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NEW ERROR ESTIMATIONS FOR THE MILNE'S QUADRATURE FORMULA IN TERMS OF AT MOST FIRST DERIVATIVES

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Abstract. Error estimations for the Milne's rule for mappings of bounded variation and for absolutely continuous mappings whose first derivatives are belong to $L_p[a,b]$ $(1 < p \leq \infty)$, are established. Some numerical applications are provided.

1. Introduction

Suppose $f : [a,b] \to \mathbb{R}$, is a four times continuously differentiable mapping on (a,b) and

$$\left\|f^{(4)}\right\|_{\infty} := \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty.$$

Then the Simpson's inequality is known as:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_{\infty}. \tag{1.1}$$

In the recent years, modern theory of inequalities is used at large and many efforts devoted to establish several generalizations of the Simpson's inequality and other inequalities for mappings of bounded variation and for monotonic, absolutely continuous and Lipschitzian mappings, as well as n-times differentiable via kernels to refine the error bounds of these inequalities. For recent results and generalizations concerning Simpson's inequality see [1]–[2], [4]–[18] and the references therein.

In terms of Newton–Cotes formulas, the Milne's formula which is of open type is parallel to the Simpson's formula which is of closed type, since they are hold under

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the same conditions. Let f as above. Then we consider the Milne's inequality as follows:

$$\left|\frac{b-a}{3}\left[2f\left(a\right)-f\left(\frac{a+b}{2}\right)+2f\left(b\right)\right]-\int_a^b f\left(t\right)dt\right| \le \frac{7\left(b-a\right)^5}{23040}\left\|f^{(4)}\right\|_\infty . \tag{1.2}$$

Indeed, Milne recommends to use the three point Newton–Cotes open formula (1.2) as a predictor and three point Newton–Cotes closed formula (1.1) as a corrector (see [3]).

The aim of this paper is to discuss the Milne's inequality for mappings of bounded variation and for absolutely continuous mappings whose first derivatives are belong to $L_p[a,b]$ $(1 < p \le \infty)$.

2. Inequalities for mappings of bounded variation

We begin with the following result:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

$$\left|\frac{b-a}{3}\left[2f\left(a\right)-f\left(\frac{a+b}{2}\right)+2f\left(b\right)\right]-\int_a^b f\left(t\right)dt\right| \le \frac{2\left(b-a\right)}{3}\cdot\bigvee_a^b\left(f\right), \tag{2.1}$$

where $\bigvee_a^b(f)$, denotes to total variation of f over $[a,b]$. The constant $\frac{2}{3}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. Define the mapping

$$k\left(t\right)=\begin{cases} t-\frac{a+2b}{3}, & t\in\left[a,\frac{a+b}{2}\right] \\ t-\frac{2a+b}{3}, & t\in\left(\frac{a+b}{2},b\right] \end{cases}.$$

Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int_a^{\frac{a+b}{2}} k\left(t\right)df\left(t\right)=-\left(\frac{b-a}{6}\right)f\left(\frac{a+b}{2}\right)+2\left(\frac{b-a}{3}\right)f\left(a\right)-\int_a^{\frac{a+b}{2}} f\left(t\right)dt,$$

and

$$\int_{\frac{a+b}{2}}^b k\left(t\right)df\left(t\right)=2\left(\frac{b-a}{3}\right)f\left(b\right)-\left(\frac{b-a}{6}\right)f\left(\frac{a+b}{2}\right)-\int_{\frac{a+b}{2}}^b f\left(t\right)dt.$$

If we add the above equalities, we get

$$\int_a^b k\left(t\right)df\left(t\right)=\frac{b-a}{3}\left[2f\left(a\right)-f\left(\frac{a+b}{2}\right)+2f\left(b\right)\right]-\int_a^b f\left(t\right)dt$$

Now, assume that $\delta_n: a=x_0^{(n)}<x_1^{(n)}<\cdots<x_{n-1}^{(n)}<x_n^{(n)}=b$, is a sequence of divisions, with $\nu\left(\delta_n\right)\to 0$ as $n\to\infty$, where $\nu\left(\delta_n\right):=\max\limits_{i\in\{0,1\dots,n-1\}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)$ and $\xi_i^{(n)}\in\left[x_i^{(n)},x_{i+1}^{(n)}\right]$.

If $s:[a,b]\to\mathbb{R}$, is a piecewise continuous on $[a,b]$, and $\nu:[a,b]\to\mathbb{R}$, is of bounded variation on $[a,b]$, then

$$\begin{aligned}
&\left|\int_a^b s(t)\,d\nu(t)\right| \\
&\leq \left|\lim_{\nu(\delta_n)\to 0}\sum_{i=0}^{n-1} s\left(\xi_i^{(n)}\right)\left[\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right]\right| \\
&\leq \lim_{\nu(\delta_n)\to 0}\sum_{i=0}^{n-1}\left|s\left(\xi_i^{(n)}\right)\right|\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\
&\leq \lim_{\nu(\delta_n)\to 0}\sum_{i=0}^{n-1}\left|s\left(\xi_i^{(n)}\right)\right|\sum_{i=0}^{n-1}\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\
&\leq \sup_{x\in[a,b]}|s(x)|\cdot\sup_{\delta_n}\sum_{i=0}^{n-1}\left|\nu\left(x_{i+1}^{(n)}\right)-\nu\left(x_i^{(n)}\right)\right| \\
&= \sup_{x\in[a,b]}|s(x)|\cdot\bigvee_a^b(\nu)\,. \qquad (2.2)
\end{aligned}$$

Applying the inequality (2.2) for $s(t)=k(t)$ as above and $\nu(t)=f(t)$, $t\in[a,b]$, we get

$$\left|\int_a^b k(t)df(t)\right|\leq\sup_{t\in[a,b]}|k(t)|\cdot\bigvee_a^b(f)\leq\frac{2(b-a)}{3}\cdot\bigvee_a^b(f)\,.$$

To show that $2/3$ is the best possible (2.1). Assume (2.1) holds with constant $C>0$, i.e.,

$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\leq C(b-a)\cdot\bigvee_a^b(f)\,. \qquad (2.3)$$

Consider the function

$$f(t)=\begin{cases}0, & t\in(a,b)\\ 1, & t=a,b\end{cases}$$

then $\int_a^b f(t)\,dt=0$ and $\bigvee_a^b(f)=2$. Using (2.3), we get

$$\frac{4}{3}(b-a)\leq 2C(b-a)\,,$$

which gives $\frac{2}{3}\leq C$, and thus $\frac{2}{3}$ is the best possible, which completes the proof. □

Therefore, we may write the following result regarding monotonic mappings:

Corollary 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a monotonous mapping on $[a,b]$. Then for all $x\in[a,b]$, we have the inequality*

(2.4)
$$\left|\frac{b-a}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\int_a^b f(t)\,dt\right|\leq\frac{2(b-a)}{3}\cdot|f(b)-f(a)|\,.$$

The following result holds for L–lipschitz mappings:

Corollary 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a L–lipschitz mapping on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le \frac{2L}{3}(b-a)^2. \tag{2.5}$$

Remark 2.1. If we assume that f is continuous differentiable on (a,b) and f' is integrable on (a,b), then we have

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le \frac{2(b-a)}{3}\|f'\|_1 \tag{2.6}$$

3. Inequalities involving derivatives belong to $L_p[a,b]$ $(1 < p \le \infty)$

The following Milne's type inequality holds for absolutely continuous mappings whose first derivatives belong to $L_\infty[a,b]$.

Theorem 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$, such that $f' \in L_1[a,b]$. If f' is bounded on $[a,b]$, i.e., $\|f'\| := \sup_{t\in[a,b]} |f'(t)| < \infty$, then we have the following inequality:*

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le \frac{5}{12}(b-a)^2\|f'\|_\infty. \tag{3.1}$$

The constant $\frac{5}{12}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. Integrating by parts

$$\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt = \int_a^b k(t) f'(t)\,dt \tag{3.2}$$

where,

$$k(t) = \begin{cases} t - \frac{a+2b}{3}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{2a+b}{3}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases}.$$

We get

$$\begin{aligned}
&\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \\
&\le \int_a^b |k(t)|\,|f(t)|\,dt \\
&= \int_a^{\frac{a+b}{2}} \left|t - \frac{a+2b}{3}\right| |f'(t)|\,dt + \int_{\frac{a+b}{2}}^b \left|t - \frac{2a+b}{3}\right| |f(t)|\,dt \\
&\le \|f'\|_\infty \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+2b}{3} - t\right) dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{2a+b}{3}\right) dt\right] \\
&= \frac{5}{12}(b-a)^2\|f'\|_\infty.
\end{aligned}$$

To show that $\frac{5}{12}$ is the best possible. Assume that (3.1) holds with constant $C > 0$, i.e.,

$$\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \le C(b-a)^2 \|f'\|_\infty . \tag{3.3}$$

Consider the function $f(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$, then $\int_a^b f(t)\,dt = \frac{(b-a)^2}{4}$ and $\|f'\|_\infty = 1$. Using (3.3), we get

$$\frac{5}{12}(b-a)^2 \le C(b-a)^2 ,$$

which gives $\frac{5}{12} \le C$, and thus $\frac{5}{12}$ is the best possible, which completes the proof. □

Next result investigate Milne's formula for absolutely continuous mappings whose first derivatives are belong to $L_p[a,b]$, $p > 1$.

Theorem 3.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If f' is belong to $L_p[a,b]$, $p > 1$, then we have the following inequality:*

$$\begin{aligned}\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \\ \le 2 \cdot \frac{\left(2^{q+1} - 2^{-q-1}\right)^{1/q}}{(q+1)^{1/q}} \cdot \left(\frac{b-a}{3}\right)^{1+\frac{1}{q}} \|f'\|_p .\end{aligned} \tag{3.4}$$

Proof. By (3.2) and using the well known Hölder inequality, we have

$$\begin{aligned}&\left|\frac{b-a}{3}\left[2f(a) - f\left(\frac{a+b}{2}\right) + 2f(b)\right] - \int_a^b f(t)\,dt\right| \\ &\le \int_a^b |k(t)|\,|f(t)|\,dt \\ &\le \left(\int_a^b |k(t)|^q\,dt\right)^{1/q} \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} \\ &= \|f'\|_p \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+2b}{3} - t\right)^q dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{2a+b}{3}\right)^q dt\right]^{1/q} \\ &= 2 \cdot \frac{\left(2^{q+1} - 2^{-q-1}\right)^{1/q}}{(q+1)^{1/q}} \cdot \left(\frac{b-a}{3}\right)^{1+\frac{1}{q}} \|f'\|_p ,\end{aligned}$$

which is required. □

Remark 3.1. One may generalizes Theorem 3.1 and gives different approaches for Theorem 3.2, by applying the Hölder inequality in a different way and we shall left the details to the interested reader.

Remark 3.2. One may write new inequalities for mappings whose $|f'|$ is convex on $[a,b]$, using the inequality

$$|f'(t)| \le \frac{t-a}{b-a}|f'(b)| + \frac{b-t}{b-a}|f'(a)| ,$$

for any $t \in [a,b]$. Also, the corresponding version for powers $|f'|^q$ $(q>1)$ may be considered by applying the well–known Hölder inequality in two different ways. We left the details to the interested reader.

4. Estimations for the error bound in the Milne's formula

Consider $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of $[a,b]$ and let $h_i = x_{i+1} - x_i$. In what follows, we point out some upper bounds for the error approximation of the Milne's formula.

$$S(f, I_n) := \sum_{i=0}^{n-1} \frac{h_i}{3} \left[2f(x_i) - f\left(\frac{x_i + x_{i+1}}{2}\right) + 2f(x_{i+1}) \right] \tag{4.1}$$

Theorem 4.1. *Assume that the assumptions of Theorem 2.1 hold. Then, we have*

$$\int_a^b f(t)\,dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (4.1) and the remainder $R(f, I_n)$ satisfies the bound

$$|R(f, I_n)| \le \frac{2}{3} \sum_{i=0}^{n-1} \left(h_i \cdot \bigvee_{x_i}^{x_{i+1}} (f) \right). \tag{4.2}$$

Proof. Applying Theorem 2.1 on the subintervals $[x_i, x_{i+1}]$, we have

$$\left| \frac{h_i}{3} \left[2f(x_i) - f\left(\frac{x_i + x_{i+1}}{2}\right) + 2f(x_{i+1}) \right] - \int_{x_i}^{x_{i+1}} f(t)dt \right| \le \frac{2}{3} h_i \cdot \bigvee_{x_i}^{x_{i+1}} (f).$$

Summing the obtained inequalities over $i = 0, \cdots, n-1$, we get,

$$\left| S(f, I_n) - \int_a^b f(t)\,dt \right| \le \frac{2}{3} \sum_{i=0}^{n-1} \left(h_i \cdot \bigvee_{x_i}^{x_{i+1}} (f) \right),$$

which is required. □

Theorem 4.2. *Assume that the assumptions of Theorem 3.1 hold. Then, we have*

$$\int_a^b f(t)\,dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (4.1) and the remainder $R(f, I_n)$ satisfies the bound

$$|R(f, I_n)| \le \frac{5}{12} (b-a) \|f'\|_\infty. \tag{4.3}$$

Proof. Applying Theorem 3.1 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \cdots, n-1$, we get the required result. We shall omit the details. □

Theorem 4.3. *Assume that the assumptions of Theorem 3.2 hold. Then, we have*

$$\int_a^b f(t)\,dt = S(f, I_n) + R(f, I_n)$$

where, $S(f, I_n)$ is given in (4.1) and the remainder $R(f, I_n)$ satisfies the bound

$$|R(f, I_n)| \leq \frac{2}{3^{\frac{q+1}{q}}} \cdot \frac{\left(2^{q+1} - 2^{-q-1}\right)^{1/q}}{(q+1)^{1/q}} \cdot \|f'\|_p \cdot \sum_{i=0}^{n-1} h_i^{\frac{q+1}{q}}. \tag{4.4}$$

Proof. Applying Theorem 3.2 on the subintervals $[x_i, x_{i+1}]$ and then summing the obtained inequalities over $i = 0, \cdots, n-1$, we get the required result. We shall omit the details. □

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