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SOME RESULTS ON THE SENSITIVITY OF SCHUR STABILITY OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. In this work, new results on the sensitivity problem of the Schur stability of linear difference equation systems with constant coefficients and scalar-linear difference equations with order k are obtained and some examples illustrating the efficiency of the theorems are given.

1. Introduction

In this article we consider the following linear system of difference equations with constant coefficients:

$$x(n+1) = Ax(n), \qquad n \in \mathbb{Z}. \tag{1.1}$$

where A is a matrix of dimensions $N \times N$. The asymptotic stability of the system (1.1) is equivalent to the asymptotic stability of the coefficient matrix A. It is well-known that with respect to Lyapunov, a matrix A is discrete-asymptotically stable if and only if the discrete-Lyapunov matrix equations $A^*XA - X + C = 0$, $C = C^* > 0$ has a solution matrix X which is positive definite matrix, i.e. $X = X^* > 0$. Moreover, this solution given by $X = \sum_{k=0}^{\infty} (A^*)^k CA^k$ and also according to the spectral criteria, a matrix A is discrete-asymptotically stable if and only if the eigenvalues of the coefficient matrix A lay in the unit disc, i.e. $|\lambda_i(A)| < 1$ for all $i = 1, 2, \ldots, N$, where λ_i $(i = 1, 2, \ldots, N)$ stands for the eigenvalues of the coefficient matrix A [1, 2, 3, 4]. Such systems are also called as Schur stable [5, 6, 7]. Throughout the study, we focus our attention to the concept of Schur stability.

In the literature, some restrictions on the perturbation matrix B are assumed to study the Schur stability of the following system

$$y(n+1) = (A+B)y(n), \qquad n \in \mathbb{Z}, \tag{1.2}$$

where A is the coefficient matrix of the Schur stable system (1.1). So called continuation are used to study the sensitivity of the ω^*- Schur stability and the Schur stabilitiy of the system (1.1) [1, 2, 4, 8].

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In this work, some results on the sensitivity of the Schur stability and the ω^*- Schur stability of the difference equation system (1.1) were presented. We have also applied the results to the delay difference equations.

2. Sensitivity of Systems

In this section, we give some results in the literature on the sensitivity of the Schur stability of the systems with constant coefficients.

Let's start with the parameter $\omega(A)$ that shows the quality of Schur stability of the system (1.1) and holds and an important place in the theory of stability.

Schur stability parameter $\omega(A)$ is defined as follows:

$$\omega(A) = ||H||; \quad H = \sum_{k=0}^{\infty} (A^*)^k A^k, \quad H = H^* > 0, \quad A^*HA - H + I = 0$$

where I is unit matrix, A^* is adjoint of the matrix A, $||A|| = \max_{\|x\|=1} \|Ax\|$ is the spectral norm of the matrix A, furthermore the norm $||x||$ is Euclidean norm for the vector $x = (x_1, x_2, \ldots, x_N)^T$. Linear difference system (1.1) is Schur stable if and only if $\omega(A) < \infty$ holds and so it is clear that the perturbed linear difference system (1.2) is Schur stable if and only if $\omega(A+B) = ||\widetilde{H}|| < \infty$ holds, where the matrix $\widetilde{H} = \sum_{k=0}^{\infty} (A^* + B^*)^k (A+B)^k$ is positive definite solution of the discrete-Lyapunov matrix equation $(A^* + B^*)\widetilde{H}(A+B) - \widetilde{H} + I = 0$. Moreover, let ω^* be the practical Schur stability parameter of the system (1.1), then the matrix A is called as practically Schur stable (ω^*- Schur stable) provided that $\omega^* > 1$ and $\omega(A) \leq \omega^*$ hold. If $\omega(A) > \omega^*$ holds, then the matrix A is called as ω^*- Schur unstable matrix [1, 3, 9].

Theorem 2.1 ([3]). *Let A be a Schur stable matrix ($\omega(A) < \infty$). If $\|B\| \leq \frac{1}{6\pi\omega(A)}$ then the matrix $A+B$ is Schur stable. Moreover, if $(2\|B\|\|A\| + \|B\|^2)\omega(A) < 0.5$ then the inequality*

$$|\omega(A+B) - \omega(A)| \leq 2\omega^2(A)(2\|A\| + \|B\|)\|B\|,$$

holds.

Corollary 2.1 ([10]). *Suppose that A is a Schur stable matrix, that is $\omega(A) < \infty$. If the matrix B satisfies $\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$, then $A+B$ is Schur stable. Moreover, the inequality*

$$|\omega(A+B) - \omega(A)| \leq \frac{(2\|A\| + \|B\|)\|B\|\omega^2(A)}{1 - (2\|A\| + \|B\|)\|B\|\omega(A)},$$

holds.

Now, considering Theorem 2.1 and Corollary 2.1 we give the continuity theorem which allows the greater perturbe than others without disturbing the Schur stability for the linear difference equation systems with constant coefficients.

Theorem 2.2 ([10]). *Suppose that A is a Schur stable matrix, that is $\omega(A) < \infty$. If the matrix B satisfies $\|B\| < \gamma$, then $A+B$ is Schur stable. Moreover, if*

$\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$, *then the following inequalities*

$$\omega(A+B) \leq \frac{\omega(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}, \quad |\omega(A+B) - \omega(A)| \leq \frac{(2\|A\|+\|B\|)\|B\|\omega^2(A)}{1-(2\|A\|+\|B\|)\|B\|\omega(A)}$$

holds, where $\gamma = \max\left\{\frac{1}{6\pi\omega(A)}, \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|\right\}$.

Corollary 2.2 ([10])**.** *Let* $\|A\| < 1$. *If the matrix* B *satisfies* $\|A\| + \|B\| < 1$ *then the matrix* $A+B$ *is Schur stable. Moreover, the following inequalities*

$$\omega(A+B) \leq \frac{1}{1-(\|A\|+\|B\|)^2}, \quad |\omega(A+B) - \omega(A)| \leq \frac{\|B\|}{1-\|A\|}\frac{1}{1-(\|A\|+\|B\|)^2}$$

holds.

Theorem 2.3 ([10])**.** *Let* A *be a* ω^*- *Schur stable matrix* $(\omega(A) \leq \omega^*)$. *If the matrix* B *satisfies* $\|B\| \leq \sqrt{\|A\|^2 + \frac{\omega^*-\omega(A)}{\omega^*\omega(A)}} - \|A\|$ *then* $A+B$ *is* ω^*- *Schur stable.*

Corollary 2.3. *Let* $\|A\| < 1$ *and* A *be a* ω^*- *Schur stable matrix* $(\omega(A) \leq \omega^*)$. *If the matrix* B *satisfies* $\|A\| + \|B\| < 1$ *and* $\|B\| \leq \sqrt{\frac{\omega^*-1}{\omega^*}} - \|A\|$ *then the matrix* $A+B$ *is* ω^*- *Schur stable.*

Proof. Let $\|A\| < 1$ and A be a ω^*- Schur stable. $\|A\| + \|B\| < 1$ and $\|B\| \leq \sqrt{\frac{\omega^*-1}{\omega^*}} - \|A\|$ are satisfied. From the second inequality

$$\begin{aligned} &\Longrightarrow \|A\| + \|B\| \leq \sqrt{\frac{\omega^*-1}{\omega^*}} \\ &\Longrightarrow (\|A\| + \|B\|)^2 \leq \frac{\omega^*-1}{\omega^*} \\ &\Longrightarrow 1 \leq \omega^*\left(1 - (\|A\| + \|B\|)^2\right) \end{aligned}$$

and therefore the inequality

$$\frac{1}{1-(\|A\|+\|B\|)^2} \leq \omega^*$$

is obtained. Since $\omega(A+B) \leq \frac{1}{1-(\|A\|+\|B\|)^2}$ is valid from Corollary 2.2, the inequality $\omega(A+B) \leq \omega^*$ is found. This completes the proof. □

3. Some Results on the Sensitivity of Scalar-Linear Difference Equations with Order k

Consider the following scalar-linear difference equations with order k

$$x(n+1) - a_0x(n) - a_1x(n-1) - ... - a_{k-1}x(n-k+1) = 0, \qquad n \geq 0. \tag{3.1}$$

The equation (3.1) can be written as

$$y(n+1) = Cy(n), \qquad n \geq 0 \tag{3.2}$$

in matrix-vector form where the matrix C is companion matrix as follows

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 \end{pmatrix}, \qquad c = (a_{k-1}, a_{k-2}, \ldots, a_0).$$

Consider the perturbation of the equation (3.1) and so, of the system (3.2)

$$z(n+1) = (C+D)\, z(n), \qquad n \geq 0, \tag{3.3}$$

and the set B_γ called as the nD$-$ball, i.e. the $n-$dimensional ball [11], where

$$D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ d_{k-1} & d_{k-2} & d_{k-3} & \cdots & d_0 \end{pmatrix}, \quad B_\gamma = \left\{ x = (x_1, x_2, \ldots, x_n) | \ \|x\| < \gamma \right\}.$$

Let

- $d = (d_{k-1}, d_{k-2}, \ldots, d_0)$,
- $\gamma(C) = \max\left\{ \frac{1}{6\pi\omega(C)}, \ \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\| \right\}$,
- $\delta_3^*(C) = \sqrt{\|C\|^2 + \frac{\omega^* - \omega(C)}{\omega^*\omega(C)}} - \|C\|$.

Theorem 3.1 ([10])**.** *Let the system (3.2) be a Schur stable (the companian matrix C is Schur stable). If the $k-$tuple $d \in B_{\gamma(C)}$, then the perturbed system (3.3) is a Schur stable.*

Theorem 3.2 ([10])**.** *Let the system (3.2) be ω^*-Schur stable (the companian matrix C is ω^*-Schur stable). If the $k-$tuple $d \in B_{\delta_3^*(C)}$, then the perturbed system (3.3) is also ω^*-Schur stable.*

Theorem 3.3. $\lim_{\omega\to\infty} B_{\gamma(C)} = \varnothing$.

Proof. The equality $\lim_{\omega\to\infty} \gamma(C) = 0$ holds, where $\lim_{\omega\to\infty} \frac{1}{6\pi\omega(C)} = 0$ and $\lim_{\omega\to\infty} \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\| = 0$.
Thus $\lim_{\omega\to\infty} \{x = (x_1, x_2, \ldots, x_n) | \, \|x\| < \gamma(C)\} = \varnothing$. □

Theorem 3.4. *The set sequence $\left\{B_{\gamma(C)}\right\}$ is increasing according to $\gamma(C)$.*

Proof. Let $x(n+1) = C_1 x(n)$ and $y(n+1) = C_2 y(n)$. If $\gamma_2(C) < \gamma_1(C)$ then the inequality $B_{\gamma_2(C)} \subset B_{\gamma_1(C)}$ holds. □

Theorem 3.5. *The set sequence $\left\{B_{\delta_3^*}\right\}$*
a) *is an increasing sequence the according to ω^*,*
b) *is a bounded sequence.*

Proof. **a)** $\omega_1^*, \ \omega_2^* \ (\omega_1^* < \omega_2^*)$ are practical Schur stability parameters of the system $y(n+1) = Cy(n)$. $\frac{1}{\omega(C)} - \frac{1}{\omega_1^*} = \frac{\omega_1^* - \omega(C)}{\omega_1^*\omega(C)} < \frac{1}{\omega(C)} - \frac{1}{\omega_2^*} = \frac{\omega_2^* - \omega(C)}{\omega_2^*\omega(C)}$ for $\omega_1^* < \omega_2^*$. Therefore $B_{\delta_{31}^*} \subset B_{\delta_{32}^*}$ for $\delta_{31}^* = \sqrt{\|C\|^2 + \frac{\omega_1^* - \omega(C)}{\omega_1^*\omega(C)}} - \|C\| < \delta_{32}^* = \sqrt{\|C\|^2 + \frac{\omega_2^* - \omega(C)}{\omega_2^*\omega(C)}} - \|C\|$.
b) $\varnothing \subset \left\{B_{\delta_3^*}\right\} \subset \left\{B_\alpha\right\}$ for $0 < \delta_3^* = \sqrt{\|C\|^2 + \frac{\omega^* - \omega(C)}{\omega^*\omega(C)}} - \|C\| < \alpha = \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\|$. □

Theorem 3.6. $\lim_{\omega\to\omega^*} B_{\delta_3^*} = \varnothing$.

Proof. Since $\lim_{\omega\to\omega^*} B_{\delta_3^*} = \varnothing$ for $\lim_{\omega\to\omega^*} \delta_3^* = \lim_{\omega\to\omega^*} \sqrt{\|C\|^2 + \frac{\omega^* - \omega(C)}{\omega^*\omega(C)}} - \|C\| = 0$ the proof is obtained. □

Theorem 3.7. $\lim_{\omega^*\to\infty} B_{\delta_3^*} = B_\alpha$ *where* $\alpha = \sqrt{\|C\|^2 + \frac{1}{\omega(C)}} - \|C\|$.

Proof. $\lim_{\omega^*\to\infty} B_{\delta_3^*} = B_\alpha$ for $\lim_{\omega^*\to\infty} \frac{\omega^*-\omega(C)}{\omega^*\omega(C)} = \frac{1}{\omega(C)}$ so the proof is completed. □

4. Numerical Examples

Example 4.1. $A_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0.5 & 9 \\ 0 & 0.1 \end{pmatrix}$.

- $\|A_1\| = 0.5$, $\omega(A_1) = \frac{4}{3}$

We have calculated $\delta^* = \sqrt{\frac{\omega^*-1}{\omega^*}} - \|A\| = 0.494987$ and $\alpha^* = \sqrt{\|A\|^2 + \frac{\omega^*-\omega(A)}{\omega^*\omega(A)}}$ $- \|A\| = 0.494987$ for $\omega^* = 100$. Let us the perturbation matrix $B = \begin{pmatrix} 0.494987 & 0 \\ 0 & 0.494987 \end{pmatrix}$ for $\delta^* = \alpha^* = 0.494987$. We have $A_1 + B$, thus we see that $\omega(A_1 + B) = 99.9913 < 100$ holds, therefore the matrix $A_1 + B$ is $100-$Schur stable matrix.

- $\|A_2\| = 9.01443$, $\omega(A_2) = 121.915$

For $\omega^* = 125$, $\alpha^* = 1.12284e-05$ and Corollary 2.3 fail to apply, therefore the values δ^* cannot be calculated. Suitable perturbation matrices for these values may be selected as $B = \begin{pmatrix} 1.12284e-05 & 0 \\ 0 & 1.12284e-05 \end{pmatrix}$. Hence we obtain $\omega(A_2 + B) = 121.919 < 125$.

Example 4.2. Consider the delay difference equations $x_{n+1} - x_n = -\frac{21}{100}x_{n-1} - \frac{1}{100}x_{n-2}$, $x_{n+1} + x_n = -\frac{1}{10}x_{n-1} - \frac{1}{100}x_{n-2}$ and $x_{n+1} - x_n = -\frac{1}{2}x_{n-1} - \frac{3}{10}x_{n-2}$. The companion matrices $C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{100} & -\frac{21}{100} & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{100} & -\frac{1}{10} & -1 \end{pmatrix}$ and $C_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{3}{10} & -\frac{1}{2} & 1 \end{pmatrix}$.

It is easy to check that $\omega(C_1) = 10.0889$, $\omega(C_2) = 16.4554$, $\omega(C_3) = 33.3264$ and $B_{\gamma_1} = \{(d_2, d_1, 0) | \, \|d\| < 0.0342529\}$, $B_{\gamma_2} = \{(d_2, d_1, 0) | \, \|d\| < 0.0212722\}$, $B_{\gamma_3} = \{(d_2, d_1, 0) | \, \|d\| < 0.0098588\}$. As is clearly seen from Figure 1, $B_{\gamma_3} \subset B_{\gamma_2} \subset B_{\gamma_1}$. Therefore Theorem 3.4 is satisfied.

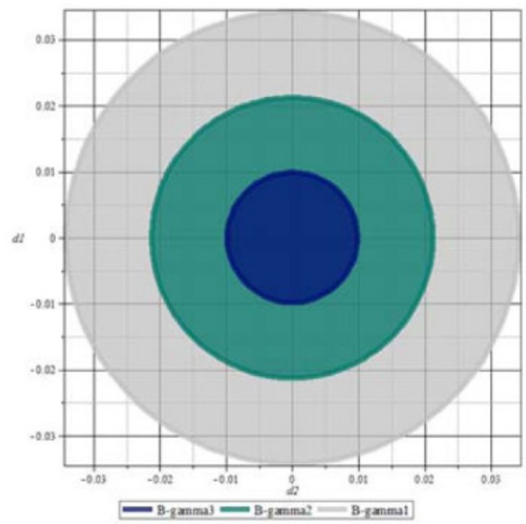

Figure 1. The regions of B_{γ_1}, B_{γ_2} and B_{γ_3}

Since $\omega(C_1) < \infty$ the equation (4) is Schur stable. Let $\omega_1^* = 15$, $\omega_2^* = 60$. It is easy to check that $B_{\delta_{31}^*} = \{(d_2, d_1, 0)| \, \|d\| < 0.0113043\}$, $B_{\delta_{32}^*} = \{(d_2, d_1, 0)| \; \|d\| < 0.0285496\}$ and it is clear that $\omega_1^* < \omega_2^* \Rightarrow B_{\delta_{31}^*} \subset B_{\delta_{32}^*}$. Therefore a option of Theorem 3.5 is satisfied.

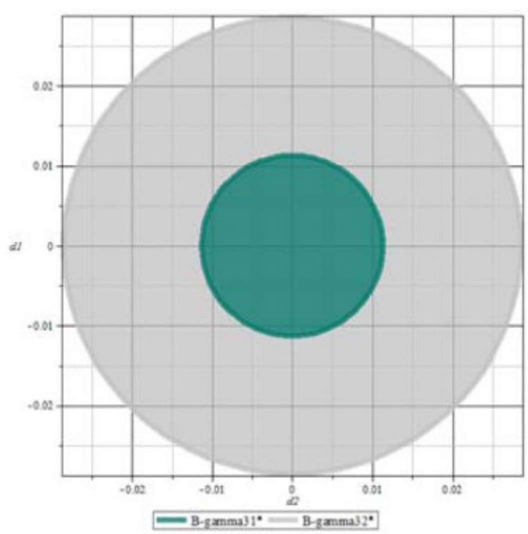

Figure 2. The regions of $B_{\delta_{31}^*}$, and $B_{\delta_{32}^*}$

Remark 4.1. The numerical examples have been computed by using matrix vector calculator MVC [12].

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