1 - \frac{1}{2p}\right)} \Phi_0(\xi) \right\|_{B(H)} \leq C \text{ for } 0 \leq \alpha < 1 - \frac{1}{2p},$$

$$\left\| A^{\alpha} S(\xi, t) \Phi_1(\xi) \right\|_{B(H)} \lesssim \left\| A^{1 - \frac{1}{2p}} \Phi_1(\xi) \right\|_{B(E)} \leq C.$$

Let $L^{\infty} = L^{\infty}(\Omega; H)$ and

$$l = \frac{s}{2}\left(1 - \frac{1}{2p}\right).$$

In view of (2.17) it is clear that

$$\left\| \mathbb{F}^{-1} C(\xi, t) A^{\alpha} D_1(\xi) D^{-1}(\xi) \hat{\varphi}(\xi) \right\|_{\Pi'_N} + \left\| \mathbb{F}^{-1} S(\xi, t) A^{\alpha} D_2(\xi) D^{-1}(\xi) \hat{\psi}(\xi) \right\|_{\Pi'_N} +$$

$$\left\| \mathbb{F}^{-1} S(\xi, t) A^{\alpha} \tilde{Q}(\xi, t) \hat{g}(\xi, \tau) \right\|_{\Pi'_N} \lesssim$$

$$\left\| \mathbb{F}^{-1} \left(1 + |\xi|^2\right)^{-\frac{l}{2}} C(\xi, t) A^{\alpha} \Phi_0 \hat{\varphi}(\xi) \right\|_{L^{\infty}} + \tag{2.19}$$

$$\left\| \mathbb{F}^{-1} \left(1 + |\xi|^2\right)^{-\frac{l}{2}} S(\xi, t) A^{\alpha} \Phi_1 \hat{\psi}(\xi) \right\|_{L^{\infty}} +$$

$$\left\| \mathbb{F}^{-1} \left(1 + |\xi|^2\right)^{-\frac{l}{2}} S(\xi, t) \left(1 + |\xi|^2\right)^{\frac{l}{2}} A^{\alpha} \tilde{Q}(\xi, t) \hat{g}(\xi, \tau) \right\|_{L^{\infty}}.$$

Then by calculating $\frac{\partial}{\partial \xi_k} \Phi_0(\xi)$, $\frac{\partial}{\partial \xi_k} \Phi_1(\xi)$, we obtain

$$A^{\alpha} \frac{\partial}{\partial \xi_k} \Phi_0(\xi) \in B(H), \; A^{\alpha} \frac{\partial}{\partial \xi_k} \Phi_1(\xi) \in B(H).$$

Let we show that $G_i(., t)$, $\Phi_i \in B_{q,1}^{n\left(\frac{1}{q} + \frac{1}{p}\right)}(\mathbb{R}^n; B(H))$ for some $q \in [1, 2]$ and for all $t \in [0, T]$, where

$$G_i(\xi, t) = \left(1 + |\xi|^2\right)^{-\frac{l}{2}} C(\xi, t) D_1(\xi) D^{-1}(\xi) \Phi_i(\xi),$$

$$\Pi_i(\xi, t) = \left(1 + |\xi|^2\right)^{-\frac{l}{2}} S(\xi, t) D_1(\xi) D^{-1}(\xi) \Phi_i(\xi), \; i = 0, 1.$$

By embedding properties of Sobolev and Besov spaces it sufficient to derive that $G_i \in W^{\sigma, q}(\mathbb{R}^n; B(H))$ for an integer σ with $\sigma \geq n\left(\frac{1}{q} + \frac{1}{p}\right)$. Indeed by by Condition 2.1 and by (2.18), (2.19) we get, G_i, $\Pi_i \in L^q(\mathbb{R}^n; B(H))$. For deriving the embeddıng relatıons G_i, $\Pi_i \in W^{\sigma, q}(\mathbb{R}^n; B(H))$, it sufficient to show

$$D_{\xi}^{\nu} G_i(., t), \; D_{\xi}^{\nu} \Pi_i(., t) \in L^{\sigma}(\mathbb{R}^n; B(H)) \text{ for } \nu = (\nu_1, \nu_2, ..., \nu_n) \text{ with } |\nu| \leq \sigma, \; t \in [0, T].$$

Indeed, by Condition 2.1 and by (2.18), (2.19) we have the following uniform estimates

$$\left\| \frac{\partial}{\partial \xi_k} G_i(., t) \right\| \lesssim \xi_k \left(1 + |\xi|^2\right)^{-\frac{l}{2} - 1} \left\| C(\xi, t) \Phi_i(\xi) \right\| +$$

$$\left(1+|\xi|^2\right)^{-\frac{l}{2}}\left[\left\|A_L C\left(t\right)\Phi_i\left(\xi\right)\frac{\partial}{\partial\xi_k}L\left(\xi\right)\right\|+\right.$$

$$\left.\left\|C\left(\xi,t\right)\frac{\partial}{\partial\xi_k}\Phi_i\left(\xi\right)\right\|\right]\leq C.$$

Then by again differentiating $G_i\left(.,t\right)$ and by using (2.18), (2.19), we get

$$G_i\left(.,t\right)\in W^{\sigma,q}\left(\mathbb{R}^n;B\left(H\right)\right)\text{ for some }q\in\left[1,2\right],\ i=0,1.$$

In a similar way, we obtain

$$\Pi_i\left(.,t\right)\in W^{\sigma,q}\left(\mathbb{R}^n;B\left(H\right)\right)\text{ for some }q\in\left[1,2\right].$$

Hence, by Fourier multiplier theorems (see e.g. [34, Theorem 4.3]), we found that the functions $G_i\left(\xi,t\right)$ and $\Pi_i\left(\xi,t\right)$ are Fourier multipliers from $L^p\left(\mathbb{R}^n;H\right)$ to $L^\infty\left(\mathbb{R}^n;H\right)$. Then by Minkowski's inequality for integrals, from (2.3), (2.16) – (2.19) and by Remake 2.3, we have

$$\left\|F^{-1}C\left(\xi,t\right)A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty}+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\hat{\psi}\left(\xi\right)\right\|_{L^\infty}\lesssim$$

$$\left\|F^{-1}C\left(\xi,t\right)\eta^{-2}\hat{\varphi}\right\|_{L^\infty}+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)\eta^{-1}\hat{\psi}\right\|_{L^\infty}\lesssim$$

$$\left[\|\varphi\|_{\mathbb{H}_{0p}}+\|\psi\|_{\mathbb{H}_{1p}}+\|g\|_{W^{s,p}}\right].\tag{2.20}$$

Moreover, by virtue of Remakes 2.1-2.3 and by reasoning as the above, we get the following estimate

$$\left\|F^{-1}A^\alpha\tilde{Q}\left(\xi,t\right)\right\|_{X_\infty}\leq C\int_0^t\left(\|g\left(.,\tau\right)\|_{W^{s,p}}+\|g\left(.,\tau\right)\|_{X_1}\right)d\tau\tag{2.21}$$

uniformly in $t\in[0,T]$. Thus, from (2.12), (2.20) and (2.21), we obtain the estimate (2.14.1).

By differentiating (2.12) in a similar way, we get the estimate (2.14.2).

Then from (2.22) and (2.23) in view of Remarks 2.1, 2.2 we obtain the estimate (2.14).

Let now, we show that problem (2.1) has a unique solution $u\in Y\left(T\right)$. Let's admit it is the opposite. So let's assume that the problem (2.1) has two solutions u_1, $u_2\in Y\left(T\right)$. Then by linearity of (2.1), we get that $\upsilon=u_1-u_2$ is also a solution of the corresponding homogenous equation

$$u_{tt}-Lu+Au=0,\ \upsilon\left(x,0\right)=0,\ \upsilon_t\left(x,0\right)=0,\ x\in\mathbb{R}^n,\ t\in\left(0,T\right).$$

Moreover, by (2.7) we have the following estimate

$$\|A^\alpha\upsilon\|_{X_\infty}\leq 0.$$

Since $N\left(A\right)=\{0\}$, the above estmate implies that $\upsilon=0$, i.e. $u_1=u_2$.

Theorem 2.2. Assume the Condition 2.1 and (2.13) are satisfied. Let $0\leq\alpha<1-\frac{1}{2p}$. Then $\varphi\in Y^{s,p}\left(A^\alpha\right)$, $\psi\in Y^{s,p}\left(A^{\alpha-\frac{1}{2}}\right)$, $g\left(.,t\right)\in Y^{s,p}$ for $t\in[0,T]$ and $g\left(x,.\right)\in L^1\left(0,T;Y^{s,p}\right)$ for $x\in\mathbb{R}^n$, we get the following estimate

$$\|A^\alpha u\|_{Y^{s,p}}\leq\tag{2.22.1}$$

$$C_0\left[\|A^\alpha\varphi\|_{Y^{s,p}}+\left\|A^{\alpha-\frac{1}{2}}\psi\right\|_{Y^{s,p}}+\int_0^t\|g\left(.,\tau\right)\|_{Y^{s,p}}d\tau\right]$$

uniformly in $t \in [0, T]$.

Moreover, for $\varphi \in Y^{s,p}\left(A^{\alpha+\frac{1}{2}}\right)$, $\psi \in Y^{s,p}\left(A^{\alpha}\right)$, $g\left(.,t\right) \in Y^{s,p}\left(A^{\frac{1}{2}}\right)$ for $t \in [0, T]$ and $g\left(x,.\right) \in L^{1}\left(0,T;Y^{s,p}\left(A^{\frac{1}{2}}\right)\right)$ for $x \in \mathbb{R}^{n}$ problem $(2.1)-(2.2)$ has a unique solution $u \in C\left([0,T];Y^{s,p}\left(A\right)\right)$ and the following uniform estimate holds

$$\left\|A^{\alpha}u_{t}\right\|_{Y^{s,p}} \leq \tag{2.22.2}$$

$$C_{0}\left[\left\|A^{\alpha+\frac{1}{2}}\varphi\right\|_{Y^{s,p}}+\left\|A^{\alpha}\psi\right\|_{Y^{s,p}}+\int\limits_{0}^{t}\left\|A^{\frac{1}{2}}g\left(.,\tau\right)\right\|_{Y^{s,p}}d\tau\right].$$

Proof. In view of (2.11) and (2.12), for proving (2.22.1) it is sufficient to show the following uniform estimate

$$\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}A^{\alpha}\hat{u}\right\|_{X_{p}}+\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}A^{\alpha}\hat{u}_{t}\right\|_{X_{p}} \leq \tag{2.23}$$

$$C_{0}\left[\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}C\left(\xi,t\right)A^{\alpha}\hat{\varphi}\right\|_{X_{p}}+\left\|\mathbb{F}^{-1}\left(1+|\xi|^{2}\right)^{\frac{s}{2}}A^{\alpha}S\left(\xi,t\right)\hat{\psi}\right\|_{X_{p}}+\right.$$

$$\left.\int\limits_{0}^{t}\left\|\left(1+|\xi|^{2}\right)^{\frac{s}{2}}A^{\alpha}\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)\right\|_{X_{p}}d\tau\right].$$

By using the Fourier multiplier theorem [34, Theorem 4.3] and by reasoning as in Theorem 2.1 we get that the operator functions

$$\left(1+|\xi|^{2}\right)^{-\frac{s}{2}}C\left(\xi,t\right),\left(1+|\xi|^{2}\right)^{-\frac{s}{2}}S\left(\xi,t\right),\left(1+|\xi|^{2}\right)^{-\frac{s}{2}}A^{\alpha}S\left(\xi,t\right)$$

are Fourier multipliers in $L^{p}\left(\mathbb{R}^{n};H\right)$ uniformly with respect to $t \in [0,T]$. Hence, for $\varphi \in Y^{s,p}\left(A^{\alpha}\right)$, $\psi \in Y^{s,p}\left(A^{\alpha-\frac{1}{2}}\right)$, $g\left(.,t\right) \in Y^{s,p}$ from (2.23) we get the estimate $(2.22.1)$.

Then by differentiating (2.12) in a similar way, for $\varphi \in Y^{s,p}\left(A^{\alpha+\frac{1}{2}}\right)$, $\psi \in Y^{s,p}\left(A^{\alpha}\right)$, $g\left(.,t\right) \in Y^{s,p}\left(A^{\frac{1}{2}}\right)$, we get the estimate $(2.22.2)$.

The uniqueness of problem is derived in a similar way as in Therem 2.1.

3. Local well posedness of IVP for nonlinear WE

In this section, we will show the local existence and uniqueness of solution of the nonlinear problem $(1.1)-(1.2)$.

For this aim we need the following lemmas. By reasoning as in [11, 20, 35], we show the following lemmas concerning the behaviour of the nonlinear term in H-valued space $Y^{s,p}$.

Here, we assume that $\Phi \subset W^{s,p}\left(\Omega;H\right) \cap L^{\infty}\left(\Omega;H\right)$ such that $f\left(u\right) \in H$ for $u \in \Phi$ and $x \in \Omega$.

Lemma 3.1. Let $s \geq 0$, $f \in C^{[s]+1}\left(\mathbb{R};H\right)$ with $f(0)=0$. Then for any $u \in Y^{s,p} \cap L^{\infty}$, we have $f(u) \in Y^{s,p} \cap X_{\infty}$. Moreover, there is some constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^{\infty}$ with $\|u\|_{X_{\infty}} \leq M$,

$$\|f(u)\|_{Y^{s,p}} \leq C\left(M\right)\|u)\|_{Y^{s,p}}. \tag{3.1}$$

Proof. For $s=0$ in view of $f(0)=0$, we get

$$f\left(u\right)=\int\limits_{0}^{1}f^{(1)}\left(\sigma u\right)d\sigma.$$

It follows that

$$\|f\left(u\right)\|_{X_p} \leq C\left(M\right)\|u\|_{X_p}.$$

If s is a positive integer, we have

$$\|f(u)\|_{Y^{s,p}} \leq C\left[\|f(u)\|_{X_p} + \sum_{k=1}^{n}\left\|\frac{\partial^s}{\partial x_k}f(u)\right\|_{X_p}\right]. \tag{3.2}$$

By calculation of derivative and applying Holder inequality, we get

$$\left\|\frac{\partial^s}{\partial x_i}f(u)\right\|_{X_p} \leq \sum_{l=1}^{s}\sum_{\alpha}\left\|f^{(l)}(u)\frac{\partial^{\beta_1}u}{\partial x_i}\frac{\partial^{\beta_2}u}{\partial x_i}...\frac{\partial^{\beta_l}u}{\partial x_i}\right\|_{X_p} \leq$$

$$\sum_{l=1}^{s}\sum_{\alpha}\left\|f^{(l)}(u)\right\|_{X_\infty}\prod_{k=1}^{l}\left\|\frac{\partial^{\beta_k}u}{\partial x_i}\right\|_{X_{p_k}}, \; i=1,2,...,n, \tag{3.3}$$

where

$$\beta = \left(\beta_1, \beta_2, , , , .\beta_l\right), \; \beta_k \geq 1, \; \beta_1 + \beta_2 + ... + \beta_l = l, \; p_k = \frac{pl}{\beta_k}.$$

Applying Gagliardo-Nirenberg's inequality in E-valued X_p spaces, we have

$$\left\|\frac{\partial^{\beta_k}u}{\partial x_i}\right\|_{X_{p_k}} \leq C\|u\|_{X_\infty}^{1-\frac{\beta_k}{l}}\left\|\frac{\partial^s u}{\partial x_i^s}\right\|_{X_p}^{\frac{\beta_k}{l}}. \tag{3.4}$$

Hence, from (3.3) and (3.4) we get

$$\left\|\frac{\partial^s}{\partial x_i}f(u)\right\|_{X_p} \leq C\left(M\right)\left\|\frac{\partial^s u}{\partial x_i^s}\right\|_{X_p}. \tag{3.5}$$

Then combining (3.2), (3.3) and (3.5) we obtain (3.1).

Let s is not integer number and $m=[s]$. From the above proof, we have

$$\|f(u)\|_{Y^{m,p}} \leq C\left(M\right)\|u)\|_{Y^{m,p}}, \; \|f(u)\|_{Y^{m+1,p}} \leq C\left(M\right)\|u)\|_{Y^{m+1,p}}.$$

Then using interpolation between $W^{m+1,p}$ and $W^{m,p}$ yields (3.1) for all $s \geq 0$.

By using Lemma 3.1 and properties of convolution operators we obtain

Corollary 3.1. Let $s \geq 0$, $f \in C^{[s]+1}\left(\mathbb{R}; H\right)$ with $f(0)=0$. Moreover, assume $\Phi \in L^\infty\left(\mathbb{R}^n; B\left(H\right)\right)$. Then for any $u \in Y^{s,p} \cap L^\infty$ we have, $f(u) \in Y^{s,p} \cap X_\infty$. Moreover, there is some constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^\infty$ with $\|u\|_{X_\infty} \leq M$,

$$\|\Phi * f(u)\|_{Y^{s,p}} \leq C\left(M\right)\|u)\|_{Y^{s,p}}.$$

Lemma 3.2. Let $s \geq 0$, $f \in C^{[s]+1}\left(\mathbb{R}; H\right)$. Then for any M there is some constant $K(M)$ depending on M such that for all u, $v \in Y^{s,p} \cap X_\infty$ with $\|u\|_{X_\infty} \leq M$, $\|v\|_{X_\infty} \leq M$, $\|u\|_{Y^{s,p}} \leq M$, $\|v\|_{Y^{s,p}} \leq M$,

$$\|f(u) - f(v\|_{Y^{s,p}} \leq K\left(M\right)\|u-v\|_{Y^{s,p}}, \; \|f(u) - f(v\|_{X_\infty} \leq K\left(M\right)\|u-v\|_{X_\infty}.$$

Corollary 3.2. Let $s > \frac{n}{2}$, $f \in C^{[s]+1}\left(\mathbb{R}; H\right)$. Then for any positive M there is a constant $K(M)$ depending on M such that for all u, $v \in Y^{s,p}$ with $\|u\|_{Y^{s,p}} \leq M$, $\|v\|_{Y^{s,p}} \leq M$,

$$\|f(u) - f(v\|_{Y^{s,p}} \leq K\left(M\right)\|u-v\|_{Y^{s,p}}.$$

Lemma 3.3. If $s > 0$, then $Y_{\infty}^{s,p}$ is an algebra. Moreover, for $f,\, g \in Y_{\infty}^{s,p}$,

$$\|fg\|_{Y^{s,p}} \leq C\left[\|f\|_{X_\infty} + \|g\|_{Y^{s,p}} + \|f\|_{Y^{s,p}} + \|g\|_{X_\infty}\right].$$

By using, the Corollary 3.1 and Lemma 3.3 we obtain

Lemma 3.4. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; H)$ and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$, $\gamma \geq 1$ be a positive integer. If $u \in Y_{\infty}^{s,p}$ and $\|u\|_{X_\infty} \leq M$, then

$$\|f(u)\|_{Y^{s,p}} \leq C(M)\left[\|u\|_{Y^{s,p}}\|u\|_{X_\infty}^{\gamma}\right],$$

$$\|f(u)\|_{X_1} \leq C(M)\|u\|_{X_p}^{p}\|u\|_{X_\infty}^{\gamma-1}.$$

Lemma 3.5. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; H)$ and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$. Moreover, let $\gamma \geq 0$ be a positive integer. If $u,\, \upsilon \in Y_{\infty}^{s,p}$, $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$ and $\|u\|_{X_\infty} \leq M$, $\|\upsilon\|_{X_\infty} \leq M$, then

$$\|f(u) - f(\upsilon)\|_{Y^{s,p}} \leq C(M)\left[\left(\|u\|_{X_\infty} - \|\upsilon\|_{X_\infty}\right)\left(\|u\|_{Y^{s,p}} + \|\upsilon\|_{Y^{s,p}}\right)\right.$$

$$\left(\|u\|_{X_\infty} + \|\upsilon\|_{X_\infty}\right)^{\gamma-1},$$

$$\|f(u) - f(\upsilon\|_{X_1} \leq C(M)\left(\|u\|_{X_\infty} + \|\upsilon\|_{X_\infty}\right)^{\gamma-1}\left(\|u\|_{X_p} + \|\upsilon\|_{X_p}\right)\|u - \upsilon\|_{X_p}.$$

Let $\mathbb{H}_0$ denotes the real interpolation space between $Y^{s,p}(A,H)$ and X_p with $\theta = \frac{1}{2p}$, i.e.

$$\mathbb{H}_{0p} = \left(Y^{s,p}(A,H), X_p\right)_{\frac{1}{2p},p}.$$

Remark 3.1. By using J. Lions-J. Peetre result (see e.g [33, § 1.8]) we obtain that the map $u \to u(t_0)$, $t_0 \in [0,T]$ is continuous and surjective from $Y^{2,s,p}(A,H)$ onto $\mathbb{H}_{0p}$ and there is a constant C_1 such that

$$\|u(t_0)\|_{\mathbb{H}_{0p}} \leq C_1\|u\|_{Y^{2,s,p}(A,H)},\ 1 \leq p \leq \infty. \tag{3.6}$$

Here,

$$Y_0 = Y_0(A) = Y_{\infty}^{s,p}(A,H).$$

First all of, we define the space $Y(T) = C([0,T]; Y_0)$ equipped with the norm defined by

$$\|u\|_{Y(T)} = \max_{t\in[0,T]}\left[\|u\|_{Y^{s,p}(A,H)} + \|u\|_{X_\infty}\right],\ u \in Y(T).$$

Condition 3.1. Assume:

(1) the Condition 2.1 holds for $s > \frac{2pn}{(2p-1)q}$, $p \in [1,\infty]$, for a $q \in [1,2]$ and $0 \leq \alpha < 1 - \frac{1}{2p}$;

(3) the function $u \to f(u)$: continuous from $u \in \mathbb{H}_{0p}$ into H, $f \in C^k(\mathbb{R}; H)$ with k an integer, $k \geq s > \frac{n}{p}$ and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$, $\gamma \geq 1$ be a positive integer.

Main aim of this section is to prove the following results:

Theorem 3.1. Let the Condition 3.1 holds. Then there exists a constant $\delta > 0$ such that for any $\varphi \in \mathbb{H}_{0p} \cap X_1(A^\alpha)$, and $\psi \in \mathbb{H}_{1p} \cap X_1(A^\alpha)$ satisfying

$$\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha\varphi\|_{X_1} + \|\psi\|_{\mathbb{H}_{1p}} + \|A^\alpha\psi\|_{X_1} \leq \delta, \tag{3.7}$$

problem (1.1) – (1.2) has a unique local strong solution $u \in C([0,\ T_0); Y_0)$, where T_0 is a maximal time interval that is appropriately relative to δ. Moreover, if

$$\sup_{t\in[0,\ T_0)}\left(\|u\|_{Y_0} + \|u_t\|_{Y_0}\right) < \infty \tag{3.8}$$

then $T_0 = \infty$.

Proof. By (2.5), ((2.6)) the problem of finding a solution u of $(1.1) - (1.2)$ is equivalent to finding a fixed point of the mapping

$$G(u) = C_1(t)\varphi(x) + S_1(t)\psi(x) + Q(u), \tag{3.9}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.6) and $Q(u)$ is a map defined by

$$Q(u) = -\int_0^t \mathbb{F}^{-1}\left[U(\xi, t-\tau)\hat{f}(u)(\xi,\tau)\right] d\tau.$$

We define the metric space

$$Q(T,A) = \left\{u \in Y(T),\ \|u\|_{Y(T)} \leq \delta + 1\right\}$$

where $\delta > 0$ satisfies (3.7) and C_0 is a constant in Theorem 2.1 and 2.2. It is easy to prove that $Q(T,A)$ is a complete metric space. From imbedding in Sobolev-Lions space $Y^{s,p}(A,H)$ (see e.g. [27] Theorem 1) and trace result (3.6) we got that $\|u\|_{X_\infty} \leq 1$ if we take that δ is enough small. For $\varphi \in Y_0(A^\alpha)$ and $\psi \in Y_1(A^\alpha)$, let

$$\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha\varphi\|_{X_1} + \|\psi\|_{\mathbb{H}_{1p}} + \|A^\alpha\psi\|_{X_1} = \delta.$$

So, we will find T and δ so that G is a contraction in $Q(T,A)$. By Theorems 2.1, 2.2 and Corollary 3.3 $f(u) \in Y_1^{s,p}$. So, problem $(1.1) - (1.2)$ has a solution satisfies the following

$$G(u)(x,t) = C_1(t)\varphi + S_1(t)\psi + Q(u), \tag{3.10}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.5) and (2.6). By assumptions, it is easy to see that the map G is well defined for $f \in C^{[s]+1}(\mathbb{H}_{0p}; H)$. First, let us prove that the map G has a unique fixed point in $Q(T,A)$. For this aim, it is sufficient to show that the operator G maps $Q(T,A)$ into $Q(T,A)$ and G is strictly contractive if δ is suitable small. In fact, by (2.7) in Theorem 2.1, Corollary 3.3 and in view of (3.7), we have

$$\|A^\alpha G(u)\|_{X_\infty} + \|A^\alpha G_t(u)\|_{X_\infty} \leq 2C_0\left[\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha\varphi\|_{X_1} + \right. \tag{3.11}$$

$$\left. \|\psi\|_{\mathbb{H}_{1p}} + \|A^\alpha\psi\|_{X_1} + \int_0^t \left(\left\|\hat{f}((u))\right\|_{Y^{s,p}} + \left\|\hat{f}((u))\right\|_{X_1}\right) d\tau\right] \leq$$

$$2C_0\delta + C\int_0^t \left(\|u(\tau)\|_{Y^{s,p}}\|u(\tau)\|_{X_\infty}^\gamma + \|u(\tau)\|_{X_p}^p\|u(\tau)\|_{X_\infty}^{\gamma-1}\right) d\tau \leq$$

$$2C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}.$$

On the other hand, by (2.17), Corollary 3.3 and (3.7), we get

$$(\|A^\alpha G(u)\|_{Y^{s,p}} + \|A^\alpha G_t(u)\|_{Y^{s,p}}) \leq \tag{3.12}$$

$$2C_0\left(\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha\varphi\|_{X_1} + \|\psi\|_{\mathbb{H}_{1p}} + \|A^\alpha\psi\|_{X_1} + \int_0^t \left\|\hat{f}((u))\right\|_{Y^{s,p}} d\tau\right) \leq$$

$$2C_0\delta + \int_0^t \left[\|u(\tau)\|_{Y^{s,p}}\|u(\tau)\|_{X_\infty}^\gamma\right] d\tau \leq 2C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}.$$

Hence, combining (3.11) with (3.12) we obtain

$$\|A^{\alpha}G(u)\|_{Y_{\infty}^{s,p}}+\|A^{\alpha}G_{t}(u)\|_{Y_{\infty}^{s,p}}\leq 4C_{0}\delta+C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}. \tag{3.13}$$

So, taking that δ is enough small such that $C(5C_{0}\delta)^{\gamma}<\frac{1}{5}$, by Theorems 2.1, 2.2 and (3.13), G maps $Q(T,A)$ into $Q(T,A)$. Now, we are going to prove that the map G is strictly contractive. Let $u_{1},\ u_{2}\in Q(T,A)$ given. From (3.10), we get

$$G(u_{1})-G(u_{2})=$$

$$\int_{0}^{t}\left[S(x,t-\tau)\left(\hat{f}(u_{1})(\tau)-\hat{f}(u_{2})(\tau)\right)\right]d\tau,\ t\in(0,T).$$

By (2.7) in Theorem 2.1 and Corollary 3.3, we have

$$\|A^{\alpha}[G(u_{1})-G(u_{2})]\|_{X_{\infty}}+\|A^{\alpha}[G(u_{1})-G(u_{2})]_{t}\|_{X_{\infty}}\leq \tag{3.14}$$

$$\int_{0}^{t}\left(\left\|\left[\hat{f}(u_{1})-\hat{f}(u_{2})\right]\right\|_{Y^{s,p}}+\left\|\left[\hat{f}(u_{1})-\hat{f}(u_{2})\right]\right\|_{X_{1}}\right)d\tau\lesssim$$

$$\int_{0}^{t}\{\|u_{1}-u_{2}\|_{X_{\infty}}(\|u_{1}\|_{Y^{s,p}}+\|u_{2}\|_{Y^{s,p}})\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma-1}+$$

$$\|u_{1}-u_{2}\|_{Y^{s,p}}\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma}+$$

$$\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma-1}\|u_{1}+u_{2}\|_{X_{p}}\|u_{1}-u_{2}\|_{X_{p}}\Big\}.$$

On the other hand, by (2.17) in Theorem 2.2, Corollary 3.3 and (3.7), we get

$$\left(\|A^{\alpha}[G(u_{1})-G(u_{2})]\|_{Y^{s,p}}+\|A^{\alpha}[G(u_{1})-G(u_{2})]_{t}\|_{Y^{s,p}}\right)\leq$$

$$C\int_{0}^{t}\left\|\hat{f}(u_{1})(\tau)-\hat{f}(u_{2})(\tau)\right\|_{Y^{s,p}}d\tau\leq \tag{3.15}$$

$$C\int_{0}^{t}\Big\{\|u_{1}-u_{2}\|_{X_{\infty}}(\|u_{1}\|_{Y^{s,p}}+\|u_{2}\|_{Y^{s,p}})\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma-1}+$$

$$\|u_{1}-u_{2}\|_{Y^{s,p}}\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma}\Big\}d\tau.$$

Combining (3.14) with (3.15) yields

$$\|G(u_{1})-G(u_{2})\|_{Y(T)}\leq \tag{3.16}$$

$$C\left(\|u_{1}\|_{Y(T)}+\|u_{2}\|_{Y(T)}\right)^{\gamma}\|u_{1}-u_{2}\|_{Y(T)}.$$

Taking δ is enough small, from (3.16) we obtain that G is strictly contractive in $Q(T,A)$. Using the contraction mapping principle, we get that $G(u)$ has a unique fixed point $u(x,t)\in Q(T,A)$ and $u(x,t)$ is the solution of $(1.1)-(1.2)$. Let us show that this solution is a unique in $Y(T)$. Let $u_{1},\ u_{2}\in Y(T)$ are two solution of $(1.1)-(1.2)$. Then for $u=u_{1}-u_{2}$, we have

$$u_{tt}-Lu+Au=[f(u_{1})-f(u_{2})]. \tag{3.17}$$

Hence, by Minkowski's inequality for integrals and by Theorem 2.2 from (3.17), we obtain

$$\|u_1 - u_2\|_{Y^{s,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{s,p}} d\tau. \tag{3.18}$$

From (3.18) and Gronwall's inequality, we have $\|u_1 - u_2\|_{Y^{s,p}} = 0$, i.e. problem $(1.1) - (1.2)$ has a unique solution in $Y(T)$. That is, we obtain the first part of the assertion. Now, let $[0\ , T_0)$ be the maximal time interval of existence for $u \in Y(T_0)$. It remains only to show that if (3.8) is satisfied, then $T_0 = \infty$. Assume contrary that, (3.8) holds and $T_0 < \infty$. For $T \in [0,\ T_0)$, we consider the following integral equation

$$\upsilon(x,t) = C_1(t)\, u(x,T) + S_1(t)\, u_t(x,T) - \tag{3.19}$$

$$\int_0^t \mathbb{F}^{-1}\left[S(t-\tau,\xi)\, \hat{f}(\upsilon)(\xi,\tau)\right] d\tau,\ t \in (0,T).$$

By virtue of (3.8) for $T' > T$, we have

$$\sup_{t \in [0,\ T)} \left(\|u\|_{Y_0} + \|u_t\|_{Y_0}\right) < \infty.$$

By reasoning as a first part of theorem and by contraction mapping principle, there is a $T^* \in (0, T_0)$ such that for each $T \in [0,\ T_0)$, the equation (3.19) has a unique solution $\upsilon \in Y(T^*)$. By reasoning as in the first part T^* can be selected independently of $T \in [0,\ T_0)$. Set $T = T_0 - \frac{T^*}{2}$ and define

$$\tilde{u}(x,t) = \begin{cases} u(x,t),\ t \in [0,T] \\ \upsilon(x,t-T),\ t \in \left[T, T_0 + \frac{T^*}{2}\right] \end{cases}.$$

By construction $\tilde{u}(x,t)$ is a solution of the problem $(1.1) - (1.2)$ on $\left[T, T_0 + \frac{T^*}{2}\right]$ and in view of local uniqueness, $\tilde{u}(x,t)$ extends u. This is against to the maximality of $[0,\ T_0)$, i.e we obtain $T_0 = \infty$.

4. Application to degenerate wave equations

Consider the problem (1.4). Let

$$X_{p,2} = L^p\left(\mathbb{R}^n; L^2(0,1)\right), Y^{s,p,2} = H^{s,p}\left(\mathbb{R}^n; L^2(0,1)\right),$$

$$Y_q^{s,p,2} = H^{s,p}\left(\mathbb{R}^n; L^2(0,1)\right) \cap L^q\left(\mathbb{R}^n; L^2(0,1)\right),$$

$$Y^{s,p,2} = H^{s,p}\left(\mathbb{R}^n; H^2(0,1), L^2(0,1)\right),\ 1 \leq p \leq \infty,$$

$$H_{0p,2} = \left(Y^{s,p}\left(A, L^2(0,1)\right) \cap X_{p.2}, X_{p,2}\right)_{\frac{1}{2p},p},$$

$$H_{1p,2} = \left(Y^{s,p}\left(A, L^2(0,1)\right) \cap X_{p,2}, X_{p,2}\right)_{\frac{1+p}{2p},p}.$$

Let $\omega_1 = \omega_1(y)$, $\omega_2 = \omega_2(y)$ be roots of equation $b_1(y)\,\omega^2 + 1 = 0$. Let

$$\mu(y) = \begin{vmatrix} (-\omega_1)^{m_1} \alpha_1 & \beta_1 \omega_1^{m_1} \\ (-\omega_2)^{m_2} \alpha_2 & \beta_2 \omega_2^{m_2} \end{vmatrix},\ A_L = \left[L(\xi) + A\right]^{\frac{1}{2}}.$$

Here,

$$H_{ip2} = \mathbb{H}_{ip}\left(L^2(0,1)\right) =$$

$$H^{s(1-\theta_i),p}\left(\mathbb{R}^n;L^2(0,1)\right)\cap L^p\left(\mathbb{R}^n;H^{2(1-\theta_i)}(0,1)\right),$$

$$\theta_i=\frac{1+ip}{2p},\ i=0,1.$$

From Theorem 3.1 we obtain the following result

Theorem 4.1. Suppose the the following conditions are satisfied:
(1) Condition 2.0 holds, $0\leq\varkappa<1-\frac{1}{p}$, $0\leq\alpha<1-\frac{1}{2p}$, $p\in[1,\infty]$ and $\mu(y)\neq0$ for all $y\in[0,1]$;
(2) $b_1\in C([0,1])$, b_0 is a bounded function on $[0,1]$, $Re\omega_k\neq0$ and $\frac{\lambda}{\omega_k}\in S(\phi_1)$ for $\phi_1\in[0\ \pi)$, $b_1(0)=b_1(1)$, $b_0(0)=b_0(1)$;
(3) $\varphi\in Y_1^{s,p,2}$, $\psi\in Y_1^{s-1,p,2}$ and $f(.,t)\in Y_1^{s,p,2}$ for $s>\frac{2pn}{(2p-1)q}$, $q\in[1,2]$ and $t\in[0,T]$.
(4) the function $u\rightarrow F(u)$ is continuous in $u\in H_{0p2}$ for x, $t\in\mathbb{R}^n\times[0,T]$; moreover $F(u)\in C^{(1)}\left(H_{0p2};L^2(0,1)\right)$.

Then there exists a constant $\delta>0$ such that for any $\varphi\in H_{0p2}$ and $\psi\in H_{1p2}$ satisfying

$$\|\varphi\|_{\mathbb{H}_{0p2}}+\|A_1^{\alpha}\varphi\|_{X_{1,2}}+\|\psi\|_{\mathbb{H}_{1p2}}+\|A_1^{\alpha}\psi\|_{X_{1,2}}\leq\delta,$$

problem (1.4) has a unique local strong solution $u\in C([0,\ T_0);Y_0(A_1))$ if T_0 is a maximal time interval that is appropriately relative to δ. Moreover, if

$$\sup_{t\in[0,\ T_0)}\left(\|u\|_{Y_0(A_1)}+\|u_t\|_{Y_0(A_1)}\right)<\infty$$

then $T_0=\infty$.

Proof. By virtue of [32], $L^2(0,1)$ is a Fourier type space. By virtue of [30], the operator A_1 defined by (1.3) is sectorial in $L^2(0,1)$ and by virtue of [24, § 3.14, 3.16] the operator $A_1^2+\omega$ is a generator of bounded cosine function in $L^2(0,1)$. Moreover, by interpolation of Banach spaces [33, § 1.3], we have

$$H_{0p,2}=\left(W^{s,p}\left(\mathbb{R}^n;H^2(0,1),L^2(0,1)\right),L^p\left(\mathbb{R}^n;L^2(0,1)\right)\right)_{\frac{1}{2p},p}=$$

$$B_{p,2}^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^n;H^{2l\left(1-\frac{1}{2p}\right)}(0,1),L^2(0,1)\right).$$

Then, by using the properties of spaces $Y^{s,p,2}$, $Y_{\infty}^{s,p,2}$, $H_{0p,2}$ we get that all conditions of Theorem 3.1 are hold, i,e., we obtain the conclusion.

As a second application, let us consider the problem (1.6). Let

$$X_{p,2}=L^p\left(\mathbb{R}^n;L^2\left(\mathbb{R}^d\right)\right),Y^{s,p,2}=H^{s,p}\left(\mathbb{R}^n;L^2\left(\mathbb{R}^d\right)\right),$$

$$Y_q^{s,p,2}=H^{s,p}\left(\mathbb{R}^n;L^2\left(\mathbb{R}^d\right)\right)\cap L^q\left(\mathbb{R}^n;L^2\left(\mathbb{R}^d\right)\right),$$

$$Y^{s,p,2}=H^{s,p}\left(\mathbb{R}^n;H^2\left(\mathbb{R}^d\right),L^2\left(\mathbb{R}^d\right)\right),\ 1\leq p\leq\infty,$$

$$H_{0p2}=\left(Y^{s,p}\left(A,L^2(0,1)\right)\cap X_{p.2},X_{p,2}\right)_{\frac{1}{2p},p},$$

$$H_{1p2}=\left(Y^{s,p}\left(A,L^2\left(\mathbb{R}^d\right)\right)\cap X_{p,2},X_{p,2}\right)_{\frac{1+p}{2p},p}.$$

Here,

$$H_{ip2}=\mathbb{H}_{ip}\left(L^2\left(\mathbb{R}^d\right)\right)=$$

$$H^{s(1-\theta_i),p}\left(\mathbb{R}^n;L^2\left(\mathbb{R}^d\right)\right)\cap L^p\left(\mathbb{R}^n;H^{2(1-\theta_i)}\left(\mathbb{R}^d\right)\right),$$

$$\theta_i=\frac{1+ip}{2p},\ i=0,1.$$

From Theorem 3.1 we obtain the following result

Theorem 4.2. Suppose the the following conditions are satisfied:
(1) Condition 2.0 holds, $0\leq\alpha<1-\frac{1}{2p}$, $p\in[1,\infty]$;
(2) $L(\xi)=\sum\limits_{|\alpha|\leq l}\hat{a}_\alpha(\xi)(i\xi)^\alpha\in S_{\varphi_1}$, $\varphi_1\in[0,\ \pi)$ for $\xi\in\mathbb{R}^n$,
$|L(\xi)|\geq C\sum\limits_{k=1}^{n}\left|\hat{a}_{\alpha(l,k)}\right||\xi_k|^l$, $\alpha(l,k)=(0,0,...,l,0,0,...,0)$, i.e $\alpha_i=0$, $i\neq k$, $\alpha_k=l$;
$\hat{a}_\alpha\in C^{(n)}(\mathbb{R}^n)$ and

$$|\xi|^{|\beta|}\ \left|D^\beta\hat{a}_\alpha(\xi)\right|\leq C_1,\ \beta_k\in\{0,1\},\ 0\leq|\beta|\leq n;$$

(3) $\varphi\in Y_1^{s,p,2}$, $\psi\in Y_1^{s-1,p,2}$ and $f(.,t)\in Y_1^{s,p,2}$ for $s>\frac{2pn}{(2p-1)q}$, $q\in[1,2]$ and $t\in[0,T]$.
(4) the function $u\rightarrow F(u)$ is continuous in $u\in H_{0p,2}$ for $x,\ t\in\mathbb{R}^n\times[0,T]$; moreover $F(u)\in C^{(1)}\left(H_{0p,2};L^2\left(\mathbb{R}^d\right)\right)$.

Then there exists a constant $\delta>0$ such that for any $\varphi\in H_{0p2}$ and $\psi\in H_{1p2}$ satisfying

$$\|\varphi\|_{\mathbb{H}_{0p2}}+\|A_2^\alpha\varphi\|_{X_{1,2}}+\|\psi\|_{\mathbb{H}_{1p2}}+\|A_1^\alpha\psi\|_{X_{1,2}}\leq\delta,$$

problem (1.6) has a unique local strong solution $u\in C([0,\ T_0);Y_0(A_2))$, where T_0 is a maximal time interval that is appropriately relative to δ. Moreover, if

$$\sup_{t\in[0,\ T_0)}\left(\|u\|_{Y_0(A_2)}+\|u_t\|_{Y_0(A_2)}\right)<\infty$$

then $T_0=\infty$.

Proof. By virtue of [32], $L^2\left(\mathbb{R}^d\right)$ is a Fourier type space. By virtue of [30], the operator A_1 defined by (1.3) is sectorial in $L^2(0,1)$ and by virtue of [24, § 3.14, 3.16] the operator $A_1^2+\omega$ is a generator of bounded cosine function in $L^2(0,1)$. Moreover, by interpolation of Banach spaces [33, § 1.3], we have

$$H_{0p,2}=\left(W^{s,p}\left(\mathbb{R}^n;H^2\left(\mathbb{R}^d\right),L^2\left(\mathbb{R}^d\right)\right),L^p\left(\mathbb{R}^n;L^2\left(\mathbb{R}^d\right)\right)\right)_{\frac{1}{2p},p}=$$

$$B_{p,2}^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^n;H^{2l\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^d\right),L^2\left(\mathbb{R}^d\right)\right).$$

Then, by using the properties of spaces $Y^{s,p,2}$, $Y_\infty^{s,p,2}$, H_{0p2} we get that all conditions of Theorem 3.1 are hold, i,e., we obtain the conclusion.

Conclusion. Here, assuming enough smoothness on the initial data in terms of interpolation spaces $H(A)$, H and the sectorial operators, the existence, uniqueness, regularity properties of solutions are established. By choosing the space H and A, the regularity properties of solutions of a wide class of wave equations in the field of physics are obtained.

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