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SIMPLICIAL AND CROSSED HOM-LIE ALGEBRA

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Abstract. We introduce the simplicial Hom-Lie algebras and determine their relations among crossed modules of Hom-Lie algebras.

1. Introduction

A Hom algebra structure is a multiplication on a vector space where the structure is twisted by a homomorphism. The structure of Hom-Lie algebra was introduced in [2]. Crossed modules were introduced by Whitehead in [7] as a model for connected homotopy 2-types. After then, crossed modules were used in many branches of mathematics such as category theory, cohomology of algebraic structures, differential geometry and in physics. This makes the crossed modules one of the fundamental algebraic gadget. For some different usage, crossed modules were defined in different categories such as Lie algebras, commutative algebras etc.([5],[3]). Also the crossed modules of Hom-Lie algebras were defined in [6]. The goal of this paper is to define simplicial Hom-Lie algebras and show their relation between the crossed modules over Hom-Lie algebras and the simplicial Hom-Lie algebras.

2. Preliminaries

In the rest of this paper $\mathbb{k}$ will be a fixed field.

Definition 2.1. ([2]) A Hom-Lie algebra is a triple space $(L, [-,-], \alpha_L)$ consisting of a $\mathbb{k}$-vector space L, a skew-symmetric bilinear map $[-,-] : L \times L \longrightarrow L$ and a $\mathbb{k}$-linear map $\alpha_L : L \longrightarrow L$ satisfying the following hom-Jacobi identity;

$$[\alpha_L(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

for all $x, y, z \in L$.

Definition 2.2. A homomorphism of Hom-Lie algebras

$$f : (L, [-,-]_L, \alpha_L) \longrightarrow (L, [-,-]_M, \alpha_M)$$

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is a linear map $f : L \longrightarrow M$ such that

$$f([x,y]_L) = [f(x), f(y)]_M \quad , \quad f \circ \alpha_L = \alpha_M \circ f,$$

for all $x, y \in L$.

Example 2.1. If we take $\alpha = id$, then every Lie algebra L forms a Hom-Lie algebra $(L, [-,-], id)$.

We have the category **HomLie** whose objects are Hom-Lie algebras and whose morphisms are Hom-Lie algebra homomorphism.

So the category **Lie** of Lie algebras is a full subcategory of **HomLie** which gives an inclusion functor **Lie** $\hookrightarrow$ **HomLie**.

From now on we use L instead of $(L, [-,-]_L, \alpha_L)$, for shortness.

3. Crossed Modules of Hom-Lie Algebras

In this section we will recall the action in **HomLie** and the definition of crossed modules from [6]. Also we will adapt some well known examples and results from crossed modules of groups to crossed modules of Hom-Lie algebras.

Definition 3.1. Let L be a Hom-Lie algebra. A Hom-representation of L is a $\Bbbk$-vector space M together with a bilinear map $\rho : L \otimes M \longrightarrow M$, $\rho(l \otimes m) = {}^{l}m$ and a $\Bbbk$-linear map $\alpha_M : M \longrightarrow M$ such that

1. ${}^{[x,y]}\alpha_M(m) = {}^{\alpha_L(x)}({}^{y}m) - {}^{\alpha_L(y)}({}^{x}m)$,
2. $\alpha_M({}^{x}m) = {}^{\alpha_L(x)}(\alpha_M(m))$,

for all $x, y \in L$ and $m \in M$.

Definition 3.2. Let L, M be Hom-Lie algebras and L has an action on M. Then we have the Hom-Lie algebra $(M \rtimes L, \alpha)$ defined on the vector space $M \oplus L$ where $\alpha : M \rtimes L \longrightarrow M \rtimes L$ is defined by $\alpha(m,l) = (\alpha_M(m), \alpha_L(l))$ and the bracket is as follows

$$[(m,l),(m',l')] = [[m,m']_M + {}^{\alpha_L(l)}m' - {}^{\alpha_L(l')}m, [l,l']_L]$$

for all $(m,l),(m',l') \in M \oplus L$.

Definition 3.3. A crossed module of Hom-Lie algebras is Hom-Lie homomorphism $\partial : M \longrightarrow L$ where M is a Hom-representation of L such that

$$\partial({}^{x}m) = [x, \partial m], \qquad {}^{\partial(m)}m' = [m, m'],$$

for all $x \in L, m, m' \in M$.

The crossed module $\partial : M \longrightarrow L$ will be denoted by (M, L, ∂).

Definition 3.4. Let $(M, L, \partial), (M', L', \partial')$ be crossed modules. A homomorphism from (M, L, ∂) to (M', L', ∂') is a pair (μ_1, μ_0) of Hom-Lie homomorphisms such that,

$$\mu_0 \partial = \partial' \mu_1 \quad \text{and} \quad \mu_1({}^{l}m) = {}^{\mu_0(l)}(\mu_1(m)),$$

for all $l \in L, m \in M$.

Consequently, we define the category of crossed modules on Hom-Lie algebras, whose objects are crossed modules of Hom-Lie algebras and whose morphisms are homomorphisms of crossed modules. This category will be denoted by **XHomLie**.

Example 3.1. Let L be a Hom-Lie algebra and I be an ideal of L. I is a Hom-representation of L thanks to the map $\rho : L \otimes I \longrightarrow I$ defined by

$$\rho(l, i) = [l, i],$$

for all $l \in L, i \in I$. This gives rise to the crossed module $(I, L, inc.)$.

Proposition 3.1. *If (M, L, ∂) is a crossed module then $\partial(M)$ is an ideal of L (This is not the case for arbitrary homomorphisms, in general).*

Proof. Since $\partial : M \longrightarrow L$ is a crossed module, we have $[l, \partial m] = \partial({}^{l}m)$ for all $l \in L, m \in M$, as required. □

Example 3.2. Let M be a $\mathbb{k}$-vector space which is also a Hom-representation of a Hom-Lie algebra L. Then $0 : M \longrightarrow L$ is a crossed module. (Here, if M chosen as an arbitrary Hom-Lie algebra, then the Peiffer condition do not satisfied, in general.)

4. Simplicial Hom-Lie Algebras

Let $\triangle$ be the category of finite ordinals. A simplicial Hom-Lie algebra **HL** is a sequence of Hom-Lie algebras

$$\mathbf{HL} = \{HL_0, HL_1, ..., HL_n, ...\}$$

together with face and degeneracy maps

$$\begin{array}{llll} d_i^n : & HL_n & \longrightarrow & HL_{n-1} \ , \ 0 \leq i \leq n \quad (n \neq 0) \\ s_i^n : & HL_n & \longrightarrow & HL_{n+1} \ , \ 0 \leq i \leq n \end{array}$$

which are Hom-Lie homomorphisms satisfying the usual simplicial identities.

4.1. **The Moore Complex.** The Moore complex **NHL** of a simplicial Hom-Lie algebra **HL** is the complex

$$\mathbf{NHL} : \cdots \longrightarrow NHL_n \xrightarrow{\partial_n} NHL_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} NHL_1 \xrightarrow{\partial_1} NHL_0$$

where $NHL_0 = HL_0$, $NHL_n = \bigcap\limits_{i=0}^{n-1} Kerd_i$ and ∂_n is the restriction of d_n to NHL_n.

We say that the Moore complex **NHL** of a simplicial Hom-Lie algebra **HL** is of length k if $NHL_n = 0$, for all $n \geq k+1$. Let $\mathbf{Simp}_{\leq k}(\mathbf{HL})$ be the category whose objects are simplicial Hom-Lie algebras with Moore complex of length k.

4.2. **Truncated Simplicial Hom-Lie Algebras.** The following terminology adapted to simplicial Hom-Lie algebras from [1]. Details of the group case can be found in [1]. A k-truncated simplicial Hom-Lie algebra is a family of Hom-Lie algebras $\{HL_0, HL_1, ..., HL_k\}$ and homomorphism $d_i : HL_n \longrightarrow HL_{n-1}$, $s_i : HL_n \longrightarrow HL_{n+1}$, for each $0 \leq i \leq n$ which satisfy the simplicial identities. We denote the category of k-truncated simplicial Hom-Lie algebras by $\mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL})$. There is a truncation functor tr_k from the category $\mathbf{Simp}(\mathbf{HL})$ to the category $\mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL})$ given by restrictions. This truncation functor has a left adjoint st_k and a right adjoint $cost_k$ called as k-skeleton and k-coskeleton respectively. These adjoints can be pictured as follows;

$$\mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL}) \quad \underset{cost_k}{\overset{tr_k}{\rightleftarrows}} \quad \mathbf{Simp}(\mathbf{HL}) \quad \underset{st_k}{\overset{tr_k}{\rightleftarrows}} \quad \mathbf{Tr}_k\mathbf{Simp}(\mathbf{HL}).$$

see [1] for details about the functors $cost_k$ and st_k.

Theorem 4.1. *The category* **XHomLie** *of crossed modules of Hom-Lie algebras is naturally equivalent to the category* $\mathbf{Simp}_{\leq 1}(\mathbf{HL})$ *of simplicial Hom-Lie algebras with Moore complex of length 1.*

Proof. Let **HL** be a simplicial Hom-Lie algebra with Moore complex of length 1. NHL_1 is a Hom-representation of NHL_0, thanks to the degenerate operator s_0^0. In fact, by using the map $\rho : NHL_0 \otimes NHL_1 \longrightarrow NHL_1$, $(x,a) \longmapsto {}^{x}a := [s_0(x), a]$, we have

$$\begin{aligned} {}^{[x,y]}\alpha_M(m) &= [[s_0x, s_0y], \alpha_M(m)] \\ &= -([\alpha_M(m), [[s_0x, s_0y]]]) \\ &= (\alpha_M s_0 x, [s_0y, m] + [\alpha_M s_0 y, [m, s_0x]]) \\ &= [s_0\alpha_L x, [s_0y, m]] - [s_0\alpha_L y, [s_0x, m]] \\ &= {}^{\alpha_{L(x)}}({}^{y}m) - {}^{\alpha_{L(y)}}({}^{x}m), \end{aligned}$$

and

$$\begin{aligned} \alpha_M({}^{x}m) &= \alpha_M[s_0(x), m] \\ &= [\alpha_M s_0 x, \alpha_M m] \\ &= [s_0\alpha_L(x), \alpha_M(m)], \end{aligned}$$

for all $x, y \in NHL_0, a \in NHL_1$.

Define $\partial := d_1 \mid_{Kerd_0}$. Then (NHL_1, NHL_0, ∂) is a crossed module. We have

$$\begin{aligned} \partial({}^{x}a) &= \partial[s_0(x), a] \\ &= [\partial s_0(x), \partial(a)] \\ &= [x, \partial(a)], \end{aligned}$$

since $d_1^1 s_0^0 = id$. On the other hand, we have

$$\begin{aligned} {}^{\partial(a)}b &= [s_0\partial(a), b] \\ &= [s_0 d_1(a), b] \\ &= [a - a + s_0 d_1(a), b] \\ &= [a, b] - [a + s_0 d_1(a), b] \\ &= [a, b] - [d_2^2 s_1^1 a + d_2^2 s_0^1 a, d_2^2 s_1^1 b] \\ &= [a, b] - d_2^2 [s_1^1 a + s_0^1 a, s_1^1 b] \\ &= [a, b], \end{aligned}$$

for all $a, b \in NHL_1$, since $d_2^2 s_1^1 = id$, $s_0^1 d_1^1 = d_2^2 s_0^1$. Consequently (NHL_1, NHL_0, ∂) is a crossed module. So we obtain the functor

$$X : \mathbf{Simp}_{\leq 1}(\mathbf{HL}) \longrightarrow \mathbf{XHomLie}$$

Conversely, let (M, L, ∂) be a crossed module. Since M is a Hom-representation of L, we have the semi-direct product $M \rtimes L$. Define the maps $d_0 : M \rtimes L \longrightarrow L$, $d_1 : M \rtimes L \longrightarrow L$ and $s_0 : L \longrightarrow M \rtimes L$ by $(m, l) \longmapsto l$, $(m, l) \longmapsto \partial(m) + l$ and $l \longmapsto (0, l)$, respectively. It can be easily showed that these maps are Hom-Lie algebra homomorphisms. So $HL_1 = M \rtimes L$ and $HL_0 = L$. Then

$$HL_1 \quad \begin{matrix} \xrightarrow{d_1} \\ \xrightarrow{d_1} \\ \xleftarrow[s_0]{} \end{matrix} \quad HL_0$$

is a 1-trunated simplicial Hom-Lie algebra. Thus we have the functor

$$T : \mathbf{XHomLie} \longrightarrow \mathbf{Tr}_1\mathbf{Simp}(\mathbf{HL})$$

By using the functor st_k, we have

$$S := st_1T : \mathbf{XHomLie} \longrightarrow \mathbf{Simp}_{\leq 1}(\mathbf{HL})$$

which gives the natural equivalance of the categories **XHomLie** and $\mathbf{Simp}_{\leq 1}(\mathbf{HL})$ with the functor X. □

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