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ON TIMELIKE PARALLEL RULED SURFACES WITH SPACELIKE RULING

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Abstract. In this paper, first, timelike parallel surfaces and their some basic properties are presented in Minkowski 3-space. Then the main theorem for timelike parallel ruled surface is given to understand how parallel surfaces of a timelike ruled surface with spacelike ruling become again a timelike ruled surface with spacelike ruling. Additionally, some basic properties of that kind ruled surface are given in Minkowski 3-space.

1. Introduction

Parallel surfaces as a subject of differential geometry have been intriguing for mathematicians throughout history and so it has been a research field. In theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which differential geometers are interested. Among these surfaces, parallel surfaces have been also studied in many papers [2, 3, 6, 8, 12, 14]. Craig had studied to find parallel of ellipsoid in [3]. Eisenhart gave a chapter for parallel surfaces in his famous *A treatise of differential geometry* [5]. Nizamoğlu stated parallel ruled surface as a curve depending on one-parameter and gave some geometric properties of such a surface [12].

A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite number of surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of point which are on the normals to M at a non-zero constant distance r from M [18].

In this paper, it has been shown that parallel surfaces of a non-developable ruled surface are not ruled surfaces by using fundamental forms, however that parallel surfaces of a timelike developable ruled surface are timelike developable ruled surface. After construction of timelike parallel ruled surface, some properties of that kind surface such as drall, striction curve and orthogonal trajectory have

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been given for timelike parallel ruled surfaces of timelike ruled surface with spacelike ruling.

2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$< d\mathbf{x}, d\mathbf{x} >= dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}_1^3$ is said to be spacelike if $< \mathbf{x}, \mathbf{x} >>0$ or $\mathbf{x} = \mathbf{0}$, timelike if $< \mathbf{x}, \mathbf{x} ><0$ and lightlike or null if $< \mathbf{x}, \mathbf{x} >=0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or lightlike vector in $\mathbb{E}_1^3$ is said to be causal. For $\mathbf{x} \in \mathbb{E}_1^3$, the norm is defined by $\|\mathbf{x}\| = \sqrt{|< \mathbf{x}, \mathbf{x} >|}$, then the vector $\mathbf{x}$ is called a spacelike unit vector if $< \mathbf{x}, \mathbf{x} >=1$ and a timelike unit vector if $< \mathbf{x}, \mathbf{x} >= -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [13]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $< \mathbf{x}, \mathbf{y} >= x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [11].

A (differentiable) one-parameter family of (straight) lines $\{\alpha(u), X(u)\}$ is a correspondence that assigns each $u \in I$ to a point $\alpha(u) \in \mathbb{R}_1^3$ and a vector $X(u) \in \mathbb{R}_1^3$, $X(u) \neq 0$, so that both $\alpha(u)$ and $X(u)$ can be differentiated in terms of the variable u. For each $u \in I$, the line L_u which passes through $\alpha(u)$ and is parallel to $X(u)$ is called the *line of the family* at u.

Given a one-parameter family of lines $\{\alpha(u), X(u)\}$, the parametrized surface

$$\varphi(u, v) = \alpha(u) + vX(u), \qquad u \in I, \qquad v \in \mathbb{R} \tag{2.1}$$

is called the ruled surface generated by the family $\{\alpha(u), X(u)\}$. The lines L_u are called the *rulings* and the curve $\alpha(u)$ is called a *directrix* of the surface φ. The normal vector of surface is denoted by $\overrightarrow{N}$. Let us take timelike ruled surface φ with timelike directrix and spacelike ruling. So the system $\{T, X, N\}$ establishes an orthonormal frame such that $T = \alpha'(u)$. Therefore

$$< T, T >= -1, \;\; < X, X >= 1, \;\; < N, N >= 1. \tag{2.2}$$

Derivative equations of the frame $\{T, X, N\}$ are

$$D_T T = aX + bN, \quad D_T X = aT + cN, \quad D_T N = bT - cX. \tag{2.3}$$

Also the reciprocal cross products of the vectors T, X, N are

$$T \wedge X = N, \;\; T \wedge N = -X, \;\; X \wedge N = -T. \tag{2.4}$$

The parameter of distribution is expressed as follows

$$\lambda = \frac{\det(\alpha', X, X')}{|X'|^2} \tag{2.5}$$

where, as usual, (α', X, X') is a short for $\langle \alpha' \wedge X, X' \rangle$ [16].

Theorem 2.1. *A surface in Minkowski 3-space is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal on the surface is a spacelike vector* [1].

Theorem 2.2. *Using standard parameters, a ruled surface is up to Lorentzian motions, uniquely determined by the following quantities:*

$$Q = \langle \alpha', X \wedge X' \rangle, \quad J = \langle X, X'' \wedge X' \rangle, \quad F = \langle \alpha', X \rangle \tag{2.6}$$

each of which is a function of u. Conversely, every choice of these three quantities uniquely determines a ruled surface [10].

Theorem 2.3. *The Gaussian K and mean H curvatures of the timelike ruled surface φ in terms of the parameters Q, J, F, D in $\mathbb{E}_1^3$ are obtained as follows:*

$$K = -\frac{Q^2}{D^4} \quad \text{and} \quad H = \frac{1}{2D^3}(-QF + Q^2 J + vQ' + v^2 J), \tag{2.7}$$

where $D = \sqrt{-Q^2 - v^2}$, respectively [4].

Theorem 2.4. *Parameter curves are lines of curvature if and only if $F = f = 0$ in $\mathbb{E}_1^3$* [11].

Theorem 2.5. *Let $\varphi(u, v)$ be a surface in $\mathbb{E}_1^3$ with the normal vector field of N. Then the shape operator S of φ is given in terms of the base $\{\varphi_u, \varphi_v\}$ by*

$$\begin{aligned} -S(\varphi_u) = N_u &= \frac{mF - lG}{EG - F^2}\varphi_u + \frac{lF - mE}{EG - F^2}\varphi_v \\ -S(\varphi_v) = N_v &= \frac{nF - mG}{EG - F^2}\varphi_u + \frac{mF - nE}{EG - F^2}\varphi_v \end{aligned} \tag{2.8}$$

[15].

The parallel surface of the timelike surface $\varphi(u, v)$, is denoted by $\varphi^r(u, v)$, is defined in $\mathbb{E}_1^3$ as follows:

$$\varphi^r(u, v) = \varphi(u, v) + rN(u, v), \tag{2.9}$$

where N is the unit normal vector of $\varphi(u, v)$ such that $\langle N, N \rangle = 1$ and $r \in R$. The coefficients of the first and second fundamental forms I^r and II^r of timelike parallel surfaces can be given in terms of the coefficients of the timelike surface's fundamental forms:

$$\begin{aligned} E^r &= E - 2rl + r^2 \langle N_u, N_u \rangle, & l^r &= l - r \langle N_u, N_u \rangle \\ F^r &= F - 2rm + r^2 \langle N_u, N_v \rangle, & m^r &= m - r \langle N_u, N_v \rangle \\ G^r &= G - 2rn + r^2 \langle N_v, N_v \rangle, & n^r &= n - r \langle N_v, N_v \rangle, \end{aligned} \tag{2.10}$$

where E, F, G are the coefficients of the first fundamental form for the surface φ and l, m, n are the coefficients of the second fundamental form for the surface φ and E^r, F^r, G^r are the coefficients of the first fundamental form for the parallel surface φ^r and l^r, m^r, n^r are the coefficients of the second fundamental form for the parallel surface φ^r [17].

Definition 2.1. Let M and M^r be two surfaces in Minkowski 3-space. The function

$$\begin{aligned} f: \quad M &\longrightarrow \quad M^r \\ p &\longrightarrow \quad f(p) = p + r\mathbf{N}_p \end{aligned}$$

is called the parallellization function between M and M^r and furthermore M^r is called parallel surface to M in $\mathbb{E}_1^3$ where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on M [6].

Theorem 2.6. *Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$,*

1. $f_*(X) = X + rS(X)$,
2. $S^r(f_*(X)) = S(X)$,
3. f preserves principal directions of curvature, that is

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at p in direction X [6].

Definition 2.2. Let M be a hypersurface of $\overline{M}$- manifold and M^r be parallel hypersurface of M in $\mathbb{E}_1^3$. If σ is a curve passing through p on M and T is the tangent vector field of σ on M, then $\sigma^r = f \circ \sigma$ is a curve passing through a point $f(p)$ on M^r and $f_*(T) \in T_{f(p)}M^r$ is a tangent of σ^r at $f(p)$. The connection D^r belongs to the parallel surface M^r of M and the vector N^r is the unit normal vector of M^r, where $\langle N^r, N^r \rangle = \varepsilon = \pm 1$, therefore the Gauss equation is as follows:

$$\overline{D}_{f_*(T)} f_*(T) = D^r_{f_*(T)} f_*(T) - \varepsilon \langle S^r(f_*(T)), f_*(T) \rangle N^r \tag{2.11}$$

[8, 13].

Definition 2.3. Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Let N^r and S^r be, respectively, the unit normal vector field and the shape operator of M^r. The Gaussian and mean curvature functions are defined as follows:

$$\begin{array}{rcll} K^r & : & M^r & \to \mathbb{R} \\ & & f(P) & \to K^r(f(P)) = \det S^r_{f(P)} \\ \\ H^r & : & M^r & \to \mathbb{R} \\ & & f(P) & \to H^r(f(P)) = \frac{1}{2}\ iz S^r_{f(P)}, \end{array} \tag{2.12}$$

where $P \in M$, $f(P) \in M^r$ and $\langle N, N \rangle = 1$, respectively [17].

Theorem 2.7. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Let N^r and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian and mean curvatures are given in terms of coefficients of fundamental forms I^r and II^r as follows:*

$$K^r = \frac{e^r g^r - f^{r2}}{E^r G^r - F^{r2}} \quad \textit{and} \quad H^r = \frac{e^r G^r - 2f^r F^r + g^r E^r}{2(E^r G^r - F^{r2})}, \tag{2.13}$$

respectively [17].

Lemma 2.1. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. The surface M is timelike one if and only if the surface M^r is timelike parallel surface* [17].

Theorem 2.8. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Then we have*

$$K^r = \frac{K}{1+2rH+r^2K} \quad \textit{and} \quad H^r = \frac{H+rK}{1+2rH+r^2K} \tag{2.14}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [17].

Corollary 2.1. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Then we have*

$$K = \frac{K^r}{1 - 2rH^r + r^2K^r} \quad and \quad H = \frac{H^r - rK^r}{1 - 2rH^r + r^2K^r} \tag{2.15}$$

where Gaussian and mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively [17].

Theorem 2.9. *Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}^3_1$. Curves on the timelike parallel surface M^r which correspond to lines of curvature on the timelike surface M are also the lines of curvature* [17].

3. Timelike parallel ruled surfaces with spacelike ruling

The timelike ruled surface M with spacelike ruling, is parameterized as

$$\varphi(u,v) = \alpha(u) + vX(u), \ \ \langle \alpha', \alpha' \rangle = -1, \ \ \langle X, X \rangle = 1, \ \ \langle X', X' \rangle = -1. \tag{3.1}$$

The normal vector of the surface M is as follows:

$$N = \alpha' \wedge X + vX' \wedge X. \tag{3.2}$$

For the normal vector of a developable ruled surface which is constant along its ruling and is independent from the parameter v, the expressions $\alpha' \wedge X$ and $X' \wedge X$ in (3.2) are linearly dependent, that is, the following equation is obtained

$$\alpha' \wedge X = \lambda X' \wedge X,$$

where $\lambda \in \mathbb{R}$. Also, from the equation (3.2), the normal vector of the surface M can be obtained as

$$N = (\lambda + v)X' \wedge X. \tag{3.3}$$

The unit normal vector of the surface becomes as follows

$$\mathbf{N} = X' \wedge X. \tag{3.4}$$

We get parallel surface of the ruled surface parameterized as $\varphi(u,v) = \alpha(u) + vX(u)$ as

$$\varphi^r(u,v) = \alpha(u) + rX'(u) \wedge X(u) + vX(u). \tag{3.5}$$

We call the surface obtained in (3.5) as the parallel ruled surface. The ruling of parallel ruled surface is

$$f_*(X) = f_*(T) \wedge N^r = (T + rbT) \wedge N = -(1 + rb)X. \tag{3.6}$$

And we also get

$$f \circ \alpha(u) = \alpha(u) + rN(u) = \alpha(u) + rX'(u) \wedge X(u). \tag{3.7}$$

The coefficient g^r of the second fundamental form II^r of the parallel surface M^r is

$$g^r = -\langle \varphi^r_v, \mathbf{N}_v \rangle = -\langle X, 0 \rangle = 0.$$

The drall of parallel ruled surface is obtained from the following formula

$$P^r = < \frac{df \circ \alpha}{du}, f'_*(X) \wedge f_*(X) > . \tag{3.8}$$

From (3.8), the value of drall is found as follows:

$$P^r = \left\langle \alpha' + rX'' \wedge X, (1 + rb)^2 X' \wedge X \right\rangle = 0 \tag{3.9}$$

Finally, the parallel ruled surface given in (3.5) is a developable ruled surface. Therefore, we can give the following theorem:

Theorem 3.1. *Let M be a timelike ruled surface with spacelike ruling and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Parallel surface of a timelike developable ruled surface is again a timelike ruled surface.*

The coefficients of the first and second fundamental forms I^r and II^r for the parallel surfaces of timelike ruled surface parameterized in (3.5) are as such:

$$\begin{aligned} E^r = {} & \langle \alpha', \alpha' \rangle + 2r \langle \alpha', X'' \wedge X \rangle + 2v \langle \alpha', X' \rangle + r^2 \langle X'' \wedge X, X'' \wedge X \rangle \\ & + 2rv \langle X', X'' \wedge X \rangle + v^2 \langle X', X' \rangle . \end{aligned} \tag{3.10}$$

Since $\langle X', X' \rangle = -1$ and $\langle X'', X' \rangle = 0$, X'' lies in the plane spanned by the vectors X and $X' \wedge X$. Therefore

$$X'' = mX + nX' \wedge X, \tag{3.11}$$

where $m, n \in \mathbb{R}$. By using (3.11), we have

$$X'' \wedge X = (mX + nX' \wedge X) \wedge X = (nX' \wedge X) \wedge X = nX'. \tag{3.12}$$

Substituting (3.12) into (3.10), the coefficients E^r, F^r and G^r of the first fundamental form I^r for the parallel surface M^r are found as

$$E^r = -1 - (rn - v)^2$$

$$F^r = \langle \varphi_u^r, \varphi_v^r \rangle = \langle \alpha' + rX'' \wedge X + vX', X \rangle = \langle \alpha', X \rangle$$

$$G^r = \langle \varphi_v^r, \varphi_v^r \rangle = \langle X, X \rangle = 1.$$

Also, the normal vector of the surface is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

Let us find the coefficients of the second fundamental form II^r. The coefficients l^r, m^r and n^r of the second fundamental form II^r for the parallel surface $M^r$are computed as follows:

$$l^r = -\langle \varphi_u^r, N_u^r \rangle = -\langle \alpha', \alpha'' \wedge X \rangle - \langle X', \alpha'' \wedge X \rangle (rn + v) + v^2 n + rvn^2$$

$$m^r = \langle -\varphi_u^r, N_v^r \rangle = -\langle \alpha', X' \wedge X \rangle$$

$$n^r = -\langle \varphi_v^r, N_v^r \rangle = -\langle X, X' \wedge X \rangle = 0.$$

Corollary 3.1. *Let M^r be a timelike parallel ruled surface of a timelike ruled surface with spacelike ruling in $\mathbb{E}_1^3$. Then the directrix and the ruling of the timelike parallel ruled surface are a timelike curve and its ruling is a timelike vector, respectively.*

Proof. The ruling of timelike parallel ruled surface M^r given in (3.5) is a spacelike vector since $\langle X, X \rangle = 1$. The causal character of the directrix is seen by the following computations:

$$\begin{aligned} \left\langle \frac{df \circ \alpha(u)}{du}, \frac{df \circ \alpha(u)}{du} \right\rangle & = \langle \alpha' + rX'' \wedge X, \alpha' + rX'' \wedge X \rangle \\ & = 1\text{-}2rn \langle \alpha', X' \rangle + r^2 n^2 \langle X', X' \rangle \langle X, X \rangle . \end{aligned} \tag{3.13}$$

By using $\langle X', X' \rangle = -1$ and $\langle X', X'' \rangle = 0$ in (3.13), it becomes

$$\left\langle \frac{df \circ \alpha(u)}{du}, \frac{df \circ \alpha(u)}{du} \right\rangle = -1 - r^2 n^2 < 0. \tag{3.14}$$

That the causal character of the directrix is timelike is seen from ☐

Theorem 3.2. *Let M be a developable timelike ruled surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Let $f_*(T)$, $f_*(X)$ and N^r be, the directrix, the ruling and the normal vector of the parallel surface M^r, respectively. Hence the reciprocal cross products of these three vectors are as follows:*

$$f_*(T) \wedge N^r = f_*(X)$$

$$f_*(T) \wedge f_*(X) = -(1+rb)^2 N^r$$

$$f_*(X) \wedge N^r = -f_*(T).$$

Proof. Frenet equations for the timelike developable ruled surface M are obtained in (2.3) by taking $c = 0$. And also the unit vectors T, X, N for the timelike developable ruled surface M are as in (2.4). By means of these information, we have the following results:

$$f_*(T) \wedge N^r = (T + rS(T)) \wedge N = (T + rbT) \wedge N = -(1+rb)X = f_*(X),$$

$$f_*(T) \wedge f_*(X) = (T + rS(T)) \wedge f_*(X) = (T + rbT) \wedge (-1-rb)X = -(1+rb)^2 N = -(1+rb)^2 N^r,$$

$$f_*(X) \wedge N^r = -(1+rb)X \wedge N = -(1+rb)T = -f_*(T).$$

☐

Theorem 3.3. *The vectors $f_*(T)$, $f_*(X), N^r$ for the timelike parallel ruled surface M^r are timelike, spacelike and spacelike vectors, respectively, while the unit vectors T, X, N for the timelike developable ruled surface with spacelike ruling M are timelike, spacelike and spacelike vectors, respectively.*

Proof. The normal vector of the timelike parallel ruled surface M^r is a spacelike vector because

$$\langle N^r, N^r \rangle = \langle N, N \rangle = 1.$$

The tangent vector field of the directrix is a timelike vector because

$$\langle f_*(T), f_*(T) \rangle = -(1 - rb)^2 < 0.$$

From (3.6), the vector $f_*(X)$ is a spacelike vector because

$$\langle f_*(X), f_*(X) \rangle = \langle -(1+rb)X, -(1+rb)X \rangle = (1+rb)^2 > 0.$$

☐

Position vector of striction curve on the timelike parallel ruled surface M^r is written as

$$\overrightarrow{O\gamma} = \overrightarrow{Of \circ \alpha} + \overrightarrow{\theta f_*(X)}. \tag{3.15}$$

By using $f_*(X) = X^r$ in (3.15), we get

$$\gamma(u) = f \circ \alpha(u) + \theta X^r(u) \text{ and } \theta = \theta(u). \tag{3.16}$$

After making the required computations in (3.16), we get the function $\theta = \theta(u)$ as follows:

$$\begin{aligned} \theta &= -\frac{\left\langle \dfrac{df \circ \alpha}{du}, \dfrac{dX^r}{du} \right\rangle}{\left\langle \dfrac{dX^r}{du}, \dfrac{dX^r}{du} \right\rangle} \\ &= \frac{\langle \alpha', X' \rangle + r \langle X'' \wedge X, X' \rangle}{(1 + rb) \langle X', X' \rangle}. \end{aligned} \tag{3.17}$$

Hence using (3.16) and (3.17), the striction curve is given as follows:

$$\gamma(u) = \alpha(u) + rX'(u) \wedge X(u) + \frac{\langle \alpha', X' \rangle + r \langle X'' \wedge X, X' \rangle}{\langle X', X' \rangle} X. \tag{3.18}$$

After some calculations, the equation (3.18) becomes

$$\gamma(u) = \alpha(u) + rX'(u) \wedge X(u) + \frac{1 + rna}{a} X. \tag{3.19}$$

Corollary 3.2. *Striction curve of the timelike parallel ruled surface is also a directrix provided that* $\langle \alpha', X' \rangle = 0$ *and* $\langle X'' \wedge X, X' \rangle = 0$.

Proof. Straightforward calculation by using (3.18). □

Corollary 3.3. *Striction curve of the timelike parallel ruled surface is also a directrix provided that* $1 + rna = 0$.

Proof. Straightforward calculation by using (3.19). □

Theorem 3.4. *The striction curve* γ *of the timelike parallel ruled surface* M^r *is a timelike curve.*

Proof. The normal vector field of the timelike parallel ruled surface M^r is

$$N^r = N = \varphi_u \wedge \varphi_v = \alpha' \wedge X + vX' \wedge X.$$

For $v = 0$, we get

$$N^r(u, 0) = \alpha'(u) \wedge X(u). \tag{3.20}$$

From (3.20), we have

$$\begin{aligned} \langle N^r(u,0), N^r(u,0) \rangle &= \langle \alpha'(u) \wedge X(u), \alpha'(u) \wedge X(u) \rangle \\ &= F^2 + 1 > 0. \end{aligned} \tag{3.21}$$

The result obtained in (3.21) means that the striction curve is timelike because the vector which is normal to it is spacelike vector. □

Theorem 3.5. *Striction curve of the timelike parallel ruled surface* M^r *does not depend on the choice of the base curve* $f \circ \alpha$.

Proof. Let $f \circ \alpha$ and ρ be two different directrices of the timelike parallel ruled surface. Then the timelike parallel ruled surface is

$$\varphi^r(u, v) = f \circ \alpha(u) + vX^r(u) = \rho(u) + sX^r(u) \tag{3.22}$$

for some function $s = s(v)$. Assume that the curves $\gamma(u)$ and $\overline{\gamma}(u)$ are the striction curves of the surfaces given in (3.22). Then as analogous to (3.18) by (3.22) we get

$$\gamma(u) - \overline{\gamma}(u) = (v - s)X^r - \frac{\langle (v - s)X^{r\prime}, X' \rangle}{\langle X', X' \rangle} X(u) = 0. \tag{3.23}$$

The proof is completed by the result obtained in (3.23). □

Theorem 3.6. *Given a timelike parallel ruled surface M^r which is parallel to developable timelike ruled surface M with spacelike ruling. There exists a unique orthogonal trajectory of M^r through each point of M. This orthogonal trajectory in terms of magnitudes of the timelike ruled surface M is as follows:*

$$\beta(s) = \alpha(s) + rX'(s) \wedge X(s) + g(s)X(s).$$

Here, the function $g(s)$ has been taken instead of $-v(1 + rb)$.

Proof. Let

$$\begin{aligned} \varphi^r : \quad & I \times J \longrightarrow \mathbb{E}^3_1 \\ & (u, v) \longrightarrow \varphi^r(u, v) = f \circ \alpha(u) + vX^r(u) \\ & \qquad\qquad\qquad\qquad = \alpha(u) + rX'(u) \wedge X(u) - v(1 + rb)X. \end{aligned} \tag{3.24}$$

An orthogonal trajectory of M^r is given by

$$\begin{array}{rccl} \beta : & \widetilde{I} & \longrightarrow & M^r \\ & s & \longrightarrow & \beta(s) = f \circ \alpha(s) + g(s)X^r(s). \end{array} \tag{3.25}$$

We may assume $\widetilde{I} \subset I$. Since

$$\langle \beta'(s), X^r(s)\rangle = \langle \alpha'(s), X(s)\rangle + g'(s) = 0, \tag{3.26}$$

we obtain

$$g(s) = -\int \langle \alpha'(s), X(s)\rangle \, ds + h,$$

where h is a real constant. Hence $h = g(s_0) - F(s_0)$, where

$$-\int \langle \alpha'(s), X(s)\rangle \, ds = F(s).$$

Therefore the orthogonal trajectory of M^r through the point P_0 is unique. Thus, we have $\widetilde{I} = I$ since the orthogonal trajectory of the surface M^r meets each one of the rulings of M^r. □

Corollary 3.4. *Let M be a timelike ruled surface with spacelike ruling and M^r be a timelike parallel surface of M in $\mathbb{E}^3_1$. The Gaussian and mean curvatures K^r and H^r in terms of the parameters Q, J, F, D are as follows:*

$$\begin{aligned} K^r &= \frac{-Q^2}{D^4 - rQFD + rQ^2JD + rvQ'D + rv^2JD - r^2Q^2} \\ H^r &= \frac{-QFD + Q^2JD + vQ'D + v^2JD - 2rQ^2}{2D^4 - 2rQFD + 2rQ^2JD + 2rvQ'D + 2rv^2JD - 2r^2Q^2}, \end{aligned} \tag{3.27}$$

respectively.

Proof. Using (2.7) in (2.14), the values of Gaussian K^r and mean curvatures H^r are obtained in (3.27). □

4. Conclusion

In this study, timelike parallel ruled surfaces have been introduced in Minkowski 3-space. Additionally timelike parallel ruled surface have been constructed by using basic features of timelike ruled surfaces with spacelike ruling. Also some properties of timelike parallel ruled surface have been given in Minkowski 3-space.

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